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An Askey–Wilson Algebra of Rank 2

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Abstract. An algebra is introduced which can be considered as a rank 2 extension of the Askey–Wilson algebra. Relations in this algebra are motivated by relations between coproducts of twisted primitive elements in the two-fold tensor product of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. It is shown that bivariate q-Racah polynomials appear as overlap coefficients of eigenvectors of generators of the algebra. Furthermore, the corresponding q-difference operators are calculated using the defining relations of the algebra, showing that it encodes the bispectral properties of the bivariate q-Racah polynomials.

Key words: Askey–Wilson algebra; q-Racah polynomials

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1 Introduction

In this paper, we study a rank 2 version of the Askey–Wilson algebra and its relation to bivariate Askey–Wilson polynomials. The Askey–Wilson algebra AW(3) was introduced by Zhedanov [35] to describe the algebraic structure underlying the Askey–Wilson polynomials [1]. The Askey–Wilson polynomials form an important family of orthogonal polynomials that are on top of the q-Askey scheme [21] of families of (q-)hypergeometric orthogonal polynomials. Every family of polynomials $\{p_n(x)\}$ in this scheme is bispectral; $p_n(x)$ is an eigenfunction of a second order differential or (q-)difference operator in x, as well as an eigenfunction of a second order difference operator in the degree n (the three-term recurrence relation). It is this bispectrality property of the Askey–Wilson polynomials that is encoded in the Askey–Wilson algebra.

Since its introduction, the Askey–Wilson algebra AW(3) has appeared in many contexts. For example, AW(3) is closely related to the rank 1 double affine Hecke algebra of type $\left(C^\vee, C\right)$, Leonard pairs and the q-Onsager algebra [2, 25, 26, 28, 31, 32]. We refer to [4] for an overview of the many interpretations of the Askey–Wilson algebra. There is also a close connection between AW(3) and the centralizer of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))^{\otimes 3}$ [6, 14, 19]. In this interpretation, intermediate Casimir operators generate AW(3), and this also explains the occurrence of q-Racah polynomials, which are essentially Askey–Wilson polynomials on a finite set, as Racah coefficients for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ [15].

The latter interpretation of AW(3) in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))^{\otimes 3}$ has led to the definition of an Askey–Wilson algebra of rank n as the algebra generated by the intermediate Casimir operators in the $(n+2)$-fold tensor product $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))^{\otimes(n+2)}$ [3, 7, 8, 9, 27]. Multivariate versions [11, 12] of Askey–Wilson polynomials, or more precisely of q-Racah polynomials, appear as overlap coefficients in this setting, which is similar to the rank 1 case. This follows from results in [13], where it is shown that the multivariate q-Racah polynomials arise as $3nj$-symbols for $(n+2)$-fold tensor product representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$. In [20], Iliev obtained q-difference operators for the multivariate Askey–Wilson polynomials, and these q-difference operators lead to a realization of a higher rank Askey–Wilson algebra [7].

In this paper, we define an algebra AW_2, a rank 2 version of the Askey–Wilson algebra, as the algebra which is, roughly speaking, generated by $\mathrm{AW}(3) \otimes \mathrm{AW}(3)$ and $\mathrm{AW}(3)$ with certain relations. This definition is motivated by the realization of $\mathrm{AW}(3)$ as a subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ from [15]: $\mathrm{AW}(3) \subseteq \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is generated by two algebra elements that are essentially Koornwinder's [24] twisted primitive elements. In [17], it is shown that Iliev's q-difference operators for the multivariate Askey–Wilson polynomials can be obtained from coproducts of twisted primitive elements in discrete series representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$. This suggests that the coproducts of twisted primitive elements generate a rank n version AW_n of the Askey–Wilson algebra in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))^{\otimes n}$. This construction is different from the construction of the rank n Askey–Wilson algebra $\mathrm{AW}(n+2)$ in [8], where a $(n+2)$-fold tensor product is used. An advantage of the construction of AW_n is that it allows to use directly representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ to construct representations of a higher rank Askey–Wilson algebra. Furthermore, the explicit relations of the generators suggest how to define the higher rank Askey–Wilson algebra without reference to the larger algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))^{\otimes n}$.

From a viewpoint of special functions, the related multivariate Askey–Wilson or q-Racah polynomials also have different interpretations depending on the construction of the algebra. In connection with $\mathrm{AW}(n+2)$ the n-variate q-Racah polynomials are overlap coefficients between $(n+1)$-variate orthogonal polynomials (i.e., one more variable), namely q-Hahn and q-Jacobi polynomials, which arise as nested Clebsch-Gordan coefficients [13]. An analogous result for AW_n is that the n-variate Askey–Wilson polynomials are overlap coefficients between n-variate orthogonal polynomials (i.e., the same number of variables), namely Al-Salam–Chihara polynomials in base q and q^{-1}, see [17], or, in the setting of $\mathcal{U}_q(\mathfrak{su}(2))$, the n-variate q-Racah polynomials are overlap coefficients for n-variate q-Krawtchouk type polynomials.

We consider the rank 2 case in this paper. We show how to obtain bivariate q-Racah polynomials and their bispectral properties from a finite-dimensional representation of our algebra AW_2, so that AW_2 encodes the bispectral properties of the bivariate q-Racah polynomials.

The organization of the paper is as follows. In Section 2, we recall well-known results on Zhedanov's Askey–Wilson algebra $\mathrm{AW}(3)$ that will be used in, and also serve as a motivation for, later sections. In particular, we give the definition of $\mathrm{AW}(3)$, we recall how q-Racah polynomials appear as overlap coefficients of eigenvectors of the two generators of $\mathrm{AW}(3)$, and we show that $\mathrm{AW}(3)$ can be realized as a subalgebra of the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Then, in Section 3, we will be ready to define AW_2, a rank 2 Askey–Wilson algebra. The relations in this algebra come from the relations between (coproducts of) twisted primitive elements and the Casimir in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \otimes \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. We also point out the connection with the rank 2 Askey–Wilson algebra introduced in [8]. In the last two sections, Sections 4 and 5, we will construct a representation of AW_2 where bivariate q-Racah polynomials, similar to the ones defined in [12], appear as overlap coefficients of the generators of AW_2 and find explicitly their difference operator. Calculations concerning the relations between coproducts of twisted primitive elements can be found in Appendices A and B.

2 Zhedanov's Askey–Wilson algebra AW(3)

In this section, we will summarize some results from the Askey–Wilson algebra $\mathrm{AW}(3)$, later in this paper also referred to as the original $\mathrm{AW}(3)$. For more details concerning these results, see [35]. Throughout this paper, we assume that $0 < q < 1$ is fixed.

2.1 The Askey–Wilson algebra

Let AW(3) be the Askey–Wilson algebra defined by Zhedanov [35]. This is the unital, associative, complex algebra generated by three generators K_0, K_1 and K_2 subject to the relations

$$\begin{aligned} &[K_0, K_1]_q = K_2, \\ &[K_1, K_2]_q = BK_1 + C_0K_0 + D_0, \\ &[K_2, K_0]_q = BK_0 + C_1K_1 + D_1, \end{aligned} \tag{2.1}$$

where $B, C_0, C_1, D_0, D_1 \in \mathbb{R}$ are the structure constants of the algebra and $[\cdot,\cdot]_q$ is the so-called q-commutator defined by

$$[X, Y]_q = qXY - q^{-1}YX.$$

By substituting the first equation of (2.1) into the second and third, AW(3) can equivalently be described as the algebra generated by two generators K_0 and K_1. To ease notation later on, we will replace K_0 by K and K_1 by L and define the following functions,

$$\sinh_q(x) := q^x - q^{-x} \qquad \text{and} \qquad \cosh_q(x) := q^x + q^{-x}.$$

Then AW(3) is the algebra generated by K, L subject to the relations

$$\cosh_q(2)KLK - K^2L - LK^2 = BK + C_1L + D_1, \tag{2.2}$$

$$\cosh_q(2)LKL - L^2K - KL^2 = BL + C_0K + D_0. \tag{2.3}$$

Moreover, AW(3) has a Casimir element Q given by

$$\begin{aligned} Q = {}&\big(q^{-1} - q^3\big)KL[K, L]_q + q^2([K, L]_q)^2 + B(KL + LK) + C_0q^2K^2 + C_1q^{-2}L^2 \\ &+ D_0\big(1 + q^2\big)K + D_1\big(1 + q^{-2}\big)L. \end{aligned} \tag{2.4}$$

2.2 Representations of AW(3)

On a finite-dimensional vector space, the generators (K, L) will form a Leonard pair, meaning that L acts as a tridiagonal operator on the eigenfunctions of K and, by symmetry of AW(3), K acts as a tridiagonal operator on the eigenfunctions of L. Moreover, the overlap coefficients of the eigenfunctions of K and L are Askey–Wilson (or q-Racah) polynomials where the parameters depend on the structure parameters B, C_0, C_1, D_0, D_1 and the dimension of the representation. We refer to [33, 34] for a description of the relation between Leonard pairs and finite-dimensional representations of AW(3). Furthermore, for a complete classification of the finite-dimensional irreducible representations of the universal Askey–Wilson algebra, a central extension of AW(3), we refer to [18].

Let V be a $(N+1)$-dimensional vector space. The sign of the constants C_0 and C_1 determine the form of the spectra of the generators L and K respectively. That is, the spectrum of K is of the form $\sinh_q(x)$ if $C_1 > 0$, $\cosh_q(x)$ if $C_1 < 0$ and q^x if $C_1 = 0$ and similarly for L and C_0. Each of the 9 cases can be treated similarly. We will focus on the case where $C_0, C_1 < 0$. Under the so-called 'quantization condition', which we will mention later, there is an irreducible $(N + 1)$-dimensional representation of AW(3). By rescaling our generators K and L, we can assign any value, while keeping the same sign, to C_0 and C_1. We will choose the canonical form of AW(3), where

$$C_0 = C_1 = -(\sinh_q(2))^2.$$

Then we have the following representation of AW(3): there exist eigenvectors $\{\psi_n\}_{n=0}^N$ of K such that

$$K\psi_n = \lambda_n\psi_n, \tag{2.5}$$
$$L\psi_n = a_n\psi_{n-1} + b_n\psi_n + a_{n+1}\psi_{n+1}. \tag{2.6}$$

The eigenvalues λ_n and coefficients a_n, b_n are given by

$$\lambda_n = \sinh_q(2n + p_0 + 1), \tag{2.7}$$
$$a_n^2 = \frac{-\prod_{k=0}^3(\sinh_q(2n+p_0) - \sinh_q(p_k))}{\cosh_q(2n+p_0)^2\cosh_q(2n+p_0+1)\cosh_q(2n+p_0-1)}, \tag{2.8}$$
$$b_n = \frac{B\lambda_n + D_1}{(\lambda_n - \lambda_{n-1})(\lambda_{n+1} - \lambda_n)}. \tag{2.9}$$

Here, $(p_k)_{k=0}^3$ are roots of the *characteristic polynomial* $\mathscr{P}\colon \mathbb{C} \to \mathbb{C}$ of AW(3) given by

$$\begin{aligned}\mathscr{P}(z) &= C_0\frac{(\sinh_q(1))^2}{(\cosh_q(1))^4}z^4 + D_0\frac{(\sinh_q(1))^2}{(\cosh_q(1))^2}z^3 \\ &\quad + \left(\frac{B^2 - (\sinh_q(1))^2Q_0}{(\cosh_q(1))^2} + \frac{\left((\sinh_q(1))^2 - 4\right)C_0C_1}{(\cosh_q(1))^4}\right)z^2 \\ &\quad + \left(BD_1 - \frac{4C_1D_0}{(\cosh_q(1))^2}\right)z + D_1^2 + C_1\frac{B^2 + 4Q_0}{(\cosh_q(1))^2} - \frac{4C_0C_1^2}{(\cosh_q(1))^4},\end{aligned} \tag{2.10}$$

where Q_0 is the representation value of the Casimir Q from (2.4). Moreover, the structure constants B, D_0 and D_1 as well as the value of the Casimir Q can be expressed in terms of the roots of the characteristic polynomial $\mathscr{P}$.

Proposition 2.1. *We have*

$$\begin{aligned}B &= (\sinh_q(1))^2\left(\sinh_q\left(\frac{p_0+p_1}{2}\right)\sinh_q\left(\frac{p_2+p_3}{2}\right) - \cosh_q\left(\frac{p_0-p_1}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right)\right),\\ D_0 &= \frac{(\sinh_q(2))^2}{\cosh_q(1)}\left(\sinh_q\left(\frac{p_0+p_1}{2}\right)\cosh_q\left(\frac{p_0-p_1}{2}\right) + \sinh_q\left(\frac{p_2+p_3}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right)\right),\\ D_1 &= \frac{(\sinh_q(2))^2}{\cosh_q(1)}\left(\sinh_q\left(\frac{p_2+p_3}{2}\right)\cosh_q\left(\frac{p_0-p_1}{2}\right) + \sinh_q\left(\frac{p_0+p_1}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right)\right),\\ Q_0 &= (\sinh_q(2))^2\left(\prod_{k=0}^3\sinh_q\left(\frac{1}{2}p_k\right) + (\sinh_q(1))^2 - \frac{(\sinh_q(1))^2B + D_1^2}{(\sinh_q(1))^2(\sinh_q(2))^2}\right).\end{aligned}$$

Proof. This is similar to [35, equation (2.4)]. We have a slightly different expression since we are in the case $C_0, C_1 < 0$ instead of $C_0, C_1 > 0$. ∎

Since our representation is finite-dimensional, we require that $a_0 = a_{N+1} = 0$. This leads to the 'quantization condition' $2(N+1) = p_1 - p_0$. When the spectrum of K is fixed, the representation is unique up to equivalence since we can calculate the matrix coefficients of L using [35, equation (1.12)] and (2.9). The eigenvalues λ_n satisfy the following recursive relations, which we will need later on,

$$\begin{aligned}&\lambda_n^2 + \lambda_{n+1}^2 = \cosh_q(2)\lambda_n\lambda_{n+1} - C_1,\\ &\cosh_q(2)\lambda_n = \lambda_{n+1} + \lambda_{n-1},\\ &\lambda_n^2 = \lambda_{n+1}\lambda_{n-1} - C_1.\end{aligned} \tag{2.11}$$

Moreover, using the symmetry of AW(3), we can interchange the roles of K and L and get a similar result. That is, there exist eigenvectors $\{\phi_n\}_{n=0}^N$ of L such that

$$L\phi_m = \mu_m\phi_m, \qquad K\phi_m = \tilde{a}_m\phi_{m-1} + \tilde{b}_n\phi_m + \tilde{a}_{m+1}\phi_{m+1}, \tag{2.12}$$

where $(\mu_m, \tilde{a}_m, \tilde{b}_m)$ can be found from the formulas for (λ_n, a_n, b_n) after interchanging $C_0 \leftrightarrow C_1$, $D_0 \leftrightarrow D_1$ and $p_k \leftrightarrow s_k$, where s_k are the roots of the polynomial (2.10) where $C_0 \leftrightarrow C_1$ and $D_0 \leftrightarrow D_1$. The p_k and s_k are linked via

$$s_0 = \frac{1}{2}\Sigma_p - p_1, \qquad s_1 = \frac{1}{2}\Sigma_p - p_0, \qquad s_2 = \frac{1}{2}\Sigma_p - p_3, \qquad s_3 = \frac{1}{2}\Sigma_p - p_1, \qquad \Sigma_p = \sum_{k=0}^{3} p_k.$$

2.3 q-Racah polynomials as overlap coefficients of the generators of AW(3)

If the structure parameters B, C_0, C_1, D_0 and D_1 are real and $a_n^2 \geq 0$, we have that λ_n, a_n, b_n are real as well. We then define an inner product on V on the basis $\{\psi_n\}_{n=0}^N$ by

$$\langle\psi_n, \psi_m\rangle = \delta_{mn}.$$

Then both K and L are self-adjoint with respect to this inner product. Consequently, the sets of eigenvectors, $\{\psi_n\}_{n=0}^N$ as well as $\{\phi_n\}_{n=0}^N$, will per definition form an orthonormal basis. Define the normalized overlap coefficients $\widehat{P}_n(m)$ to be

$$\widehat{P}_n(m) = \frac{\langle\phi_m, \psi_n\rangle}{\langle\phi_m, \psi_0\rangle}, \qquad n, m = 0, \dots, N.$$

Since

$$\begin{aligned} \mu_m\langle\phi_m, \psi_n\rangle &= \langle L\phi_m, \psi_n\rangle = \langle\phi_m, L\psi_n\rangle \\ &= a_n\langle\phi_m, \psi_{n-1}\rangle + b_n\langle\phi_m, \psi_n\rangle + a_{n+1}\langle\phi_m, \psi_{n+1}\rangle, \end{aligned} \tag{2.13}$$

the overlap coefficients satisfy

$$\mu_m\widehat{P}_n(m) = a_n\widehat{P}_{n-1}(m) + b_n\widehat{P}_n(m) + a_{n+1}\widehat{P}_{n+1}(m). \tag{2.14}$$

Together with the initial condition $\widehat{P}_0(m) = 1$ and the convention $\widehat{P}_{-1}(m) = 0$, (2.14) generates polynomials $\big(\widehat{P}_n\big)_{n=0}^N$ in the variable μ_m, which can be shown [35] to be q-Racah polynomials with parameters that depend on $(p_k)_{k=0}^3$. The q-Racah polynomials $\{R_n\}_{n=0}^N$ are defined [21] by a q-hypergeometric series,

$$R_n(y_j; \alpha, \beta, \gamma, \delta; q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-j}, \gamma\delta q^{j+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix}; q; q\right], \tag{2.15}$$

where

$$y_j = q^{-j} + \gamma\delta q^{j+1}.$$

Then we have

$$\widehat{P}_n(m) = \frac{1}{\sqrt{h_n}} R_n\big(y_m; \alpha, \beta, \gamma, \delta; q^2\big), \tag{2.16}$$

where h_n are normalizing constants and

$$\alpha = -q^{p_0+p_2}, \qquad \beta = q^{p_0-p_2}, \qquad \gamma = q^{p_0-p_1}, \qquad \delta = -q^{p_2+p_3}. \tag{2.17}$$

We will rescale the overlap coefficients $\widehat{P}_n(m)$ such that it becomes a q-Racah polynomial without the normalizing constant $\sqrt{1/h_n}$, which will be convenient later on. That is, we define

$$P_n(m) = \frac{\langle \phi_m, \psi_n \rangle}{\langle \phi_m, \psi_0 \rangle} \frac{\langle \phi_0, \psi_0 \rangle}{\langle \phi_0, \psi_n \rangle}. \tag{2.18}$$

This gives on the one hand

$$P_n(0) = 1.$$

On the other hand, by (2.16), we have

$$P_n(0) = \frac{1}{\sqrt{h_n}} R_n\big(y_0; \alpha, \beta, \gamma, \delta; q^2\big) \frac{\langle \phi_0, \psi_0 \rangle}{\langle \phi_0, \psi_n \rangle} = \frac{1}{\sqrt{h_n}} \frac{\langle \phi_0, \psi_0 \rangle}{\langle \phi_0, \psi_n \rangle},$$

since by the definition of the q-hypergeometric series we have

$$R_n\big(y_0; \alpha, \beta, \gamma, \delta; q^2\big) = 1.$$

Therefore,

$$\frac{\langle \phi_0, \psi_0 \rangle}{\langle \phi_0, \psi_n \rangle} = \sqrt{h_n}$$

and thus

$$P_n(m) = R_n\big(y_m; \alpha, \beta, \gamma, \delta; q^2\big).$$

Since $\{\phi_m\}_{m=0}^N$ is an orthonormal basis, we have

$$\psi_n = \sum_{m=0}^N \langle \psi_n, \phi_m \rangle \, \phi_m.$$

Orthogonality in the degree of the q-Racah polynomials comes from $\{\psi_n\}_{n=0}^N$ being orthonormal,

$$\delta_{n,n'} = \langle \psi_{n'}, \psi_n \rangle = \sum_{m=0}^N \langle \psi_{n'}, \phi_m \rangle \langle \phi_m, \psi_n \rangle = \sum_{m=0}^N \tilde{w}(m,n) P_n(m) \overline{P_{n'}(m)}, \tag{2.19}$$

where $\tilde{w}(m,n)$ is the weight function given by

$$\tilde{w}(m,n) = \frac{|\langle \phi_m, \psi_0 \rangle \langle \phi_0, \psi_n \rangle|^2}{|\langle \phi_0, \psi_0 \rangle|^2}.$$

Since the orthogonality for q-Racah polynomials is unique, we have (see, e.g., [21]),

$$\tilde{w}(m,n) := \tilde{w}(m,n,p_0,p_1,p_2,p_3;q) = \frac{\rho\big(m,\alpha,\beta,\gamma,\delta;q^2\big)}{h_n(\alpha,\beta,\gamma,\delta;q^2)}, \tag{2.20}$$

where

$$\rho(m,\alpha,\beta,\gamma,\delta;q) = \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_m \big(1-\gamma\delta q^{2m+1}\big)}{\big(q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q\big)_m (\alpha\beta q)^m (1-\gamma\delta q)}, \tag{2.21}$$

and

$$\begin{aligned} h_n(\alpha,\beta,\gamma,\delta;q) = {} & \frac{\big(\alpha^{-1}, \beta^{-1}\gamma, \alpha^{-1}\delta, \beta^{-1}, \gamma\delta q^2; q\big)_\infty}{\big(\alpha^{-1}\beta^{-1}q^{-1}, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q\big)_\infty} \\ & \times \frac{(1-\alpha\beta q)(\gamma\delta q)^n \big(q, \alpha\beta\gamma^{-1}q, \alpha\delta^{-1}q, \beta q; q\big)_n}{\big(1-\alpha\beta q^{2n+1}\big)(\alpha q, \alpha\beta q, \beta\delta q, \gamma q; q)_n}. \end{aligned} \tag{2.22}$$

In (2.20), the roots $(p_k)_{k=0}^3$ are related to $(\alpha,\beta,\gamma,\delta)$ via (2.17).

As can be seen, the parameters of the q-Racah polynomials can be simply computed from the roots $(p_k)_{k=0}^3$. In Section 2.5, we will reparametrize the structure constants B, C_0, C_1, D_0, D_1 such that $(p_k)_{k=0}^3$ can be easily determined from the structure parameters.

2.4 AW(3) as subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$

Let $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ be the unital, associative, complex algebra generated by[1] $\hat{K}$, $\hat{E}$, $\hat{F}$, $\hat{K}^{-1}$ subject to the relations

$$\hat{K}\hat{K}^{-1} = 1 = \hat{K}^{-1}\hat{K}, \qquad \hat{K}\hat{E} = q\hat{E}\hat{K}, \qquad \hat{K}\hat{F} = q^{-1}\hat{F}\hat{K}, \qquad \hat{E}\hat{F} - \hat{F}\hat{E} = \frac{\hat{K}^2 - \hat{K}^{-2}}{q - q^{-1}}.$$

The quantum algebra $\mathcal{U}_q$ has a comultiplication $\Delta\colon \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ defined on the generators by

$$\begin{aligned} &\Delta(\hat{K}) = \hat{K} \otimes \hat{K}, && \Delta(\hat{E}) = \hat{K} \otimes \hat{E} + \hat{E} \otimes \hat{K}^{-1}, \\ &\Delta(\hat{K}^{-1}) = \hat{K}^{-1} \otimes \hat{K}^{-1}, && \Delta(\hat{F}) = \hat{K} \otimes \hat{F} + \hat{F} \otimes \hat{K}^{-1}. \end{aligned}$$

Moreover, $\mathcal{U}_q$ has a Casimir element given by

$$\Omega = q^{-1}\hat{K}^2 + q\hat{K}^{-2} + (\sinh_q(1))^2\hat{E}\hat{F} = q\hat{K}^2 + q^{-1}\hat{K}^{-2} + (\sinh_q(1))^2\hat{F}\hat{E},$$

which commutes with all elements in $\mathcal{U}_q$. Note that $\Delta(\Omega)$ is not a central element of $\mathcal{U}_q \otimes \mathcal{U}_q$. Actually, we will see in Section 5 that it acts simultaneously as a three-term operator on the eigenvectors of $1 \otimes Y_K$ as well as $Y_L \otimes 1$. Two elements of $\mathcal{U}_q$, which are closely related to Koornwinder's twisted primitive elements [24], play an important role in this paper. Let $a_{\hat{E}}, a_{\hat{F}}, a_s, b_{\hat{E}}, b_{\hat{F}}, b_t \in \mathbb{C}$, then we define (suggestively) the following elements in $\mathcal{U}_q$,

$$\begin{aligned} Y_K &= q^{\frac{1}{2}} a_{\hat{E}} \hat{E}\hat{K} + q^{-\frac{1}{2}} a_{\hat{F}} \hat{F}\hat{K} + a_s \hat{K}^2, \\ Y_L &= q^{-\frac{1}{2}} b_{\hat{E}} \hat{E}\hat{K}^{-1} + q^{\frac{1}{2}} b_{\hat{F}} \hat{F}\hat{K}^{-1} + b_t \hat{K}^{-2}. \end{aligned}$$

They behave quite well with respect to coproduct,[2]

$$\begin{aligned} \Delta(Y_K) &= K^2 \otimes Y_K + \big(Y_K - a_s\hat{K}^2\big) \otimes 1, \\ \Delta(Y_L) &= 1 \otimes \big(Y_L - b_t\hat{K}^{-2}\big) + Y_L \otimes K^{-2}. \end{aligned} \tag{2.23}$$

From (2.23) one can see that $1 \otimes Y_K$ commutes with $\Delta(Y_K)$ and $Y_L \otimes 1$ commutes with $\Delta(Y_L)$.

Note that Y_K and Y_L relate to the twisted primitive elements $Y_{s,u}$ and $\tilde{Y}_t$ from [17] by adding a constant such that the unital term cancels[3] and taking $a_{\hat{E}} = u$, $a_{\hat{F}} = -u^{-1}$ and $b_{\hat{E}} = b_{\hat{F}} = 1$. Thus, shifting the eigenvalues by a constant while keeping the same eigenvectors.

The elements Y_K and Y_L satisfy the AW-relations [15], i.e., they satisfy both (2.2) and (2.3).

Theorem 2.2. *The elements $Y_K, Y_L \in \mathcal{U}_q$ satisfy the AW-relations* (2.2) *and* (2.3) *with structure parameters*

$$\begin{aligned} B &= (\sinh_q(1))^2(a_s b_t - \theta\Omega), \\ C_0 &= -(\cosh_q(1))^2 b_{\hat{E}} b_{\hat{F}}, \\ C_1 &= -(\cosh_q(1))^2 a_{\hat{E}} a_{\hat{F}}, \\ D_0 &= \cosh_q(1)\big(a_s\Omega b_{\hat{E}} b_{\hat{F}} + (\sinh_q(1))^2 b_t\theta\big), \\ D_1 &= \cosh_q(1)\big(b_t\Omega a_{\hat{E}} a_{\hat{F}} + (\sinh_q(1))^2 a_s\theta\big), \end{aligned} \tag{2.24}$$

where $\theta = -(\sinh_q(1))^{-2}\big(a_{\hat{E}} b_{\hat{F}} + a_{\hat{F}} b_{\hat{E}}\big)$.

[1] Note that the $\hat{K} \in \mathcal{U}_q$ is different from $K \in \mathrm{AW}(3)$.

[2] The subspaces $\mathrm{span}\{Y_K, 1\}$ and $\mathrm{span}\{Y_L, 1\}$ satisfy the right and left co-ideal property respectively.

[3] $Y_{s,u}$ and $\tilde{Y}_t$ satisfy more general AW-relations, which are more complicated to work with but are in essence the same object.

Remark 2.3. Note that B, D_0, D_1 are no longer constants due to the appearance of the Casimir Ω. However, they remain central elements of $\mathcal{U}_q$. Therefore, Y_K and Y_L formally do not generate an AW(3) algebra. However, they do generate a universal Askey–Wilson algebra [29], which is the central extension of AW(3) where B, C_0, C_1, D_0, D_1 are central elements of the algebra instead of constants.

The representation of almost any instance of AW(3) can be realized by the pair (Y_K, Y_L). The cases where either D_0 or D_1 is the only non-zero parameter have to be excluded. Since in an irreducible representation, Ω is a constant, we define the algebra $\mathcal{U}_q(\Omega_0)$, where we add an extra relation to $\mathcal{U}_q$ where Ω is equal to some $\Omega_0 \in \mathbb{C}$. That is,

$$\Omega = \Omega_0 \cdot 1, \qquad \Omega_0 \in \mathbb{C}, \qquad 1 \in \mathcal{U}_q(\Omega_0).$$

Using this, we can show that almost every instance of AW(3) is homomorphic to a subalgebra of $\mathcal{U}_q(\Omega_0)$.

Proposition 2.4. *Let $(B, C_0, C_1, D_0, D_1) \in \mathbb{R}^5$ be structure constants of* AW(3) *such that D_0 or D_1 is not the only non-zero constant. Then there exist $Y_K, Y_L \in \mathcal{U}_q(\Omega_0)$ such that* AW(3) *is homomorphic to the subalgebra of $\mathcal{U}_q(\Omega_0)$ generated by Y_K and Y_L.*

Proof. Let $B, C_0, C_1, D_0, D_1 \in \mathbb{R}$ such that D_0 or D_1 is not the only non-zero constant. We will show, using Theorem 2.2, that there exist $Y_K, Y_L \in \mathcal{U}_q(\Omega_0)$ which satisfy the AW-relation with parameters B, C_0, C_1, D_0, D_1. Let us fix $\Omega_0 \neq 0$ and take

$$b_{\hat{E}} = -\frac{C_0}{(\cosh_q(1))^2 b_{\hat{F}}}, \qquad a_{\hat{F}} = -\frac{C_1}{(\cosh_q(1))^2 a_{\hat{E}}} \tag{2.25}$$

and

$$\theta = -(\sinh_q(1))^{-2}\big(a_{\hat{E}} b_{\hat{F}} + a_{\hat{F}} b_{\hat{E}}\big) = -(\sinh_q(1))^{-2}\left(a_{\hat{E}} b_{\hat{F}} + \frac{C_0 C_1}{(\cosh_q(1))^4 a_{\hat{E}} b_{\hat{F}}}\right). \tag{2.26}$$

For arbitrary C_0, C_1, θ, we can find $a_{\hat{E}}$, $a_{\hat{F}}$, $b_{\hat{E}}$, $b_{\hat{F}}$ that satisfy (2.25) and (2.26). Therefore, we have to solve the following three equations

$$\begin{aligned}
B &= (\sinh_q(1))^2 (a_s b_t - \theta\Omega_0), \\
D_0 &= \cosh_q(1)\left(-\frac{C_0 a_s \Omega_0}{(\cosh_q(1))^2} + (\sinh_q(1))^2 \theta b_t\right), \\
D_1 &= -\cosh_q(1)\left(-\frac{C_1 b_t \Omega_0}{(\cosh_q(1))^2} + (\sinh_q(1))^2 \theta a_s\right),
\end{aligned}$$

where we have three variables θ, a_s, b_t. Let us assume that $C_0, C_1 \neq 0$. Then we can rewrite above equations to the following form,

$$x_1 - a_1 x_2 x_3 = b_1, \tag{2.27}$$
$$x_2 - a_2 x_1 x_3 = b_2, \tag{2.28}$$
$$x_3 - a_3 x_1 x_2 = b_3, \tag{2.29}$$

with three variables $x_1 = \theta$, $x_2 = a_s$, $x_3 = b_t$ and where $a_1, a_2, a_3 \neq 0$, since $C_0, C_1, \Omega_0 \neq 0$. We will show the above system is consistent. We can eliminate x_1 by substituting (2.27) into (2.28). We obtain

$$x_2 - a_2(b_1 + a_1 x_2 x_3) x_3 = b_2, \qquad x_3 - a_3(b_1 + a_1 x_2 x_3) x_2 = b_3,$$

which we can rewrite to the system

$$x_2 - c_1 x_2 (x_3)^2 = d_1, \tag{2.30}$$
$$x_3 - c_2 (x_2)^2 x_3 = d_2, \tag{2.31}$$

where $c_1, c_2 \neq 0$, since a_1, a_2, a_3 are non-zero. The first equation implies

$$x_2 = \frac{d_1}{1 - c_1 (x_3)^2}. \tag{2.32}$$

If $x_3 \neq \pm (c_1)^{-1/2}$, we can substitute this into the second. We then get

$$\frac{(x_3 - d_2)\big(1 - c_1(x_3)^2\big)^2 - c_2 (d_1)^2 x_3}{(1 - c_1(x_3)^2)^2} = 0. \tag{2.33}$$

Since $c_1 \neq 0$, the numerator is a polynomial of degree 5 in x_3. It always has five (in general complex) roots, since $\mathbb{C}$ is algebraically closed. If it has a root which is not of the form $x_3 = \pm (c_1)^{-1/2}$, we can use (2.32) to find x_2 and (2.27) to find x_1 and we are done. If it has a solution of the form $x_3 = \pm (c_1)^{-1/2}$, it follows from (2.33) that

$$\pm c_2 (d_1)^2 (c_1)^{-1/2} = 0.$$

Since $c_1, c_2 \neq 0$, this implies $d_1 = 0$. However, in that case the system (2.30) and (2.31) can be solved by taking $x_2 = 0$ and $x_3 = d_2$.

If $C_0 = 0$, $C_1 = 0$ or both, the system (2.27)–(2.29) becomes

$$x_1 - a_1 x_2 x_3 = b_1, \qquad (1 - \delta_{C_0 0}) x_2 - a_2 x_1 x_3 = b_2, \qquad (1 - \delta_{C_1 0}) x_3 - a_3 x_1 x_2 = b_3.$$

This can be solved similarly as before, the only two exceptions being $C_0 = C_1 = b_1 = b_2 = 0$, $b_3 \neq 0$ and $C_0 = C_1 = b_1 = b_3 = 0$, $b_2 \neq 0$. If we are in one of those cases, say the first, we obtain

$$x_1 - a_1 x_2 x_3 = 0, \tag{2.34}$$
$$-a_2 x_1 x_3 = 0, \tag{2.35}$$
$$-a_3 x_1 x_2 = b_3. \tag{2.36}$$

The second equation (2.35) demands x_1 or x_3 to be zero, while the third one (2.36) requires x_1 and x_2 to be non-zero. Therefore, we need $x_3 = 0$. However, then (2.34) tells us that $x_1 = 0$ has to hold, which violates (2.36). This corresponds to the case $B = C_0 = C_1 = D_0 = 0$ and $D_1 \neq 0$, for which no Y_K, Y_L exist that generate the corresponding AW(3). ■

2.5 Reparametrizing AW(3)

In this subsection we will redefine the structure parameters (B, C_0, C_1, D_0, D_1) of AW(3) into a way that is similar to (2.24). We introduce new structure parameters $(A_0, \dots, A_5)$ by

$$\begin{aligned} &B = (\sinh_q(1))^2 (A_0 A_1 - A_2 A_3), \qquad C_0 = -(\cosh_q(1))^2 A_4, \qquad C_1 = -(\cosh_q(1))^2 A_5, \\ &D_0 = \cosh_q(1)(A_0 A_3 A_4 + (\sinh_q(1))^2 A_1 A_2), \\ &D_1 = \cosh_q(1)(A_1 A_3 A_5 + (\sinh_q(1))^2 A_0 A_2). \end{aligned} \tag{2.37}$$

Compared with (2.24), one should think of the correspondence

$$a_s = A_0, \qquad b_t = A_1, \qquad \theta = A_2, \qquad \Omega = A_3.$$

This notation is convenient for two reasons. First of all, similarly as in AW(3) we have that the overlap coefficients of Y_K and Y_L are univariate Askey–Wilson polynomials [17] with parameters depending explicitly on a_s, b_t, Ω and θ. Looking at how these parameters appear in (2.24), we will now see that the overlap coefficients of K and L are q-Racah polynomials that depend on $(A_k)_{k=0}^3$ in a simple way.

Secondly, we will see in the rank 2 Askey–Wilson algebra defined later on in Section 3, that B, D_0 and D_1 are not necessarily central elements anymore, but B, D_0, D_1 have precisely the structure of (2.37).

To see the convenience of the reparametrized structure parameters, let us consider a finite-dimensional representation from Section 2.2, where $N+1$ is the dimension of the representation and $C_0 = C_1 = -(\sinh_q(2))^2$. Let $(\alpha_k)_{k=0}^3$ be real numbers and let

$$\begin{aligned} &A_0 = \sinh_q(\alpha_0), \qquad A_1 = \sinh_q(\alpha_1), \qquad A_2 = \cosh_q(\alpha_2),\\ &A_3 = \cosh_q(\alpha_3), \qquad A_4 = A_5 = (\sinh_q(1))^2. \end{aligned} \tag{2.38}$$

Then one can show that $(p_k)_{k=0}^3$ and $(\alpha_k)_{k=0}^3$ relate via a simple linear transformation.

Proposition 2.5. *Let $C_0 = C_1 = -(\sinh_q(2))^2$, then we have*

$$p_0 = \alpha_0 + \alpha_3, \qquad p_1 = \alpha_0 - \alpha_3, \qquad p_2 = \alpha_1 + \alpha_2, \qquad p_3 = \alpha_1 - \alpha_2, \tag{2.39}$$

where $\alpha_3 = -N-1$ with $N \in \mathbb{N}$ the dimension of the representation and $\alpha_0 + \alpha_3$ is the starting parameter of the spectrum of K.

Proof. The structure parameters B, D_0 and D_1 can be written in terms of the parameters $(p_k)_{k=0}^3$, see Proposition 2.1. Together with requiring that the spectrum of K is of the form

$$\sinh_q(2n + p_0), \qquad n = 0, \dots, N,$$

and the quantization condition $p_1 - p_0 = 2(N+1)$ this uniquely determines the roots $(p_k)_{k=0}^3$. We can easily compare the expressions for B, D_0, D_1 from Proposition 2.1 with (2.37), using that $\sinh_q(2) = \cosh_q(1)\sinh_q(1)$. This gives

$$p_0 + p_1 = 2\alpha_0, \qquad p_2 + p_3 = \alpha_1, \qquad p_0 - p_1 = 2\alpha_3, \qquad p_2 - p_3 = 2\alpha_2.$$

Rewriting this leads to (2.39). ■

Furthermore, the condition for finite-dimensional representations given by $p_1 - p_0 = 2(N+1)$, now leads to $\alpha_3 = -(N+1)$. Consequently, substituting this into (2.7) gives that the spectrum of K can be written as

$$\lambda_{n(k)} = \sinh_q(2n(k) + \alpha_0), \qquad n(k) = k - \frac{N}{2}, \qquad k \in \{0, \dots, N\}, \tag{2.40}$$

where k will be the degree of the resulting q-Racah polynomial. Similarly, the spectrum of L is given by

$$\mu_{m(j)} = \sinh_q(2m(j) + \alpha_1), \qquad m(j) = j - \frac{N}{2}, \qquad j \in \{0, \dots, N\}, \tag{2.41}$$

where j will determine the variable of the q-Racah polynomial. Since the form of the spectra will remain the same in the rank 2 case, we define

$$\lambda_x = \sinh_q(2x + \alpha_0) \qquad \text{and} \qquad \mu_x = \sinh_q(2x + \alpha_1). \tag{2.42}$$

The parameters A_4 and A_5 are similar to the roles of C_0 and C_1. That is, by rescaling the generators K and L we can assign any value, while keeping the same sign, to A_4 and A_5. This sign determines the form of the spectrum: positive leads to $\sinh_q$, while negative gives $\cosh_q$.

In summary, we have the following interpretation of the new structure parameters:

- $\alpha_0 \in \mathbb{R}$ determines the spectrum of K,
- $\alpha_1 \in \mathbb{R}$ determines the spectrum of L,
- $\alpha_3 \in \mathbb{Z}_+$ determines the dimension of the representation,
- The sign of A_4 determines the form of the spectrum of L,
- The sign of A_5 determines the form of the spectrum of K.

The overlap coefficients of the eigenfunctions of K and L are q-Racah polynomials with parameters that depend in a simple way on $(p_k)_{k=0}^3$ and the $(p_k)_{k=0}^3$ are in turn related to the new structure parameters $(\alpha_k)_{k=0}^3$ via (2.39). Therefore, we see that the q-Racah polynomials arising from the overlap coefficients of K and L can be simply computed from $(\alpha_k)_{k=0}^3$.

Corollary 2.6. *Let $P_{n(k)}(m(j))$ be the overlap coefficients from* (2.18)*. If $\alpha_3 = -N-1$ with $N \in \mathbb{N}$, we have*

$$P_{n(k)}(m(j)) = R_k\big(y_j; \alpha, \beta, \gamma, \delta; q^2\big), \qquad k, j = 0, \dots, N.$$

Here, $R_k(y_j; \alpha, \beta, \gamma, \delta; q)$ are the q-Racah polynomials from (2.15) *and*

$$\alpha = -q^{\alpha_0+\alpha_1+\alpha_2+\alpha_3}, \qquad \beta = q^{\alpha_0-\alpha_1-\alpha_2+\alpha_3}, \qquad \gamma = q^{2\alpha_3}, \qquad \delta = -q^{2\alpha_1}. \tag{2.43}$$

Proof. This is just (2.17) combined with (2.39). ■

Similar to (2.19), we have

$$\delta_{n(k),n(k')} = \sum_{j=0}^{N} w(j,k) P_{n(k)}(m(j)) \overline{P_{n(k')}(m(j))},$$

where

$$w(j,k) := w(j,k,\alpha_0,\alpha_1,\alpha_2,\alpha_3;q) = \frac{\rho\big(j,\alpha,\beta,\gamma,\delta;q^2\big)}{h_n\big(\alpha,\beta,\gamma,\delta : q^2\big)}. \tag{2.44}$$

Here, the parameters $(\alpha_k)_{k=0}^3$ are related to $(\alpha,\beta,\gamma,\delta)$ via (2.43) and the functions ρ and h_n are given in (2.21) and (2.22) respectively.

Moreover, a_n and b_n from (2.6) depend explicitly on α_0, α_1, α_2, N. This can be seen by substituting (2.37) and (2.38) into formula (2.9) for b_n and (2.39) into formula (2.8) for a_n. We can simplify the expression for b_n even more by using

$$\lambda_n - \lambda_{n-1} = \sinh_q(1)\cosh_q(2n+\alpha_0-1). \tag{2.45}$$

Because of the result after substitution, for $\boldsymbol{\alpha} = (\alpha_0,\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^4$ we define

$$a_n^2(\boldsymbol{\alpha}) = \frac{-\prod_{k=0}^3(\sinh_q(2n+\alpha_0-1) - \sinh_q(p_k))}{\cosh_q(2n+\alpha_0-1)^2\cosh_q(2n+\alpha_0)\cosh_q(2n+\alpha_0-2)}, \tag{2.46}$$

$$b_n(\boldsymbol{\alpha}) = \frac{(A_1 - A_2A_3)\lambda_n + \cosh_q(1)(A_1A_3 + A_0A_2)}{\cosh_q(2n+\alpha_0-1)\cosh_q(2n+\alpha_0+1)}, \tag{2.47}$$

where $(p_k)_{k=0}^3$ and $(A_k)_{k=0}^3$ depend on $(\alpha_k)_{k=0}^3$ via (2.39) and (2.38) respectively. When $C_0 > 0$ and $C_1 < 0$, the formulas for a_n^2 and b_n change slightly. For a_n^2, we have to remove the minus sign in front of the product in the numerator and interchange $\alpha_1 \leftrightarrow \alpha_2$. The formula for b_n changes in the sense that $(A_k)_{k=0}^3$ depend on $(\alpha_k)_{k=0}^3$ in a different way. Therefore we define

$$\tilde{a}_n^2(\tilde{\alpha}_0,\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3) = -a_n^2(\tilde{\alpha}_0,\tilde{\alpha}_2,\tilde{\alpha}_1,\tilde{\alpha}_3), \tag{2.48}$$

$$\tilde{b}_n(\boldsymbol{\alpha}) = \frac{(\tilde{A}_1 - \tilde{A}_2\tilde{A}_3)\lambda_n + \cosh_q(1)(\tilde{A}_1\tilde{A}_3 + \tilde{A}_0\tilde{A}_2)}{\cosh_q(2n+\alpha_0-1)\cosh_q(2n+\alpha_0+1)}, \tag{2.49}$$

where

$$\tilde{A}_0 = \sinh_q(\alpha_0), \qquad \tilde{A}_1 = \cosh_q(\alpha_1), \qquad \tilde{A}_2 = \sinh_q(\alpha_2), \qquad \tilde{A}_3 = \cosh_q(\alpha_3).$$

To end this section let us make the connection with the Askey–Wilson algebra as defined in [4]. Let us write

$$\Lambda_1 = A_1, \qquad \Lambda_2 = A_3, \qquad \Lambda_3 = A_0, \qquad \Lambda_{12} = L, \qquad \Lambda_{23} = K, \qquad \Lambda_{123} = -A_2,$$

and set $A_4 = A_5 = -(\sinh_q(1))^2$. Define

$$\Lambda_{13} = -\frac{[\Lambda_{12}, \Lambda_{23}]_q}{\sinh_q(2)} + \frac{\Lambda_1\Lambda_3 + \Lambda_2\Lambda_{123}}{\cosh_q(1)},$$

then the AW-relations (2.2) and (2.3) can be written as

$$\Lambda_{12} = -\frac{[\Lambda_{23}, \Lambda_{13}]_q}{\sinh_q(2)} + \frac{\Lambda_2\Lambda_3 + \Lambda_1\Lambda_{123}}{\cosh_q(1)}, \qquad \Lambda_{23} = -\frac{[\Lambda_{13}, \Lambda_{12}]_q}{\sinh_q(2)} + \frac{\Lambda_1\Lambda_2 + \Lambda_3\Lambda_{123}}{\cosh_q(1)}.$$

Moreover, for the Askey–Wilson algebra inside $\mathcal{U}_q$ generated by Y_L and Y_k it follows from [30, Theorem 2.17] that the corresponding Casimir element Q (2.4) of the Askey–Wilson algebra can be simplified to

$$Q = (\cosh_q(1))^2 - \Lambda_{123}^2 - \Lambda_1^2 - \Lambda_2^2 - \Lambda_3^2 - \Lambda_{123}\Lambda_1\Lambda_2\Lambda_3.$$

This implies that Y_L and Y_K generate a *special* Askey–Wilson algebra, as defined in [4, Section 2.1], inside $\mathcal{U}_q$.

3 Construction of the rank 2 Askey–Wilson algebra AW_2

3.1 Coproducts of Y_K and Y_L and the Askey–Wilson algebra

In [17], it was shown that the overlap coefficients of coproducts of Y_K and Y_L are multivariate Askey–Wilson polynomials. Moreover, q-difference operators for which these multivariate Askey–Wilson polynomials are eigenfunctions can be realized by these coproducts of twisted primitive elements. One can check that these coproducts also satisfy the AW-relations (Theorem 3.1). This then will be our motivation for the defining relations of the rank 2 Askey–Wilson algebra AW_2.

Let us consider the following elements in $\mathcal{U}_q \otimes \mathcal{U}_q$,

$$1 \otimes Y_K, \quad \Delta(Y_K), \quad Y_L \otimes 1, \quad \Delta(Y_L) \quad \text{and} \quad \Delta(\Omega).$$

We want to know how these elements relate to each other. Two easy observations are that $1 \otimes Y_K$ and $Y_L \otimes 1$ commute and that $\Delta(Y_K)$ and $\Delta(Y_L)$ satisfy the AW-relations, since Δ is an algebra homomorphism. That is,

$$\begin{aligned}
&\cosh_q(2)\Delta(Y_L)\Delta(Y_K)\Delta(Y_L) - \Delta(Y_L)^2\Delta(Y_K) - \Delta(Y_K)\Delta(Y_L)^2 \\
&\qquad = \Delta(B)\Delta(Y_L) + C_0\Delta(Y_K) + \Delta(D_0), \\
&\cosh_q(2)\Delta(Y_K)\Delta(Y_L)\Delta(Y_K) - \Delta(Y_K)^2\Delta(Y_L) - \Delta(Y_L)\Delta(Y_K)^2 \\
&\qquad = \Delta(B)\Delta(Y_K) + C_1\Delta(Y_L) + \Delta(D_1),
\end{aligned}$$

where B, C_0, C_1, D_0, D_1 can be found in (2.24). Notice that C_0 and C_1 are constants, thus

$$\Delta(C_0) = C_0(1 \otimes 1) \quad \text{and} \quad \Delta(C_1) = C_1(1 \otimes 1).$$

However, $\Delta(B)$, $\Delta(D_0)$ and $\Delta(D_1)$ are not central elements of $\mathcal{U}_q \otimes \mathcal{U}_q$ anymore due to the appearance of $\Delta(\Omega)$. They do not even commute with our four generators, since $\Delta(\Omega)$ does not commute with $1 \otimes Y_K$ or $Y_L \otimes 1$. We do see that $\Delta(\Omega)$ 'locally' commutes, i.e., it commutes with $\Delta(Y_K)$ and $\Delta(Y_L)$, the elements of the AW-relations it appears in as a structure parameter. Also, we already noted that $1 \otimes Y_K$ and $\Delta(Y_K)$ commute and $Y_L \otimes 1$ and $\Delta(Y_L)$ as well because of (2.23).

Let us now study how $Y_L \otimes 1$ and $\Delta(Y_K)$ relate. It turns out that $(Y_L \otimes 1, \Delta(Y_K))$ satisfy the AW(3) relations (2.2) and (2.3) where the structure 'parameters' contain the 'locally' commuting operators $\Delta(Y_L)$ and $1 \otimes Y_K$. A similar result is (non surprisingly) true for $(1 \otimes Y_K, \Delta(Y_L))$. Also, both couples $(\Delta(\Omega), 1 \otimes Y_K)$ and $(\Delta(\Omega), Y_L \otimes 1)$ satisfy the AW-relations. This is summarized in the next theorem.

Theorem 3.1. *Each pair of the elements* $1 \otimes Y_K, \Delta(Y_K), Y_L \otimes 1, \Delta(Y_L), \Delta(\Omega) \in \mathcal{U}_q \otimes \mathcal{U}_q$ *either commutes or satisfies the AW-relations* (2.2) *and* (2.3)*. The pairs of non-commuting elements satisfy these relations with the structure parameters in Table* 1*.*

Generator 1	Generator 2	A_0	A_1	A_2	A_3	A_4	A_5
$\Delta(Y_K)$	$\Delta(Y_L)$	a_s	b_t	θ	$\Delta(\Omega)$	$b_{\hat{E}} b_{\hat{F}}$	$a_{\hat{E}} a_{\hat{F}}$
$1 \otimes Y_K$	$\Delta(Y_L)$	a_s	$Y_L \otimes 1$	θ	$1 \otimes \Omega$	$b_{\hat{E}} b_{\hat{F}}$	$a_{\hat{E}} a_{\hat{F}}$
$\Delta(Y_K)$	$Y_L \otimes 1$	$1 \otimes Y_K$	b_t	θ	$\Omega \otimes 1$	$b_{\hat{E}} b_{\hat{F}}$	$a_{\hat{E}} a_{\hat{F}}$
$1 \otimes Y_K$	$\Delta(\Omega)$	a_s	$\Omega \otimes 1$	$-\Delta(Y_K)$	$1 \otimes \Omega$	$-\sinh_q(1)^2$	$a_{\hat{E}} a_{\hat{F}}$
$\Delta(\Omega)$	$Y_L \otimes 1$	$1 \otimes \Omega$	b_t	$-\Delta(Y_L)$	$\Omega \otimes 1$	$b_{\hat{E}} b_{\hat{F}}$	$-\sinh_q(1)^2$

Table 1. Askey–Wilson algebra relations of twisted primitive elements.

Proof. This is a direct computation in $\mathcal{U}_q \otimes \mathcal{U}_q$. For readability, these calculations can be found in Appendix A. ■

Remark 3.2. Let us consider the tensor product of two irreducible representations of $\mathcal{U}_q$. Clebsch-Gordan coefficients for a basis of eigenvectors of $\Delta(Y_K)$ which are also eigenvectors of $1 \otimes Y_K$, can be obtained from diagonalizing $\Delta(\Omega)$ acting on such eigenvectors. From Table 1, we see that $\Delta(\Omega)$ and $1 \otimes Y_K$ generate an Askey–Wilson algebra, and note that the structure 'parameters' $A_1 = \Omega \otimes 1$, $A_2 = \Delta(Y_K)$ and $A_3 = 1 \otimes \Omega$ act as multiplication operators in this case. As a consequence, see Section 2.3, the Clebsch-Gordan coefficients are essentially q-Racah polynomials. Using this reasoning in the case of tensor products of (infinite-dimensional) irreducible representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$, one expects the Clebsch-Gordan coefficients for simultaneous eigenvectors of $\Delta(Y_K)$ and $1 \otimes Y_K$ to be eigenfunctions of an Askey–Wilson q-difference operator. Indeed, in the case of a tensor product of discrete series representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$ the Clebsch-Gordan are essentially Askey–Wilson polynomials, see [16, 23].

3.2 Defining the rank 2 Askey–Wilson algebra AW_2

Inspired by Theorem 3.1, we will define the rank 2 Askey–Wilson algebra AW_2. Constructing this algebra will be done without any dependence on $\mathcal{U}_q$. We will introduce elements in AW_2 which resemble Y_K, Y_L, Ω in $\mathcal{U}_q$ as well as their coproducts. Defining a comultiplication is not needed in AW_2. This construction of AW_2 will proceed in three steps. First, we will set up two original AW(3) algebras and define the algebra $\mathscr{A}$ to be their tensor product. Then, we let $\mathrm{AW}_1(M_2)$ be the Askey–Wilson algebra that resembles the first row of Table 1. Lastly, we will combine these algebras and define AW_2 as the algebra generated[4] by the two algebras $\mathscr{A}$ and $\mathrm{AW}_1(M_2)$ subject to the Askey–Wilson algebra relations that appear in rows 2–5 of Table 1.

To have a short notation for when two elements of an algebra satisfy the AW-relations, we will introduce the function $\mathscr{AW}$.

Definition 3.3. Let A be an algebra over $\mathbb{C}$ and A^n be the n-ary Cartesian product of A. Define the function $\mathscr{AW}: A^8 \to A^2$ by

$$\mathscr{AW}(K, L \mid A_0, A_1, A_2, A_3, A_4, A_5) = \begin{pmatrix} \cosh_q(2)KLK - K^2L - LK^2 \\ \cosh_q(2)LKL - L^2K - KL^2 \end{pmatrix}$$
$$- \begin{pmatrix} (\sinh_q(1))^2(A_0A_1 - A_2A_3)K - (\cosh_q(1))^2A_5L + \cosh_q(1)(A_1A_3A_5 + (\sinh_q(1))^2A_0A_2) \\ (\sinh_q(1))^2(A_0A_1 - A_2A_3)L - (\cosh_q(1))^2A_4K + \cosh_q(1)(A_0A_3A_4 + (\sinh_q(1))^2A_1A_2) \end{pmatrix}$$

for $K, L, A_0, A_1, A_2, A_3, A_4, A_5 \in A$. We then call K, L the generators and $(A_k)_{k=0}^5$ the structure parameters. Moreover, we say that the structure parameters are locally central if $(A_k)_{k=0}^5$ commute with each other as well as with the generators K and L of that relation. That is,

$$[A_k, A_l] = [A_k, K] = [A_k, L] = 0 \qquad \text{for all} \quad k, l \in \{0, 1, 2, 3, 4, 5\},$$

where $[\cdot, \cdot]$ is the regular commutation bracket defined by

$$[X, Y] = XY - YX.$$

Remark 3.4. We have that $\mathscr{AW}(K, L|A_0, A_1, A_2, A_3, A_4, A_5) = 0$ if and only if two elements K, L of an algebra satisfy the AW-relations with parameters $A_0, \dots, A_5$.

From now on, fix constants $A_0, A_1, A_2, A_{N_1}, A_{N_2}, \sigma_K, \sigma_L \in \mathbb{R}$. Later on, A_{N_i} will determine the dimension of the vector space V_i by taking

$$A_{N_i} = \cosh_q(-N_i - 1).$$

Let us first introduce the two original AW(3) algebras. For $i \in \{1, 2\}$, let $\mathrm{AW}_1(A_{N_i})$ be the algebra generated by K_1 and L_1 subject to the relations

$$\mathscr{AW}(K_1, L_1 \mid A_0, A_1, A_2, A_{N_i}, \sigma_L, \sigma_K) = 0.$$

Then we let

$$\mathscr{A} = \mathrm{AW}_1(A_{N_1}) \otimes \mathrm{AW}_1(A_{N_2}).$$

Secondly, we need the Askey–Wilson algebra $\mathrm{AW}_1(M_2)$. This is the algebra generated by K_2, L_2 and M_2 subject to the relations

$$\mathscr{AW}(K_2, L_2 \mid A_0, A_1, A_2, M_2, \sigma_L, \sigma_K) = 0, \tag{3.1}$$

where M_2 is a central element of $\mathrm{AW}_1(M_2)$. In the setting of Section 3.1, M_2 would be the coproduct of the Casimir of $\mathcal{U}_q$, which is central with respect to algebra $\Delta(\mathcal{U}_q)$. Now we are ready to define AW_2, where the relations are motivated by the commutation relations of $\mathcal{U}_q \otimes \mathcal{U}_q$ in rows 2–5 of Table 1.

[4]This is the free product algebra of the algebras $\mathscr{A}$ and $\mathscr{B}$ subject to the extra relations.

Definition 3.5. Let AW_2 be the unital, associative, complex algebra generated by the algebras $\mathscr{A}$ and $\mathrm{AW}_1(M_2)$ subject to the relations

$$\mathscr{AW}(1\otimes K_1, L_2 \mid A_0, L_1\otimes 1, A_2, A_{N_2}, \sigma_L, \sigma_K) = 0, \tag{3.2}$$
$$\mathscr{AW}(K_2, L_1\otimes 1 \mid 1\otimes K_1, A_1, A_2, A_{N_1}, \sigma_L, \sigma_K) = 0, \tag{3.3}$$
$$\mathscr{AW}\big(1\otimes K_1, M_2 \mid A_0, A_{N_1}, K_2, A_{N_2}, -\sinh_q(1)^2, \sigma_K\big) = 0, \tag{3.4}$$
$$\mathscr{AW}\big(M_2, L_1\otimes 1 \mid A_{N_2}, A_1, L_2, A_{N_1}, \sigma_L, -\sinh_q(1)^2\big) = 0, \tag{3.5}$$

where all structure parameters are locally central.

Remark 3.6. Since $1\otimes K_1$ is locally central in (3.3), it commutes with K_2. Similarly, $L_1\otimes 1$ commutes with L_2. Also, M_2 commutes with both K_2 and L_2.

The correspondence between elements in $\mathcal{U}_q\otimes\mathcal{U}_q$ and AW_2 can be found in Table 2.

Algebra	Generators					Central elements						
$\mathcal{U}_q\otimes\mathcal{U}_q$	$1\otimes Y_K$	$\Delta(Y_K)$	$Y_L\otimes 1$	$\Delta(Y_L)$	$\Delta(\Omega)$	a_s	b_t	θ	$\Omega\otimes 1$	$1\otimes\Omega$	b_Eb_F	a_Ea_F
AW_2	$1\otimes K_1$	K_2	$L_1\otimes 1$	L_2	M_2	A_0	A_1	A_2	A_{N_1}	A_{N_2}	σ_L	σ_K

Table 2. Correspondence between $\mathcal{U}_q\otimes\mathcal{U}_q$ and the generators of AW_2.

Remark 3.7. There exists a homomorphism from AW_2 to the rank 2 Askey–Wilson algebra $\mathrm{AW}(4)$ defined in [8]. Indeed, let Λ_A, $A\subseteq\{1,2,3,4\}$ be defined as in [8, Section 2.3]. It can be checked that the Λ's satisfy the AW_2-relations if we set

$$\begin{aligned}
&\Lambda_{\{1\}} = A_1, \qquad \Lambda_{\{2\}} = A_{N_1}, \qquad \Lambda_{\{3\}} = A_{N_2}, \qquad \Lambda_{\{4\}} = A_0,\\
&\Lambda_{\{1,2\}} = L_1\otimes 1, \qquad \Lambda_{\{2,3\}} = M_2, \qquad \Lambda_{\{3,4\}} = 1\otimes K_1,\\
&\Lambda_{\{1,2,3\}} = L_2, \qquad \Lambda_{\{1,2,3,4\}} = A_2, \qquad \Lambda_{\{2,3,4\}} = K_2.
\end{aligned} \tag{3.6}$$

However, this correspondence does not give an isomorphism since $\mathrm{AW}(4)$ has more relations. Namely, the relations between Λ_A, Λ_B and Λ_C where (A,B,C) is any cyclic permutation of

$$(\{1,3\},\{3,4\},\{1,4\}) \qquad \text{or} \qquad (\{1,2\},\{2,4\},\{1,4\}), \tag{3.7}$$

i.e., the relations where two Λ's have 'holes' in it. An interesting question is whether above relations hold in the subalgebra of $\mathcal{U}_q\otimes\mathcal{U}_q$ generated by twisted primitive elements, the Casimir Ω and their coproducts. That is, we can use Table 2 and (3.6) to obtain a correspondence between that subalgebra of $\mathcal{U}_q\otimes\mathcal{U}_q$ and $\mathrm{AW}(4)$. It can be show that the extra relations (3.7) indeed hold there, which requires large computations in $\mathcal{U}_q\otimes\mathcal{U}_q$. For one of the relations we have written out parts of the computation in Appendix B.

The relations in [8] suggest how the relations of a rank N Askey–Wilson algebra inside $\mathcal{U}_q^{\otimes N}$ may look. This would simplify checking relations in $\mathcal{U}_q^{\otimes N}$, which is done in Appendix A for the 'easy' case $N=2$. Let us define $\Delta^n\colon \mathcal{U}_q^{\otimes n}\to\mathcal{U}_q^{\otimes n+1}$ recursively by

$$\Delta^n = \big(1^{\otimes n-1}\otimes\Delta\big)\Delta^{n-1},$$

with Δ^0 the identity on $\mathcal{U}_q$. Note that by the coassociativity of Δ, this is equal to

$$\Delta^n = \big(\Delta\otimes 1^{\otimes n-1}\big)\Delta^{n-1}.$$

Denote by $[i:j]$ the set of consecutive integers $\{i, i+1, \dots, j-1, j\}$. Then we conjecture the following correspondence between coproducts of twisted primitive elements and Casimirs in $\mathcal{U}_q^{\otimes N}$ and Λ_A's, $A \subseteq \{1, \dots, N+2\}$, in $\mathrm{AW}(N+2)$,

$$\Lambda_{\{1\}} = a_\sigma, \qquad \Lambda_{\{N+2\}} = b_t, \qquad \Lambda_{[1:N+2]} = \theta,$$
$$\Lambda_{[1:n+1]} = \Delta^{n-1}(Y_L) \otimes 1^{\otimes N-n}, \qquad n = 0, \dots, N,$$
$$\Lambda_{[i:j]} = 1^{\otimes i-2} \otimes \Delta^{j-i}(\Omega) \otimes 1^{\otimes N+1-j}, \qquad 2 \le i \le j \le N+1,$$
$$\Lambda_{[n+1:N+2]} = 1^{\otimes n-1} \otimes \Delta^{N-n}(Y_K), \qquad n = 1, \dots, N.$$

3.3 Finite-dimensional representations of $\mathbf{AW_2}$

By Theorem 3.1, the elements $1 \otimes Y_K$, $Y_L \otimes 1$, $\Delta(Y_K)$, $\Delta(Y_L)$ and $\Delta(\Omega)$ in $\mathcal{U}_q^{\otimes 2}$ satisfy all relations of AW_2. Therefore, we automatically get existence of a representation of AW_2 from the representation of $\mathcal{U}_q^{\otimes 2}$ if we take A_{N_1} and A_{N_2} to be the constants of the representation value of $\Omega \otimes 1$ and $1 \otimes \Omega$ in $\mathcal{U}_q^{\otimes 2}$.

Corollary 3.8. *Let V_1 and V_2 be vector spaces with $\dim(V_k) = N_k + 1$ for $k = 1, 2$. If $A_{N_1} = \cosh_q(N_1 + 1)$ and $A_{N_2} = \cosh_q(N_2 + 1)$, there exists a representation of AW_2 on $V_1 \otimes V_2$.*

Moreover, we can show that there exists a representation of AW_2, where $1 \otimes K_1$, K_2, $L_1 \otimes 1$, L_2 and M_2 are self-adjoint in the case $\sigma_K, \sigma_L \ge 0$ using the representation theory of $\mathcal{U}_q^{\otimes 2}$ and the $*$-structures of the Hopf algebra $\mathcal{U}_q$.

Proposition 3.9. *Let V_1, V_2, A_{N_1} and A_{N_2} as in Corollary 3.8, $A_0, A_1, A_2 \in \mathbb{R}$ and let $\sigma_K, \sigma_L \ge 0$. Then there exist a representation of AW_2 on $V_1 \otimes V_2$ and an inner product on $V_1 \otimes V_2$ such that $1 \otimes K_1$, K_2, $L_1 \otimes 1$, L_2 and M_2 are self-adjoint.*

Proof. Let $*$ be any $*$-structure on $\mathcal{U}_q$, then we have

$$Y_K^* = q^{-\frac{1}{2}} \overline{a}_{\hat{E}} \hat{K}^* \hat{E}^* + q^{\frac{1}{2}} \overline{a}_{\hat{F}} \hat{K}^* \hat{F}^* + \overline{a}_s \big(\hat{K}^*\big)^2,$$
$$Y_L^* = q^{\frac{1}{2}} \overline{b}_{\hat{E}} \big(\hat{K}^{-1}\big)^* \hat{E}^* + q^{-\frac{1}{2}} \overline{b}_{\hat{F}} \big(\hat{K}^{-1}\big)^* \hat{F}^* + \overline{b}_t \big(\big(\hat{K}^{-1}\big)^*\big)^2.$$

There are 2 inequivalent $*$-structures on $\mathcal{U}_q$ in the case $0 < q < 1$,

(i) $\mathcal{U}_q(\mathfrak{su}(2))$ defined by $\hat{K}^* = \hat{K}$, $\hat{E}^* = \hat{F}$, $\hat{F}^* = \hat{E}$,

(ii) $\mathcal{U}_q(\mathfrak{su}(1{,}1))$ defined by $\hat{K}^* = \hat{K}$, $\hat{E}^* = -\hat{F}$, $\hat{F}^* = -\hat{E}$.

Since $\mathcal{U}_q(\mathfrak{su}(1{,}1))$ has no irreducible finite-dimensional representations except the trivial one, we will focus on $\mathcal{U}_q(\mathfrak{su}(2))$. The Casimir Ω is self-adjoint in $\mathcal{U}_q(\mathfrak{su}(2))$, which implies $\Delta(\Omega)$ is as well. In order for Y_K and Y_L to be self-adjoint in $\mathcal{U}_q(\mathfrak{su}(2))$, we require $\overline{a}_{\hat{E}} = a_{\hat{F}}$, $\overline{b}_{\hat{E}} = b_{\hat{F}}$ and $a_s, b_t \in \mathbb{R}$. Then we have, using Table 2,

$$\sigma_K = |a_{\hat{E}}|^2 \ge 0 \qquad \text{and} \qquad \sigma_L = |b_{\hat{E}}|^2 \ge 0$$

and $A_0, A_1, A_2 \in \mathbb{R}$. For each dimension $N + 1 \in \mathbb{N}$, the algebra $\mathcal{U}_q(\mathfrak{su}(2))$ has an irreducible $*$-representation, see, e.g., [22]. This in turn defines a $*$-representation on $\mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2))$. By Theorem 3.1, this is automatically a representation of AW_2 in which $1 \otimes K_1$, K_2, $L_1 \otimes 1$, L_2 and M_2 are self-adjoint. ■

Remark 3.10. In the following sections we use the above defined finite-dimensional representation of AW_2. For existence of this representation we referred to $*$-representations of $\mathcal{U}_q$. However, let us stress that the calculations in the following sections only make use of the AW_2-relations, and other than for existence of the representations, no representation theory of $\mathcal{U}_q$ is needed.

4 Bivariate Askey–Wilson polynomials as overlap coefficients of AW_2

We will now show that the overlap coefficients of eigenvector of K_2 and L_2 are bivariate q-Racah polynomials similar to the ones defined by Gasper and Rahman [12]. We will construct a representation of AW_2 on a tensor product of vector spaces V_1 and V_2, which have dimensions N_1+1 and N_2+1 respectively.

A representation of AW_2 on $V_1 \otimes V_2$ exists by Corollary 3.8. Note that $\mathrm{AW}_1(A_{N_1}) \otimes \mathrm{AW}_1(A_{N_2})$ has a natural representation on $V_1 \otimes V_2$ coming from the representations from $\mathrm{AW}_1(A_{N_1})$ on V_1 and $\mathrm{AW}_1(A_{N_2})$ on V_2 given by (2.5)–(2.9). We will extend this representation of $\mathrm{AW}_1(A_{N_1}) \otimes \mathrm{AW}_1(A_{N_2})$ to AW_2. Let us fix $\sigma_L = \sigma_K = (\sinh_q(1))^2$ and $A_{N_i} = \cosh_q(-N_i - 1)$. Then we are in the same setting[5] as before. Then, by (2.5) and (2.40) we get a basis for V_2 given by $\{\psi_{n_2(k)}\}_{k=0}^{N_2}$, where

$$n_2(k) = k - \frac{1}{2}N_2.$$

These are the eigenvectors of $K_1 \in \mathrm{AW}_1(A_{N_2})$ corresponding to eigenvalue $\lambda_{n_2(k)}$. We will often write

$$\psi_{n_2} := \psi_{n_2(k)} \qquad \text{and} \qquad \lambda_{n_2} := \lambda_{n_2(k)}$$

when the index k is not relevant.

Similarly, by (2.12) and (2.41) there is a basis $\{\phi_{m_1(j)}\}_{j=0}^{N_1}$ for V_1, which are the eigenvectors of $L_1 \in \mathrm{AW}_1(A_{N_1})$ with eigenvalue $\mu_{m_1(j)}$. Here

$$m_1(j) = j - \frac{1}{2}N_1.$$

Similar to before, we will write

$$\phi_{m_1} := \phi_{m_1(j)} \qquad \text{and} \qquad \mu_{m_1} := \mu_{m_1(j)}$$

when the index j is not relevant.

Thus, if $v_1 \in V_1$ and $v_2 \in V_2$, we have

$$\begin{aligned}
&(1 \otimes K_1)(v_1 \otimes \psi_{n_2}) = \lambda_{n_2} v_1 \otimes \psi_{n_2},\\
&(L_1 \otimes 1)(\phi_{m_1} \otimes v_2) = \mu_{m_1} \phi_{m_1} \otimes v_2,
\end{aligned}$$

where λ_k and μ_k are given in (2.42). Using these eigenvectors, we can define an inner product on V_1 and V_2 on the basis elements by

$$\langle \phi_{m_1}, \phi_{m_1'} \rangle_{V_1} = \delta_{m_1 m_1'} \qquad \text{and} \qquad \langle \psi_{n_2}, \psi_{n_2'} \rangle_{V_2} = \delta_{n_2 n_2'}.$$

This in turn defines an inner product on $V_1 \otimes V_2$ by

$$\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle_{V_1 \otimes V_2} = \langle v_1, v_1' \rangle_{V_1} \langle v_2, v_2' \rangle_{V_2}, \qquad v_1, v_1' \in V_1, \quad v_2, v_2' \in V_2.$$

With respect to this inner product, both $L_1 \otimes 1$ and $1 \otimes K_1$ are self-adjoint. Also, we can use ψ_{n_2} and ϕ_{m_1} to construct eigenvectors of K_2 and L_2 respectively. Let us start with K_2, for which we can find an eigenvector coming from the eigenvector ψ_{n_2} of K_1.

Proposition 4.1. *For each ψ_{n_2}, there exists $\xi \in V_1$ such that $\xi \otimes \psi_{n_2}$ is an eigenvector of K_2.*

[5] That is, $C_0 = C_1 = -(\sinh_q(2))^2$.

Proof. Let λ_{n_2} be the eigenvalue corresponding to eigenvector ψ_{n_2}, i.e.,

$$K_1\psi_{n_2} = \lambda_{n_2}\psi_{n_2}.$$

Since K_2 and $1 \otimes K_1$ commute, we have for any vector $v \in V_1$,

$$(1 \otimes K_1)K_2(v \otimes \psi_{n_2}) = \lambda_{n_2}K_2(v \otimes \psi_{n_2}).$$

Thus $K_2(v \otimes \psi_{n_2})$ is also an eigenvector of $1 \otimes K_1$ with eigenvalue λ_{n_2}. Since the eigenspaces of K_1 in V_2 are one-dimensional, there exists a unique $w \in V_1$ such that

$$K_2(v \otimes \psi_{n_2}) = w \otimes \psi_{n_2}.$$

Define A to be the operator on V_1 that resembles the action of K_2 on the left side of the above tensor product, i.e.,

$$Av = w.$$

Since K_2 is a linear operator, A is as well. Therefore, A has an eigenvector, say ξ with eigenvalue λ'. Then we have

$$K_2(\xi \otimes \psi_{n_2}) = (A\xi) \otimes \psi_{n_2} = \lambda'\xi \otimes \psi_{n_2}. \qquad \blacksquare$$

Since K_2 and $L_1 \otimes 1$ satisfy the AW relation (3.3), we get a similar 'ladder' property as in the original AW(3). Namely we can use $L_1 \otimes 1$ to create a ladder of eigenvectors for K_2 from the eigenvector $\xi \otimes \psi_{n_2}$. Moreover, the 'Leonard pair' property still holds. That is, $L_1 \otimes 1$ acts as a three-term operator on the eigenvectors of K_2. The proof is similar to the original AW(3), where one has to realize that the non-constant term '$1 \otimes K_1$' that appears in the AW-relation (3.3) must be treated as the constant λ_{n_2}: the eigenvalue of how it acts on $\xi \otimes \psi_{n_2}$.

Proposition 4.2. *For each ψ_{n_2}, there is a basis for V_1 given by*

$$\big\{\psi^{n_2}_{n_1(k)}\big\}_{k=0}^{N_1},$$

such that $\psi^{n_2}_{n_1} \otimes \psi_{n_2}$ is an eigenvector of K_2 with eigenvalue λ_{n_1},

$$K_2\big(\psi^{n_2}_{n_1} \otimes \psi_{n_2}\big) = \lambda_{n_1}\psi^{n_2}_{n_1} \otimes \psi_{n_2}.$$

Here,

$$n_1 := n_1(k) = k + n_2 - \frac{1}{2}N_1.$$

Moreover, $L_1 \otimes 1$ acts as a three-term operator on these eigenvectors,

$$\begin{aligned}(L_1 \otimes 1)\big(\psi^{n_2}_{n_1} \otimes \psi_{n_2}\big) = {}& a_{n_1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)\psi^{n_2}_{n_1-1} \otimes \psi_{n_2} + b_{n_1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)\psi^{n_2}_{n_1} \otimes \psi_{n_2}\\ &+ a_{n_1+1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)\psi^{n_2}_{n_1+1} \otimes \psi_{n_2},\end{aligned}$$

where $\boldsymbol{\alpha}^{n_2}_{L_1} = (\alpha_0 + 2n_2, \alpha_1, \alpha_2, -N_1 - 1)$ and the functions a_n and b_n are given by (2.46) and (2.47).

Proof. The proof is similar to the construction of the representation of the original AW(3), see [35]. Let ψ_{n_2} be an eigenvector of K_1. By the previous proposition, there exists an eigenvector $\psi^{n_2}_p \otimes \psi_{n_2}$ of K_2, say with eigenvalue $\lambda(p)$,

$$K_2\psi^{n_2}_p \otimes \psi_{n_2} = \lambda(p)\psi^{n_2}_p \otimes \psi_{n_2}. \tag{4.1}$$

Define

$$A = [K_2, L_1 \otimes 1]_q,$$

which plays the role of the third generator of the original AW(3). We will use the ladder property of the Askey–Wilson algebra (3.3). That is, we will show there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\psi_s^{n_2} \otimes \psi_{n_2} = (\alpha K_2 + \beta(L_1 \otimes 1) + \gamma A)\, \psi_p^{n_2} \otimes \psi_{n_2}$$

is also an eigenvector of K_2 with a different eigenvalue $\lambda(s)$. Requiring

$$K_2 \psi_s^{n_2} \otimes \psi_{n_2} = \lambda(s) \psi_s^{n_2} \otimes \psi_{n_2}$$

implies

$$\big(\alpha K_2^2 + \beta K_2 (L_1 \otimes 1) + \gamma K_2 A\big) \psi_p^{n_2} \otimes \psi_{n_2} = \lambda(s)(\alpha K_2 + \beta(L_1 \otimes 1) + \gamma A) \psi_p^{n_2} \otimes \psi_{n_2}.$$

Using (4.1) and the algebra relations (3.3), this leads to

$$\begin{aligned} 0 = \psi_p^{n_2} \otimes \psi_{n_2} \big[& \alpha \lambda(p)^2 + \beta\big(q^{-2}\lambda(p)(L_1 \otimes 1) + q^{-1} A\big) \\ & + \gamma\big(q^2 \lambda(p) A - q\lambda(p)\hat{B} - q\hat{C}_1(L_1 \otimes 1) - q\hat{D}_1\big) - \lambda(s)\alpha\lambda(p) + \beta(L_1 \otimes 1) + \gamma A\big]. \end{aligned}$$

Here, $\hat{B}$, $\hat{C}_1$, $\hat{D}_1$ are the parameters of the Askey–Wilson algebra (3.3) such that

$$\cosh_q(2) K_2 (L_1 \otimes 1) K_2 - K_2^2 (L_1 \otimes 1) - (L_1 \otimes 1) K_2^2 = \hat{B} K_2 + \hat{C}_1 (L_1 \otimes 1) + \hat{D}_1.$$

That is,

$$\begin{aligned} & \hat{B} = (\sinh_q(1))^2 (A_1(1 \otimes K_1) - A_2 A_{N_1}), \qquad \hat{C}_1 = -(\cosh_q(1))^2 \sigma_K, \\ & \hat{D}_1 = \cosh_q(1\big(A_1 A_{N_1} \sigma_K + (\sinh_q(1))^2 A_2 (1 \otimes K_1)\big). \end{aligned}$$

Since

$$(1 \otimes K_1) \psi_{n_1}^{n_2} \otimes \psi_{n_2} = \lambda_{n_2} \psi_{n_1}^{n_2} \otimes \psi_{n_2},$$

$1 \otimes K_1$ acts as the constant λ_{n_2} on $\psi_p^{n_2} \otimes \psi_{n_2}$. Therefore, we can interpret $\hat{B}$, $\hat{C}_1$ and $\hat{D}_1$ as constants and proceed as Zhedanov in [35]. That is, we can find a ladder of eigenvectors of K_2

$$\psi_{n_1}^{n_2} \otimes \psi_{n_2}, \qquad n_1 = n_2 - \frac{1}{2} N_1, n_2 - \frac{1}{2} N_1 + 1, \dots, n_2 + \frac{1}{2} N_1$$

with eigenvalues λ_{n_1}. Also, $L_1 \otimes 1$ acts as a three-term operator on these eigenvectors,

$$(L_1 \otimes 1)\big(\psi_{n_1}^{n_2} \otimes \psi_{n_2}\big) = a_{n_1} \psi_{n_1-1}^{n_2} \otimes \psi_{n_2} + b_{n_1} \psi_{n_1}^{n_2} \otimes \psi_{n_2} + a_{n_1+1} \psi_{n_1+1}^{n_2} \otimes \psi_{n_2}.$$

Let us find the coefficients a_{n_1} and b_{n_1}. Since $1 \otimes K_1$ can be interpreted here as

$$\lambda_{n_2} = \sinh_q(\alpha_0 + 2n_2),$$

we are in the setting (2.38) where we have to replace α_0 by $\alpha_0 + 2n_2$. Therefore, a_{n_1} and b_{n_1} can be found using the formulas (2.46) and (2.47) with $\boldsymbol{\alpha}_{L_1}^{n_2} = (\alpha_0 + 2n_2, \alpha_1, \alpha_2, -N_1 - 1)$. ■

Remark 4.3. Note that the label n_2 of the eigenvector of $1 \otimes K_1$ gets into the parameter of the three-term recurrence relation for the left side of the eigenvectors for K_2. We will see that in the bivariate q-Racah polynomials this leads to the degree of the right q-Racah polynomial getting into the parameters of the left q-Racah polynomial.

By symmetry, we get a similar result for L_2 and $1\otimes K_1$.

Corollary 4.4. *For each ϕ_{m_1}, there is a basis for V_2 given by*

$$\big\{\phi^{m_1}_{m_2(j)}\big\}_{j=0}^{N_2},$$

such that $\phi_{m_1}\otimes\phi^{m_1}_{m_2}$ is an eigenvector of L_2 with eigenvalue μ_{m_2},

$$L_2\big(\phi_{m_1}\otimes\phi^{m_1}_{m_2}\big)=\mu_{m_2}\phi_{m_1}\otimes\phi^{m_1}_{m_2}.$$

Here we have

$$m_2:=m_2(j)=j+m_1-\frac{1}{2}N_2.$$

Moreover, $1\otimes K_1$ acts as a three-term operator on these eigenvectors,

$$\begin{aligned}(1\otimes K_1)\big(\phi_{m_1}\otimes\phi^{m_1}_{m_2}\big)={}&a_{m_2}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\phi_{m_1}\otimes\phi^{m_1}_{m_2-1}+b_{m_2}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\phi_{m_1}\otimes\phi^{m_1}_{m_2}\\&+a_{m_2+1}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\phi_{m_1}\otimes\phi^{m_1}_{m_2+1},\end{aligned}$$

where $\boldsymbol{\alpha}^{m_1}_{K_1}=(\alpha_1+2m_1,\alpha_0,\alpha_2,-N_2-1)$ and the functions a_m and b_m are again given by (2.46) and (2.47).

Proof. From (2.37) and (2.38), we see that interchanging the roles of K and L is the same as interchanging $\alpha_0\leftrightarrow\alpha_1$ and $\sigma_L\leftrightarrow\sigma_K$. Therefore, this result is just Proposition 4.2 together with these substitutions. ■

We are now ready to prove the main theorem of this section. Define the overlap coefficients

$$P_{n_1,n_2}(m_1,m_2)=C(n_1,n_2,m_1,m_2)\big\langle\phi_{m_1}\otimes\phi^{m_1}_{m_2},\psi^{n_2}_{n_1}\otimes\psi_{n_2}\big\rangle_{V_1\otimes V_2},\tag{4.2}$$

where C is the normalizing function defined by

$$C(n_1,n_2,m_1,m_2)=\frac{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\big\rangle_{V_1}\big\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(0)}\big\rangle_{V_2}}{\big\langle\phi_{m_1},\psi^{n_2}_{n_1(0)}\big\rangle_{V_1}\big\langle\phi^{m_1}_{m_2},\psi_{n_2(0)}\big\rangle_{V_2}\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\big\rangle_{V_1}\big\langle\phi^{m_1}_{m_2(0)},\psi_{n_2}\big\rangle_{V_2}}.\tag{4.3}$$

Then we can show that $P_{n_1,n_2}(m_1,m_2)$ are bivariate q-Racah polynomials. This comes from the observation that just as in the univariate case, $\big\langle\phi_{m_1},\psi^{n_2}_{n_1}\big\rangle$ and $\big\langle\phi^{m_1}_{m_2},\psi_{n_2}\big\rangle$ are q-Racah polynomials with parameters $\boldsymbol{\alpha}^{n_2}_{L_1}$ and $\boldsymbol{\alpha}^{m_2}_{K_1}$ respectively instead of $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ in (2.43). There is essentially only one difference in the proof of the bivariate case compared to the univariate case. The overlap coefficient $\big\langle\phi^{m_1}_{m_2},\psi_{n_2}\big\rangle$ is a q-Racah polynomial of degree j_2 in the variable k_2, i.e., the degree and variable have switched. We can interchange these by a simple change in parameters.

Theorem 4.5. *We have*

$$P_{n_1(k_1),n_2(k_2)}\big(\mu_{m_1(j_1)},\mu_{m_2(j_2)}\big)=R_{k_1,k_2}\big(y_{j_1},y_{j_2};\alpha_0,\alpha_1,\alpha_2,N_1,N_2;q^2\big),$$

where the right-hand side is a product of two q-Racah polynomials given by

$$\begin{aligned}&R_{k_1}\big(y_{j_1};-q^{\alpha_0+2k_2-N_2+\alpha_1+\alpha_2-N_1-1},q^{\alpha_0+2k_2-N_2-\alpha_1-\alpha_2-N_1-1},q^{-2N_1-2},-q^{2\alpha_1};q^2\big)\\&\quad\times R_{k_2}\big(y_{j_2};-q^{\alpha_0+\alpha_1+2j_1-N_1-\alpha_2-N_2-1},q^{\alpha_0-\alpha_1-2j_1+N_1+\alpha_2-N_2-1},\\&\qquad q^{-2N_2-2},-q^{2\alpha_1+4j_1-2N_1};q^2\big),\end{aligned}\tag{4.4}$$

with $y_{j_1}=q^{-2j_1}-q^{2\alpha_1-2N_1+2j_j}$ and $y_{j_2}=q^{-2j_2}-q^{2\alpha_1-2N_1-2N_2+4j_1+2j_2}$.

Proof. The proof is similar to the original AW(3), we only have to do it twice. It is convenient to split $P_{n_1,n_2}(m_1,m_2)$ into the two parts concerning V_1 and V_2 and analyze both parts separately. We have

$$P_{n_1,n_2}(m_1,m_2)=\frac{\langle\phi_{m_1},\psi^{n_2}_{n_1}\rangle_{V_1}\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\rangle_{V_1}}{\langle\phi_{m_1},\psi^{n_2}_{n_1(0)}\rangle_{V_1}\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\rangle_{V_1}}\times\frac{\langle\phi^{m_1}_{m_2},\psi_{n_2}\rangle_{V_2}\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(0)}\rangle_{V_2}}{\langle\phi^{m_1}_{m_2},\psi_{n_2(0)}\rangle_{V_2}\langle\phi^{m_1}_{m_2(0)},\psi_{n_2}\rangle_{V_2}}. \tag{4.5}$$

For notational convenience, we will define q-Racah polynomials in terms of $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$,

$$r_k(j;\boldsymbol{\alpha})=R_k\big(y_j;-q^{\alpha_0-\alpha_1+\alpha_2+\alpha_3},q^{\alpha_0-\alpha_1-\alpha_2+\alpha_3},q^{2\alpha_3},-q^{-2\alpha_1};q^2\big).$$

We will show that the part concerning V_1 of (4.5) is equal to

$$P^{V_1}_{n_1(k_1)}(m_1(j_1);n_2):=\frac{\langle\phi_{m_1(j_1)},\psi^{n_2}_{n_1(k_1)}\rangle_{V_1}\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\rangle_{V_1}}{\langle\phi_{m_1(j_1)},\psi^{n_2}_{n_1(0)}\rangle_{V_1}\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(k_1)}\rangle_{V_1}}=r_{k_1}\big(j_1;\boldsymbol{\alpha}^{n_2}_{L_1}\big), \tag{4.6}$$

where $\boldsymbol{\alpha}^{n_2}_{L_1}=(\alpha_0+2n_2,\alpha_1,\alpha_2,-N_1-1)$. For the part concerning V_2 of (4.5) we have

$$\begin{aligned}P^{V_2}_{n_2(k_2)}(m_2(j_2);m_1)&:=\frac{\langle\phi^{m_1}_{m_2(j_2)},\psi_{n_2(k_2)}\rangle_{V_2}\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(0)}\rangle_{V_2}}{\langle\phi^{m_1}_{m_2(j_2)},\psi_{n_2(0)}\rangle_{V_2}\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(k_2)}\rangle_{V_2}}\\&=r_{k_2}(j_2;\alpha_0,\alpha_1+2m_1,-\alpha_2,-N_2-1),\end{aligned} \tag{4.7}$$

which comes from $\boldsymbol{\alpha}^{n_1}_{K_1}$ where $\alpha_0\leftrightarrow\alpha_1+2m_1$ and $\alpha_2\leftrightarrow-\alpha_2$, which is necessary to interchange degree and variable of the q-Racah polynomial.

Note that

$$P^{V_1}_{n_1(0)}(m_1;n_2)=1=P^{V_1}_{n_1}(m_1(0);n_2)\qquad\text{and}\qquad P^{V_2}_{n_2(0)}(m_2;m_1)=1=P^{V_2}_{n_2}(m_2(0);m_1).$$

Let us start with $P^{V_1}_{n_1}(m_1;n_2)$. We know that the self adjoint operator $L_1\otimes 1$ acts as multiplication by μ_{m_1} on $\phi_{m_1}\otimes\phi^{m_1}_{m_2}$ and as a three-term operator (Proposition 4.2) on $\psi^{n_2}_{n_1}\otimes\psi_{n_2}$. It only acts non-trivially on V_1, thus we can look at the action on the left side of the inner product. Entirely similar to (2.13) we get, using a slight abuse of notation,

$$\begin{aligned}\mu_{m_1}\langle\phi_{m_1},\psi^{n_2}_{n_1}\rangle_{V_1}&=\langle L_1\phi_{m_1},\psi^{n_2}_{n_1}\rangle_{V_1}=\langle\phi_{m_1},L_1\psi^{n_2}_{n_1}\rangle_{V_1}\\&=a_{n_1}(\boldsymbol{\alpha}^{n_2}_{L_1})\langle\phi_{m_1},\psi^{n_2}_{n_1-1}\rangle_{V_1}+b_{n_1}(\boldsymbol{\alpha}^{n_2}_{L_1})\langle\phi_{m_1},\psi^{n_2}_{n_1}\rangle_{V_1}\\&\quad+a_{n_1+1}(\boldsymbol{\alpha}^{n_2}_{L_1})\langle\phi_{m_1},\psi^{n_2}_{n_1+1}\rangle_{V_1},\end{aligned}$$

where $\boldsymbol{\alpha}^{n_2}_{L_1}$ can be found in Proposition 4.2 and $a_{n_1}(\boldsymbol{\alpha})$, $b_{n_1}(\boldsymbol{\alpha})$ are given by (2.46), (2.47). We want to obtain a three-term recurrence relation for $P^{n_2}_{n_1}(m_1)$ from this equation. However, this will not work because of the appearance of n_1 in the normalizing factor $\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\rangle_{V_1}$. To solve this, we define

$$p^{V_1}_{n_1}(m_1;n_2):=\frac{\langle\phi_{m_1},\psi^{n_2}_{n_1}\rangle_{V_1}}{\langle\phi_{m_1},\psi^{n_2}_{n_1(0)}\rangle_{V_1}},$$

which is only the left part of $P^{V_1}_{n_1}(m_1;n_2)$ in (4.6). Now we have

$$\begin{aligned}\mu_{m_1}p^{V_1}_{n_1}(m_1;n_2)&=a_{n_1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)p^{V_1}_{n_1-1}(m_1;n_2)+b_{n_1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)p^{V_1}_{n_1}(m_1;n_2)\\&\quad+a_{n_1+1}\big(\boldsymbol{\alpha}^{n_2}_{L_1}\big)p^{V_1}_{n_1+1}(m_1;n_2).\end{aligned}$$

Together with the initial conditions

$$p^{V_1}_{n_1(0)}(m_1;n_2)=1 \qquad \text{and} \qquad p^{V_1}_{n_1(-1)}(m_1;n_2)=0,$$

this three-term recurrence relation generates a set $\{p^{V_1}_{n_1(k_1)}(m_1;n_2)\}_{k_1=0}^{N_1}$ of polynomials in the variable μ_{m_1}. Similarly as in the original AW(3), one can show that this coincides with the recurrence relation for q-Racah polynomials with parameters $\boldsymbol{\alpha}^{n_2}_{L_1}$. Therefore,

$$p^{V_1}_{n_1(k_1)}(m_1(j_1);n_2)=h(n_1,n_2)r_{k_1}\big(j_1;\boldsymbol{\alpha}^{n_2}_{L_1}\big), \tag{4.8}$$

where h is some normalization function. To prove (4.6), we have to show that $1/h$ is exactly the right-hand side of $P^{V_1}_{n_1}(m_1;n_2)$, i.e.,

$$\frac{1}{h(n_1,n_2)}=\frac{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\big\rangle_{V_1}}{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\big\rangle_{V_1}}. \tag{4.9}$$

Since $P^{V_1}_{n_1}(m_1(0);n_2)=1$, we can use (4.8) to obtain

$$1=p^{V_1}_{n_1(k_1)}(m_1(0);n_2)\frac{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\big\rangle_{V_1}}{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\big\rangle_{V_1}}=h(n_1,n_2)r_{k_1}\big(0;\boldsymbol{\alpha}^{n_2}_{L_1}\big)\frac{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1(0)}\big\rangle_{V_1}}{\big\langle\phi_{m_1(0)},\psi^{n_2}_{n_1}\big\rangle_{V_1}}.$$

By definition of the q-hypergeometric series, we also have

$$r_{k_1}\big(0;\boldsymbol{\alpha}^{n_2}_{L_1}\big)=1,$$

which proves our claim (4.9).

Let us now turn to $P^{V_2}_{n_2}(m_2;m_1)$, which is slightly more subtle because we do not know how $1\otimes L_1$ acts on $\phi_{m_1}\otimes\phi^{m_1}_{m_2}$. However, since we do know that $1\otimes K_1$ acts as a three-term operator on $\phi_{m_1}\otimes\phi^{m_1}_{m_2}$, we get a three-term recurrence relation in the variable μ_{m_2} instead of the degree n_2. That is, the roles of 'K' and 'L' have interchanged: we use $1\otimes K_1$ as multiplication and three-term operator instead of $L_1\otimes 1$. Using the symmetry of AW_2 we can see that the overlap coefficients of $\phi^{m_1}_{m_2}$ and ψ_{n_2} are q-Racah polynomials of degree m_2 in the variable n_2.

That is, Corollary 4.4 tells us that

$$\begin{aligned}\lambda_{n_2}\big\langle\phi^{m_1}_{m_2},\psi_{n_2}\big\rangle_{V_2}&=\big\langle\phi^{m_1}_{m_2},K_1\psi_{n_2}\big\rangle_{V_2}=\big\langle K_1\phi^{m_1}_{m_2},\psi_{n_2}\big\rangle_{V_2}\\&=a_{m_2}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\big\langle\phi^{m_1}_{m_2-1},\psi_{n_2}\big\rangle_{V_2}+b_{m_2}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\big\langle\phi^{m_1}_{m_2},\psi_{n_2}\big\rangle_{V_2}+a_{m_2+1}\big(\boldsymbol{\alpha}^{m_1}_{K_1}\big)\big\langle\phi^{m_1}_{m_2+1},\psi_{n_2}\big\rangle_{V_2},\end{aligned}$$

Therefore,

$$\frac{\big\langle\phi^{m_1}_{m_2(j_2)},\psi_{n_2(k_2)}\big\rangle_{V_2}}{\big\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(k_2)}\big\rangle_{V_2}}=g(m_1,m_2(j_2))r_{j_2}\big(k_2;\boldsymbol{\alpha}^{m_1}_{K_1}\big),$$

for some normalizing function g. Using the duality property of q-Racah polynomials, which can directly be seen from the q-hypergeometric series (2.15), we can swap degree and variable by interchanging $\alpha_0\leftrightarrow\alpha_1$ and $\alpha_2\leftrightarrow-\alpha_2$. In terms of our original definition of the q-Racah polynomials (2.15), this interchanges $\alpha\beta\leftrightarrow\gamma\delta$ and $\beta\delta\leftrightarrow\alpha$, while keeping γ the same. Therefore,

$$\frac{\big\langle\phi^{m_1}_{m_2(j_1)},\psi_{n_2(k_2)}\big\rangle_{V_2}}{\big\langle\phi^{m_1}_{m_2(0)},\psi_{n_2(k_2)}\big\rangle_{V_2}}=g(m_1,m_2)r_{k_2}(j_2,\boldsymbol{\beta}),$$

where $\boldsymbol{\beta} = (\alpha_0, \alpha_1 + 2m_1, -\alpha_2, -N_2 - 1)$. Consequently,

$$\begin{aligned} P^{V_2}_{n_2(k_2)}(m_2(j_2); m_1) &= \frac{\langle \phi^{m_1}_{m_2(j_2)}, \psi_{n_2(k_2)} \rangle_{V_2}}{\langle \phi^{m_1}_{m_2(0)}, \psi_{n_2(k_2)} \rangle_{V_2}} \frac{\langle \phi^{m_1}_{m_2(0)}, \psi_{n_2(0)} \rangle_{V_2}}{\langle \phi^{m_1}_{m_2(j_2)}, \psi_{n_2(0)} \rangle_{V_2}} \\ &= g(m_1, m_2) r_{k_2}(j_2, \boldsymbol{\beta}) \frac{\langle \phi^{m_1}_{m_2(0)}, \psi_{n_2(0)} \rangle_{V_2}}{\langle \phi^{m_1}_{m_2(j_2)}, \psi_{n_2(0)} \rangle_{V_2}}. \end{aligned} \tag{4.10}$$

Using that

$$1 = P^{V_2}_{n_2(0)}(m_2(j_2); m_1) = g(m_1, m_2) r_0(j_2; \boldsymbol{\beta}) \frac{\langle \phi^{m_1}_{m_2(0)}, \psi_{n_2(0)} \rangle_{V_2}}{\langle \phi^{m_1}_{m_2(j_2)}, \psi_{n_2(0)} \rangle_{V_2}}$$

and

$$r_0(j_2, \boldsymbol{\beta}) = 1,$$

we get

$$\frac{g(m_1, m_2)}{\langle \phi^{m_1}_{m_2}, \psi_{n_2(0)} \rangle_{V_2}} = \frac{1}{\langle \phi^{m_1}_{m_2(0)}, \psi_{n_2(0)} \rangle_{V_2}}.$$

Substituting this into (4.10) gives

$$P^{V_2}_{n_2(k_2)}(m_2(j_2); m_1) = r_{k_2}(j_2, \boldsymbol{\beta}),$$

which proves the theorem. ■

Remark 4.6. Note that the bivariate q-Racah polynomials in the theorem above are formally not polynomials but rational functions. This is because of the appearance of j_1 in the parameters of the right q-Racah polynomial R_{k_2}. This can be solved easily by putting an appropriate factor in front.

Remark 4.7. Orthogonality from the bivariate q-Racah polynomials arises similarly as in the univariate case. Let

$$\mathbf{N} = \{(l_1, l_2) \colon l_1 = 0, \dots, N_1 \text{ and } l_2 = 0, \dots, N_2\}.$$

From $\{\phi_{m_1(j_1)} \otimes \phi^{m_1(j_1)}_{m_2(j_2)}\}_{(j_1,j_2)\in\mathbf{N}}$ and $\{\psi^{n_2(k_2)}_{n_1(k_1)} \otimes \psi_{n_2(k_2)}\}_{(k_1,k_2)\in\mathbf{N}}$ both being orthonormal, we obtain

$$\delta_{k_1,k_1'}\delta_{k_2,k_2'} = \delta_{n_1(k_1),n_1(k_1')}\delta_{n_2(k_2),n_2(k_2')} = \langle \psi^{n_2(k_2')}_{n_1(k_1')}, \psi^{n_2(k_2)}_{n_1(k_1)} \rangle_{V_1} \langle \psi_{n_2(k_2')}, \psi_{n_2(k_2)} \rangle_{V_2}. \tag{4.11}$$

Using that

$$\psi^{n_2(k_2')}_{n_1(k_1')} = \sum_{j_1=0}^{N_1} \langle \psi^{n_2(k_2')}_{n_1(k_1')}, \phi_{m_1(j_1)} \rangle \phi_{m_1(j_1)}$$

and

$$\psi_{n_2(k_2')} = \sum_{j_2=0}^{N_2} \langle \psi_{n_2(k_2')}, \phi^{m_1(j_1)}_{m_2(j_2)} \rangle \phi^{m_1(j_1)}_{m_2(j_2)},$$

we obtain that (4.11) is equal to

$$\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}\left\langle\psi_{n_1(k_1')}^{n_2(k_2')},\phi_{m_1(j_1)}\right\rangle_{V_1}\left\langle\phi_{m_1(j_1)},\psi_{n_1(k_1)}^{n_2(k_2)}\right\rangle_{V_1}\left\langle\psi_{n_2(k_2')},\phi_{m_2(j_2)}^{m_1(j_1)}\right\rangle_{V_2}\left\langle\phi_{m_2(j_2)}^{m_1(j_1)},\psi_{n_2(k_2)}\right\rangle_{V_2}.$$

Rearranging this expression gives

$$\begin{aligned}\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}&\left\langle\phi_{m_1(j_1)}\otimes\phi_{m_2(j_2)}^{m_1(j_1)},\psi_{n_1(k_1)}^{n_2(k_2)}\otimes\psi_{n_2(k_2)}\right\rangle_{V_1\otimes V_2}\\&\times\overline{\left\langle\phi_{m_1(j_1)}\otimes\phi_{m_2(j_2)}^{m_1(j_1)},\psi_{n_1(k_1')}^{n_2(k_2')}\otimes\psi_{n_2(k_2')}\right\rangle}_{V_1\otimes V_2}.\end{aligned}$$

Therefore, using (4.2), we obtain the orthogonality

$$\begin{aligned}\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2}&w_2(j_1,j_2,k_1,k_2)P_{n_1(k_1),n_2(k_2)}(m_1(j_1),m_2(j_2))\overline{P_{n_1(k_1'),n_2(k_2')}(m_1(j_1),m_2(j_2))}\\&=\delta_{k_1,k_1'}\delta_{k_2,k_2'},\end{aligned}$$

where the weight function is given by

$$w_2(j_1,j_2,k_1,k_2)=\frac{1}{|C(n_1(k_1),n_2(k_2),m_1(j_1),m_2(j_2)|^2}.$$

The normalizing function C can be found in (4.3). This weight function is just a product of the weight functions of the two univariate q-Racah polynomials in (4.4),

$$\begin{aligned}w_2(j_1,j_2,k_1,k_2)&:=w_2(j_1,j_2,k_1,k_2,\alpha_0,\alpha_1,\alpha_2,\alpha_3;q)\\&=w(j_1,k_1,\boldsymbol{\alpha}_{L_1}^{n_2(k_2)};q)w(j_2,k_2,\boldsymbol{\beta};q),\end{aligned}\tag{4.12}$$

where w is the weight function for q-Racah polynomials given in (2.44), $\boldsymbol{\alpha}_{L_1}^{n_2(k_2)}$ is given in Proposition 4.2 and $\boldsymbol{\beta}=(\alpha_0,\alpha_1+2m_1(j_1),-\alpha_2,-N_2-1)$. This can be seen from analyzing both q-Racah polynomials (4.6) and (4.7) from $P_{n_1,n_2}(m_1,m_2)$. Let us show this for $P_{n_1(k_1)}^{V_1}(m_1,m_2)$ in (4.6). Entirely similar to the univariate case (2.19) we have

$$\begin{aligned}\delta_{k_1,k_1'}&=\left\langle\psi_{n_1(k_1')}^{n_2(k_2)},\psi_{n_1(k_1)}^{n_2(k_2)}\right\rangle_{V_1}=\sum_{j_1=0}^{N_1}\left\langle\psi_{n_1(k_1')}^{n_2(k_2)},\phi_{m_1(j_1)}\right\rangle_{V_1}\left\langle\phi_{m_1(j_1)},\psi_{n_1(k_1)}^{n_2(k_2)}\right\rangle_{V_1}\\&=\sum_{j_1=0}^{N_1}w(j_1,k_1,\boldsymbol{\alpha}_{L_1}^{n_2(k_2)};q)P_{n_1(k_1)}^{V_1}(m_1(j_1);m_2)\overline{P_{n_1(k_1')}^{V_1}(m_1(j_1);m_2)}.\end{aligned}$$

Similarly, one finds that

$$\delta_{k_2,k_2'}=\sum_{j_2=0}^{N_2}w(j_2,k_2,\boldsymbol{\beta};q)P_{n_2(k_2)}^{V_2}(m_2(j_2);n_1)\overline{P_{n_2(k_2')}^{V_2}(m_2(j_2);n_1)},$$

which proves (4.12).

Remark 4.8. The bivariate q-Racah polynomials (4.4) are the same, up to a factor in front of the two q-hypergeometric series, as found by Gasper and Rahman [12] after a change of parameters, variables and degrees. In the notation of [12, equation (2.6)], looking only at the q-hypergeometric series we have

$$\begin{aligned}
&R_{n_1,n_2}(x_1,x_2;a_1,a_2,a_3,b,N;q)\\
&\quad= r_{n_1}\big(x_1;b,a_2/q,a_1q^{2x_2},x_2;q\big)r_{n_2}\big(x_2-n_1;ba_2q^{n_1},a_3/q,a_1a_2q^{N+n_1},N-n_1;q\big)\\
&\quad= {}_4\phi_3\left[\begin{matrix}q^{-n_1},\ ba_2q^{n_1},\ q^{-x_1},\ a_1q^{x_1}\\ bq,\ a_1a_2q^{x_2},\ q^{-x_2}\end{matrix};q;q\right]\\
&\quad\times\ {}_4\phi_3\left[\begin{matrix}q^{-n_2},\ ba_2a_3q^{2n_1+n_2},\ q^{n_1-x_2},\ a_1a_2q^{n_1+x_2}\\ ba_2q^{2n_1+1},\ a_1a_2a_3q^{n_1+N},\ q^{n_1-N}\end{matrix};q;q\right].
\end{aligned}\tag{4.13}$$

Let us now make a change of parameters, variables and degrees. First, define

$$\alpha=\alpha_0+\alpha_1+\alpha_2 \qquad\text{and}\qquad \hat\alpha=\alpha_0-\alpha_1+\alpha_2.$$

Then take parameters

$$\begin{aligned}
&a_1=-q^{2\alpha_0+2\alpha_2+2},\qquad a_2=q^{-2N_2},\qquad a_3=q^{-2N_1},\qquad b=-q^{2\alpha_0},\\
&2N=-\hat\alpha+N_1+N_2-1,
\end{aligned}$$

variables

$$2x_1=2j_1+2j_2-\hat\alpha-N_1-N_2-1,\qquad 2x_2=2j_1-\hat\alpha-N_1+N_2-1,$$

and degrees $n_1=k_2$ and $n_2=k_1$. If we also take q^2 instead of q, we obtain that (4.13) is equal to

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix}q^{-2k_2},\ -q^{2k_2+2\alpha_0-2N_2},\ q^{-2j_1-2j_2+\hat\alpha+N_1+N_2+1},\ -q^{2j_1+2j_2+\alpha-N_1-N_2+1}\\ -q^{2\alpha_0+2},\ -q^{2j_1+\alpha-N_1-N_2-1},\ q^{-2j_1+\hat\alpha+N_1-N_2+1}\end{matrix};q^2;q^2\right]\\
&\times{}_4\phi_3\left[\begin{matrix}q^{-2k_1},\ -q^{4k_2+2k_1+2\alpha_0-2N_1-2N_2},\ q^{2k_2-2j_1+\hat\alpha+N_1-N_2+1},\ -q^{2k_2+2j_1+\alpha-N_1-N_2-1}\\ -q^{4k_2+2\alpha_0-2N_2+2},\ -q^{2k_2+\alpha-N_1-N_2-1},\ q^{2k_2+\hat\alpha-N_1-N_2+1}\end{matrix};q^2;q^2\right].
\end{aligned}\tag{4.14}$$

Then apply to both q-Racah polynomials (4.14) Sears' transformation formula for a balanced ${}_4\phi_3$ [10, equation (III.15)],

$${}_4\phi_3\left[\begin{matrix}q^{-n},\ \alpha,\ \beta,\ \gamma\\ \delta,\ \varepsilon,\ \zeta\end{matrix};q;q\right]=\frac{\big(\alpha^{-1}\delta,\alpha^{-1}\zeta;q\big)_n}{(\delta,\zeta;q)_n}\,{}_4\phi_3\left[\begin{matrix}q^{-n},\ \alpha,\ \beta^{-1}\varepsilon,\ \gamma^{-1}\varepsilon\\ \alpha\delta^{-1}q^{1-n},\ \epsilon,\ \alpha\zeta^{-1}q^{1-n}\end{matrix};q;q\right].$$

Afer this transformation, our product of q-Racah polynomials (4.4) is the same as the bivariate q-Racah polynomial defined by Gasper and Rahman [12] up to a factor in front of the q-hypergeometric series.

5 AW$_2$ and q-difference operators for bivariate q-Racah polynomials

Until now, the structure parameters of an AW-relation could be interpreted as a constant. Either it was a central element of AW$_2$ or it acted as multiplication by an eigenvalue. In this way, one generator of the AW-relation worked as a three-term operator on the eigenvectors of the other generator. In this section, we will see that this is no longer true for all generators that satisfy an AW-relation. We will show that L_2 acts as a 9-term operator on the eigenvectors $\psi_{n_1}^{n_2}\otimes\psi_{n_2}$. Crucial here is the following observation. Suppose we have elements K, L and $(A_k)_{k=0}^3$ in an algebra and $A_4, A_5\in\mathbb{R}$ such that

$$\mathscr{AW}(K,L\mid A_0,A_1,A_2,A_3,A_4,A_5)=0$$

and $(A_k)_{k=0}^3$ are locally central. Then we will show that K still acts as a three-term operator on the eigenspaces of L. Also, the form of the spectrum of K and L is similar to what we have seen before. This result is proven in the following lemma.

Lemma 5.1. *Let $\mathscr{A}$ be an algebra, K, L and $(A_k)_{k=0}^3$ elements in $\mathscr{A}$ and $A_4, A_5 \in \mathbb{R}$. Suppose*

$$\mathscr{AW}(K, L \mid A_0, A_1, A_2, A_3, A_4, A_5) = 0$$

and $(A_k)_{k=0}^3$ are locally central. Then we have the following:

(i) *The spectra of K and L have the same hyperbolic form as in the original* AW(3). *In particular, if $A_4 = A_5 = (\sinh_q(1))^2$ we have*

$$\lambda_n(p_0) = \sinh_q(2n + p_0 + 1)$$

for some $p_0 \in \mathbb{R}$.

(ii) *Denote the eigenspace corresponding to $\lambda_n(p_0)$ of K by $\Psi(n)$. Then L acts as a three-term operator on the eigenspaces of K. That is,*

$$L\Psi(n) \subset \Psi(n-1) \cup \Psi(n) \cup \Psi(n+1).$$

Here the eigenspace $\Psi(n)$ is defined as subspace of vectors which have eigenvalue $\lambda_n(p_0)$ and

$$L\Psi(n) = \{Lv \colon v \in \Psi(n)\}.$$

Proof. We proceed similarly as in the original algebra AW(3) and in the proof of Proposition 4.2, only with a slight adjustment, since in the current setting $(A_k)_{k=0}^3$ cannot all be interpreted as constants. Let B, C_0, C_1, D_0, D_1 as in (2.37) and let ψ_n be an eigenvector of K. We want to find a new eigenvector ψ_s of K. Take

$$\psi_s = (\beta L + \gamma[K, L]_q + \delta B + \varepsilon D_1)\psi_n$$

for some constants β, γ, δ, ε that still need to be determined. Requiring ψ_s to be an eigenvector of K with a different eigenvalue leads to

$$\lambda_n^2 + \lambda_s^2 - \cosh_q(2)\lambda_n\lambda_s + C_1 = 0 \tag{5.1}$$

and

$$-q\gamma(\lambda_n B + D_1) + (\lambda_n - \lambda_s)\delta B + \varepsilon(\lambda_n - \lambda_s)D_1 = 0.$$

This second equation is equivalent to

$$(\delta(\lambda_n - \lambda_s) - q\gamma\lambda_n)B + (\varepsilon(\lambda_n - \lambda_s) - q\gamma)D_1 = 0, \tag{5.2}$$

which has a nontrivial solution for any nonzero γ and $\lambda_s \neq \lambda_n$. Equation (5.1) is the same as in the original AW(3) and forces the spectrum of K to be of the same hyperbolic form, proving (i).

For (ii), let $\psi_n \in \Psi(n)$. Since (5.2) can be solved for all n and $s \in \{n-1, n+1\}$, there exists $\psi_{n+1} \in \Psi(n+1)$ and $\psi_{n-1} \in \Psi(n-1)$ and constants β^+, γ^+, δ^+, ε^+, β^-, γ^-, δ^-, ε^- such that

$$\begin{aligned} \psi_{n+1} &= \big(\beta^+ L + \gamma^+[K, L]_q + \delta^+ B + \varepsilon^+ D_1\big)\psi_n, \\ \psi_{n-1} &= \big(\beta^- L + \gamma^-[K, L]_q + \delta^- B + \varepsilon^- D_1\big)\psi_n. \end{aligned} \tag{5.3}$$

By substitution,[6] we can eliminate $[K, L]_q$ from (5.3) and obtain

$$L\psi_n = a_n\psi_{n-1} + b_n\psi_{n+1} + c_n B\psi_n + d_n D_1\psi_n,$$

for suitable constants a_n, b_n, c_n, d_n. Since B and D_1 commute with K, it leaves its eigenspaces intact. Therefore,

$$B\psi_n, D_1\psi_n \in \Psi(n),$$

which proves (ii). ■

Remark 5.2. In the original AW(3), the eigenspaces of K and L were one-dimensional. However, this need not be true in general as we already saw in AW_2. There, the eigenspaces of K_2 and L_2 can have dimensions up to $(N_1 + 1) + (N_2 + 1)$.

We can use Lemma 5.1 twice for L_2, since it appears in two AW-relations: (3.1) and (3.2). Consequently, L_2 acts on $\psi^{n_2}_{n_1} \otimes \psi_{n_2}$ as a three term operator in the eigenspaces of both n_1 and n_2, which means L_2 is a 9-term operator.

Proposition 5.3. *L_2 acts as a 9-term operator on $\psi^{n_2}_{n_1} \otimes \psi_{n_2}$. That is, there exists constants a_{n_1,n_2}, b_{n_1,n_2}, c_{n_1,n_2}, d_{n_1,n_2} and e_{n_1,n_2} such that*

$$\begin{aligned} L_2\big(\psi^{n_2}_{n_1} \otimes \psi_{n_2}\big) = {} & a_{n_1,n_2}\ \psi^{n_2-1}_{n_1-1} \otimes \psi_{n_2-1} + b_{n_1,n_2}\ \psi^{n_2}_{n_1-1} \otimes \psi_{n_2} + c_{n_1,n_2}\ \psi^{n_2+1}_{n_1-1} \otimes \psi_{n_2+1} \\ & + d_{n_1,n_2}\ \psi^{n_2-1}_{n_1} \otimes \psi_{n_2-1} + e_{n_1,n_2}\ \psi^{n_2}_{n_1} \otimes \psi_{n_2} + d_{n_1,n_2+1}\ \psi^{n_2+1}_{n_1} \otimes \psi_{n_2+1} \\ & + c_{n_1+1,n_2-1}\ \psi^{n_2-1}_{n_1+1} \otimes \psi_{n_2-1} + b_{n_1+1,n_2}\ \psi^{n_2}_{n_1+1} \otimes \psi_{n_2} \\ & + a_{n_1+1,n_2+1}\ \psi^{n_2+1}_{n_1+1} \otimes \psi_{n_2+1}, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} & n_2 \in \left\{-\frac{N_2}{2}, -\frac{N_2}{2}+1, \dots, \frac{N_2}{2}-1, \frac{N_2}{2}\right\}, \\ & n_1 \in \left\{n_2 - \frac{N_1}{2}, n_2 - \frac{N_1}{2}+1, \dots, n_2 + \frac{N_1}{2}-1, n_2 + \frac{N_1}{2}\right\}. \end{aligned} \tag{5.5}$$

Proof. We apply Lemma 5.1 twice and then we will see that L_2 has to be a 9-term operator. We will work with simultaneous eigenvectors of both $1 \otimes K_1$ and K_2 given by

$$\psi^{n_2}_{n_1} \otimes \psi_{n_2},$$

which corresponds, a bit confusingly, to either the eigenvalue λ_{n_1} for K_2 or λ_{n_2} for K_1. Moreover, n_1, n_2 are as in (5.5). Fix n_1 and n_2 and denote the eigenspace corresponding to λ_{n_1} by $\Psi_{K_2}(n_1)$ and to λ_{n_2} by $\Psi_{K_1}(n_2)$. First of all, since (3.2) holds, Lemma 5.1 tells us that

$$L_2\Psi_{K_1}(n_2) \subset \Psi_{K_1}(n_2 - 1) \cup \Psi_{K_1}(n_2) \cup \Psi_{K_1}(n_2 + 1).$$

Said differently, there exists $v_1, v_2, v_3 \in V_1$ such that

$$L_2\big(\psi^{n_2}_{n_1} \otimes \psi_{n_2}\big) = v_1 \otimes \psi_{n_2-1} + v_2 \otimes \psi_{n_2} + v_3 \otimes \psi_{n_2+1}. \tag{5.6}$$

Secondly, because (3.1) holds, Lemma 5.1 tells us that

$$L_2\Psi_{K_2}(n_1) \subset \Psi_{K_2}(n_1 - 1) \cup \Psi_{K_2}(n_1) \cup \Psi_{K_2}(n_1 + 1).$$

[6] If $\lambda_{n-1} \neq \lambda_{n+1}$, the matrix $\left(\begin{smallmatrix}\beta^- & \gamma^- \\ \beta^+ & \gamma^+\end{smallmatrix}\right)$ is non-singular and we can always solve (5.3) for $L\psi_n$ while eliminating $[K, L]_q$.

Combining this with (5.6) gives that

$$
\begin{aligned}
&v_1 \otimes \psi_{n_2-1} \in \Psi_{K_2}(n_1-1) \cup \Psi_{K_2}(n_1) \cup \Psi_{K_2}(n_1+1),\\
&v_2 \otimes \psi_{n_2} \in \Psi_{K_2}(n_1-1) \cup \Psi_{K_2}(n_1) \cup \Psi_{K_2}(n_1+1),\\
&v_3 \otimes \psi_{n_2+1} \in \Psi_{K_2}(n_1-1) \cup \Psi_{K_2}(n_1) \cup \Psi_{K_2}(n_1+1).
\end{aligned}
$$

Therefore, there exist constants $(c_k)_{k=0}^8$ such that

$$
\begin{aligned}
&v_1 = c_0\psi_{n_1-1}^{n_2-1} \otimes \psi_{n_2-1} + c_1\psi_{n_1}^{n_2-1} \otimes \psi_{n_2-1} + c_2\psi_{n_1+1}^{n_2-1} \otimes \psi_{n_2-1},\\
&v_2 = c_3\psi_{n_1-1}^{n_2} \otimes \psi_{n_2} + c_4\psi_{n_1}^{n_2} \otimes \psi_{n_2} + c_5\psi_{n_1+1}^{n_2} \otimes \psi_{n_2},\\
&v_3 = c_5\psi_{n_1-1}^{n_2+1} \otimes \psi_{n_2+1} + c_7\psi_{n_1}^{n_2+1} \otimes \psi_{n_2+1} + c_8\psi_{n_1+1}^{n_2+1} \otimes \psi_{n_2+1}.
\end{aligned}
$$

By Proposition 3.9, we can require L_2 to be self-adjoint, which gives the form (5.4). ■

To compute the coefficients of (5.4), we need to know how M_2 acts on $\psi_{n_1}^{n_2} \otimes \psi_{n_2}$. This is the subject of the next proposition, which is also of interest of its own. Since M_2 and $1 \otimes K_1$ satisfy the AW-relations (3.4) and M_2 commutes with K_2, we can show that M_2 acts as a three-term operator on the eigenvectors $\psi_{n_1}^{n_2} \otimes \psi_{n_2}$.

Proposition 5.4. *We have*

$$
\begin{aligned}
M_2\big(\psi_{n_1}^{n_2} \otimes \psi_{n_2}\big) = {}& \tilde{a}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\psi_{n_1}^{n_2-1} \otimes \psi_{n_2-1} + \tilde{b}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\psi_{n_1}^{n_2} \otimes \psi_{n_2}\\
&+ \tilde{a}_{n_2+1}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\psi_{n_1}^{n_2+1} \otimes \psi_{n_2+1},
\end{aligned}
$$

where $\boldsymbol{\alpha}_{M_2}^{n_1} = (\alpha_0, -N_1-1, -\alpha_0-2n_1, -N_2-1)$. *The coefficients* $\tilde{a}_n^2$ *and* $\tilde{b}_n$ *can be found from* (2.48) *and* (2.49)*, respectively.*

Proof. Denote by $\Psi_{K_1}(n_2)$ the eigenspace of $1 \otimes K_1$ corresponding to the eigenvalue λ_{n_2}. From Lemma 5.1(ii) and the AW-relations (3.4), we know that

$$
M_2\big(\psi_{n_1}^{n_2} \otimes \psi_{n_2}\big) \in \Psi(n_2-1) \cup \Psi(n_2) \cup \Psi(n_2+1). \tag{5.7}
$$

Since M_2 and K_2 commute, the eigenspaces of K_2, denoted by $\Psi_{K_2}(n_1)$ for the eigenvalue λ_{n_1}, are invariant under the action of M_2,

$$
M_2\Psi_{K_2}(n_1) \subset \Psi_{K_2}(n_1).
$$

Combining this with (5.7) implies that there exists constants c_{n_2-1}, c_{n_2} and c_{n_2+1} such that

$$
M_2\big(\psi_{n_1}^{n_2} \otimes \psi_{n_2}\big) = c_{n_2-1}\psi_{n_1}^{n_2-1} \otimes \psi_{n_2-1} + c_{n_2}\psi_{n_1}^{n_2} \otimes \psi_{n_2} + c_{n_2+1}\psi_{n_1}^{n_2+1} \otimes \psi_{n_2+1}.
$$

It remains to find the constants c_{n_2-1}, c_{n_2} and c_{n_2+1}. Observe that we can now interpret K_2 in the AW-relations (3.4) as the 'constant' λ_{n_1}. Therefore, we can repeat the proof of Proposition 4.2. Note that we are in the setting $C_0 > 0$ and $C_1 < 0$. Therefore, we have to use the formulas (2.48) and (2.49). We find

$$
c_{n_2-1} = \tilde{a}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big), \qquad c_{n_2} = \tilde{b}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big) \qquad \text{and} \qquad c_{n_2+1} = \tilde{a}_{n_2+1}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big).
$$
■

Let us now compute L_2 explicitly.

Theorem 5.5. *The coefficients in* (5.4) *of the 9-term operator* L_2 *are given by*

$$
a_{n_1,n_2} = \frac{a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)d_{n_1-1,n_2} - a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2-1}\big)d_{n_1,n_2}}{b_{n_1-1}\big(\boldsymbol{\alpha}_{L_1}^{n_2-1}\big) - b_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)},
$$

$$
\begin{aligned}
b_{n_1,n_2} &= \frac{a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)\big(A_0\lambda_{n_2} + \cosh_q(1)A_3^{(2)}\big)}{\cosh_q(2n_2+\alpha_0-1)\cosh_q(2n_2+\alpha_0+1)},\\
c_{n_1,n_2} &= \frac{a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)d_{n_1-1,n_2+1} - a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2+1}\big)d_{n_1,n_2+1}}{b_{n_1-1}\big(\boldsymbol{\alpha}_{L_1}^{n_2+1}\big) - b_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)},\\
d_{n_1,n_2} &= \frac{\tilde{a}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)(\cosh_q(1)A_1 - A_2\lambda_{n_1})}{\cosh_q(2n_1+\alpha_0-1)\cosh_q(2n_1+\alpha_0+1)},\\
e_{n_1,n_2} &= \frac{b_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)(\cosh_q(1)A_1 - A_2\lambda_{n_1}) + A_0(A_1\lambda_{n_1} + \cosh_q(1)A_2)}{\cosh_q(2n_1+\alpha_0-1)\cosh_q(2n_1+\alpha_0+1)}.
\end{aligned}
$$

Proof. Computing b_{n_1,n_2}, d_{n_1,n_2}, e_{n_1,n_2} is similar to calculating b_n in the original AW(3). The 'corner' terms a_{n_1,n_2}, c_{n_1,n_2} can be deduced from the commutativity of L_2 and $L_1 \otimes 1$ and the expressions for b_{n_1,n_2}, d_{n_1,n_2}, e_{n_1,n_2}. Also note that we do not need to worry about consistency of the coefficients because of Proposition 3.9.

Let us start with d_{n_1,n_2} and e_{n_1,n_2}. We can apply both sides of the first AW-relation of (3.1) to $\psi_{n_1}^{n_2} \otimes \psi_{n_2}$,

$$
\begin{aligned}
&\big(\cosh_q(2)K_2L_2K_2 - K_2^2L_2 - L_2K_2^2\big)\psi_{n_1}^{n_2} \otimes \psi_{n_2}\\
&\qquad = \big[(\sinh_q(1))^2(A_0A_1 - A_2M_2)K_2 - (\sinh_q(2))^2L_2\\
&\qquad\quad + \cosh_q(1)(\sinh_q(1))^2(A_1M_2 + A_0A_2)\big]\psi_{n_1}^{n_2} \otimes \psi_{n_2}.
\end{aligned}
$$

If we use Proposition 5.4 and

$$
K_2\psi_{n_1}^{n_2} \otimes \psi_{n_2} = \lambda_{n_1}\psi_{n_1}^{n_2} \otimes \psi_{n_2},
$$

we get 9 equations. Each one coming from the term in front of one of the eigenvectors $\psi_{n_1+i}^{n_2+j} \otimes \psi_{n_2+j}$ with $i,j \in \{-1,0,1\}$. Similar to the original AW(3), one can show that the 6 equations coming from $\psi_{n_1\pm 1}^{n_2+j} \otimes \psi_{n_2+j}$ are satisfied automatically using (2.11). Looking at the terms in front of $\psi_{n_1}^{n_2+j} \otimes \psi_{n_2+j}$ for $j \in \{-1,0,1\}$ gives

$$
c_{n_1}d_{n_1,n_2} = (\sinh_q(1))^2\big(\cosh_q(1)A_1\tilde{a}_{n_2}(\boldsymbol{\alpha}_{M_2}^{n_1} - A_2\tilde{a}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\lambda_{n_1})\big), \tag{5.8}
$$
$$
c_{n_1}e_{n_1,n_2} = (\sinh_q(1))^2\big(\big(A_0A_1 - A_2\tilde{b}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\big)\lambda_{n_1} + \cosh_q(1)(A_1\tilde{b}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big) + A_0A_2)\big), \tag{5.9}
$$
$$
c_{n_1}d_{n_1,n_2+1} = (\sinh_q(1))^2\big(\cosh_q(1)A_1\tilde{a}_{n_2+1}\big(\boldsymbol{\alpha}_{M_2}^{n_1} - A_2\tilde{a}_{n_2+1}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)\lambda_{n_1}\big)\big), \tag{5.10}
$$

where

$$
\begin{aligned}
c_{n_1} &= \lambda_{n_1}^2\,(\cosh_q(2) - 2) + \cosh_q(2)^2\\
&= (\sinh_q(1))^2\cosh_q(2n_1+\alpha_0-1)\cosh_q(2n_1+\alpha_0+1),
\end{aligned}
$$

using (2.11) and (2.45). It is easy to see that (5.10) is just (5.8) with n_2+1 instead of n_2. The coefficients d_{n_1,n_2} and e_{n_1,n_2} can be computed from (5.8) and (5.9),

$$
\begin{aligned}
d_{n_1,n_2} &= \frac{\tilde{a}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)(\cosh_q(1)A_1 - A_2\lambda_{n_1})}{\cosh_q(2n_1+\alpha_0-1)\cosh_q(2n_1+\alpha_0+1)},\\
e_{n_1,n_2} &= \frac{\tilde{b}_{n_2}\big(\boldsymbol{\alpha}_{M_2}^{n_1}\big)(\cosh_q(1)A_1 - A_2\lambda_{n_1}) + A_0(A_1\lambda_{n_1} + \cosh_q(1)A_2)}{\cosh_q(2n_1+\alpha_0-1)\cosh_q(2n_1+\alpha_0+1)}.
\end{aligned} \tag{5.11}
$$

Similarly, we can apply both sides of the first AW-relation of (3.2) to $\psi_{n_1}^{n_2} \otimes \psi_{n_2}$ to compute b_{n_1,n_2} and e_{n_1,n_2}. Instead of Proposition 5.4, we will use Proposition 4.2. Equating the terms in front of each of the eigenvectors $\psi_{n_1+i}^{n_2+j} \otimes \psi_{n_2+j}$ with $i,j \in \{-1,0,1\}$ again gives 9 equations, where now the 6 equations coming from $\psi_{n_1+i}^{n_2\pm1} \otimes \psi_{n_2\pm1}$, $i \in \{-1,0,1\}$, are satisfied automatically. The other 3 lead in a similar way as before to

$$
\begin{aligned}
b_{n_1,n_2} &= \frac{a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)\big(A_0\lambda_{n_2} + \cosh_q(1)A_3^{(2)}\big)}{\cosh_q(2n_2+\alpha_0-1)\cosh_q(2n_2+\alpha_0+1)}, \\
e_{n_1,n_2} &= \frac{b_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)\big(A_0\lambda_{n_2} + \cosh_q(1)A_3^{(2)}\big) - A_2\big(A_3^{(2)}\lambda_{n_2} - \cosh_q(1)A_0\big)}{\cosh_q(2n_2+\alpha_0-1)\cosh_q(2n_2+\alpha_0+1)}.
\end{aligned} \tag{5.12}
$$

By Proposition 3.9, we know that both expressions (5.11) and (5.12) are consistent. One can also check this by doing a simple, quite tedious computation.

Let us now compute the corner terms a_{n_1,n_2} and c_{n_1,n_2}. Since L_2 and $L_1 \otimes 1$ commute, we have

$$
L_2(L_1\otimes 1)\psi_{n_1}^{n_2}\otimes\psi_{n_2} = (L_1\otimes 1)L_2\psi_{n_1}^{n_2}\otimes\psi_{n_2}.
$$

Working this out gives 15 equations for each of the eigenvectors $\psi_{n_1+i}^{n_2+j} \otimes \psi_{n_2+j}$, where $i \in \{-2,-1,0,1,2\}$ and $j \in \{-1,0,1\}$. Let us first focus on the eigenvector $\psi_{n_1-1}^{n_2-1} \otimes \psi_{n_2-1}$ for which there are two possibilities to get terms in front. Comparing these terms on both sides gives

$$
a_{n_1,n_2}b_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big) + d_{n_1-1,n_2}a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big) = b_{n_1-1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)a_{n_1,n_2} + a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2-1}\big)d_{n_1,n_2}.
$$

This is equivalent to

$$
a_{n_1,n_2} = \frac{a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)d_{n_1-1,n_2} - a_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2-1}\big)d_{n_1,n_2}}{b_{n_1-1}\big(\boldsymbol{\alpha}_{L_1}^{n_2-1}\big) - b_{n_1}\big(\boldsymbol{\alpha}_{L_1}^{n_2}\big)}.
$$

Doing the same for $\psi_{n_1-1}^{n_2+1} \otimes \psi_{n_2+1}$ gives the expression for c_{n_1,n_2}. ■

Since L_2 acts as a 9-term operator on the basis $\psi_{n_1}^{n_2} \otimes \psi_{n_2}$ and as a multiplication operator on the basis $\phi_{m_1} \otimes \phi_{m_2}^{m_1}$, it follows immediately that the overlap coefficient $P_{n_1,n_2}(m_1,m_2)$ defined by (4.2) satisfies a 9-term recurrence relation, or equivalently, it is an eigenfunction of a 9-term difference operator in the variables n_1 and n_2. Similarly, from the action of $L_1 \otimes 1$ it follows that $P_{n_1,n_2}(m_1,m_2)$ is also an eigenfunction of a 3-term difference operator. In this way we recover Iliev's difference operators for the bivariate q-Racah polynomials, see [20, Proposition 4.5 and Remark 2.3] in case $d=2$. So AW_2 encodes the bispectral properties of Gasper and Rahman's (or Tratnik-type) bivariate q-Racah polynomials.

Remark 5.6. A recent paper [5] studies the rank 2 Racah algebra, which can be considered as a $q=1$ version of a rank 2 Askey–Wilson algebra. The Tratnik-type bivariate Racah polynomials appear as overlap coefficients, but also other overlap coefficients resembling $9j$-symbols are studied. These are shown to be 'Griffith-like' bivariate Racah polynomials, which are different from the bivariate Tratnik-type Racah polynomials, and the algebraic setting provides difference operators for these polynomials. This shows that that the rank 2 Racah algebra not only encodes the spectral properties of the Tratnik-type bivariate Racah polynomials, but also of other bivariate functions. It would be very interesting to see if similar 'Griffith-like' bivariate q-Racah polynomials can be obtained in the setting of the algebra AW_2. Presumably this requires the extra relations (3.7) that are present in AW_2 inside $\mathcal{U}_q \otimes \mathcal{U}_q$, but not in the general setup of Definition 3.5.

A Proof of Theorem 3.1

Let Y_K, Y_L, Ω as in Section 2.4 and B, C_0, C_1, D_0, D_1 as in Theorem 2.2. We will show that each pair of the elements $1\otimes Y_K, \Delta(Y_K), Y_L\otimes 1, \Delta(Y_L), \Delta(\Omega) \in \mathcal{U}_q\otimes\mathcal{U}_q$ either commutes or satisfies the AW-relations (2.2) and (2.3). The pairs of non-commuting elements satisfy these relations with the structure parameters in Table 3, where $\theta = -\sinh_q(1)^{-2}(a_{\hat{E}}b_{\hat{F}})$.

Generator 1	Generator 2	A_0	A_1	A_2	A_3	A_4	A_5
$\Delta(Y_K)$	$\Delta(Y_L)$	a_s	b_t	θ	$\Delta(\Omega)$	$b_{\hat{E}}b_{\hat{F}}$	$a_{\hat{E}}a_{\hat{F}}$
$1\otimes Y_K$	$\Delta(Y_L)$	a_s	$Y_L\otimes 1$	θ	$1\otimes\Omega$	$b_{\hat{E}}b_{\hat{F}}$	$a_{\hat{E}}a_{\hat{F}}$
$\Delta(Y_K)$	$Y_L\otimes 1$	$1\otimes Y_K$	b_t	θ	$\Omega\otimes 1$	$b_{\hat{E}}b_{\hat{F}}$	$a_{\hat{E}}a_{\hat{F}}$
$1\otimes Y_K$	$\Delta(\Omega)$	a_s	$\Omega\otimes 1$	$-\Delta(Y_K)$	$1\otimes\Omega$	$-(\sinh_q(1))^2$	$a_{\hat{E}}a_{\hat{F}}$
$\Delta(\Omega)$	$Y_L\otimes 1$	$1\otimes\Omega$	b_t	$-\Delta(Y_L)$	$\Omega\otimes 1$	$b_{\hat{E}}b_{\hat{F}}$	$-(\sinh_q(1))^2$

Table 3. Askey–Wilson algebra relations of twisted primitive elements.

The commuting part of the proof was already discussed in Section 2.4 as well as the first row of Table 3. We will show rows three and four. The others follow by symmetry. For elements a, b, c in an algebra A, define

$$f_A(a,b,c) := \cosh_q(2)abc - bca - cab,$$

coming from the AW(3) relations (2.2) and (2.3). That is,

$$f_{\mathcal{U}_q}(Y_L,Y_K,Y_L) = BY_L + C_0Y_K + D_0,$$
$$f_{\mathcal{U}_q}(Y_K,Y_L,Y_K) = BY_K + C_1Y_L + D_1.$$

Let us start with showing the AW-relation between $\Delta(Y_K)$ and $Y_L\otimes 1$. Explicitly calculating gives

$$\begin{aligned} f_{\mathcal{U}_q^{\otimes 2}}(\Delta(Y_K), Y_L\otimes 1, \Delta(Y_K)) &= f_{\mathcal{U}_q}\big(\hat{K}^2, Y_L, \hat{K}^2\big)\otimes Y_K^2 + f\big(\tilde{Y_K}, Y_L, \tilde{Y_K}\big)\otimes 1 \\ &\quad + \big[f_{\mathcal{U}_q}\big(\hat{K}^2, Y_L, \tilde{Y_K}\big) + f_{\mathcal{U}_q}\big(\tilde{Y_K}, Y_L, \hat{K}^2\big)\big]\otimes Y_K. \end{aligned}$$

Using (2.24) with $a_{\hat{E}} = a_{\hat{F}} = 0$, we obtain

$$\begin{aligned} f_{\mathcal{U}_q^{\otimes 2}}(\Delta(Y_K), Y_L\otimes 1, \Delta(Y_K)) &= (\sinh_q(1))^2 b_t\hat{K}^2\otimes Y_K^2 + (\tilde{B}\tilde{Y_K} + C_1Y_L + \tilde{D})\otimes 1 \\ &\quad + \big[f_{\mathcal{U}_q}\big(\hat{K}^2, Y_L, \tilde{Y_K}\big) + f_{\mathcal{U}_q}\big(\tilde{Y_K}, Y_L, \hat{K}^2\big)\big]\otimes Y_K, \end{aligned} \tag{A.1}$$

where $\tilde{B} = \big(a_{\hat{E}}b_{\hat{F}} + a_{\hat{F}}b_{\hat{E}}\big)\Omega$ and $\tilde{D}_1 = \cosh_q(1)a_{\hat{E}}a_{\hat{F}}b_t\Omega$. We have

$$\begin{aligned} & f_{\mathcal{U}_q}\big(\hat{K}^2, Y_L, \tilde{Y_K}\big) + f_{\mathcal{U}_q}\big(\tilde{Y_K}, Y_L, \hat{K}^2\big) \\ &\quad = f\big(\tilde{Y_K} + \hat{K}^2, Y_L, \tilde{Y_K} + \hat{K}^2\big) - f\big(\tilde{Y_K}, Y_L, \tilde{Y_K}\big) - f\big(\hat{K}^2, Y_L, \hat{K}^2\big) \\ &\quad = (\sinh_q(1))^2 b_t\tilde{Y_K} + \tilde{B}\hat{K}^2 + \tilde{D}, \end{aligned}$$

where $\tilde{D} = -\cosh_q(1)\big(a_{\hat{E}}b_{\hat{F}} + a_{\hat{F}}b_{\hat{E}}\big)$. Therefore, (A.1) becomes

$$\begin{aligned} & f_{\mathcal{U}_q^{\otimes 2}}(\Delta(Y_K), Y_L\otimes 1, \Delta(Y_K)) \\ &\quad = \big(\tilde{B}\otimes 1\big)\Delta(Y_K) + C_1Y_L\otimes 1 + \tilde{D}_1 + (\sinh_q(1))^2 b_t\Delta(Y_K)(1\otimes Y_K) + \tilde{D}(1\otimes Y_K). \end{aligned}$$

For the other relation of $Y_L\otimes 1$ and $\Delta(Y_K)$, we obtain

$$f_{\mathcal{U}_q^{\otimes 2}}(Y_L\otimes 1, \Delta(Y_K), Y_L\otimes 1) = f_{\mathcal{U}_q}(Y_L, \hat{K}^2, Y_L)\otimes Y_K + f_{\mathcal{U}_q}(Y_L, \tilde{Y_K}, Y_L)\otimes 1$$

$$
\begin{aligned}
&= \big((\sinh_q(1))^2 b_t Y_L + C_0\hat{K}^2 + \tilde{D}_0\big)\otimes Y_K\\
&\quad + \big(\tilde{B}Y_L + C_0\tilde{Y_K} + \tilde{D}b_t\big)\otimes 1\\
&= \big(\tilde{B}\otimes 1\big)(Y_L\otimes 1) + C_0\Delta(Y_K) + \tilde{D}b_t\\
&\quad + (\sinh_q(1))^2 b_t (Y_L\otimes 1)(1\otimes Y_K) + (\tilde{D}_0\otimes 1)(1\otimes Y_K),
\end{aligned}
$$

where $\tilde{D}_0 = \cosh_q(1) b_{\hat{E}} b_{\hat{F}}\Omega$.

Let us now show the AW-relation between $\Delta(\Omega)$ and $1\otimes Y_K$. Let us first calculate the coproduct of the Casimir of $\mathcal{U}_q$. Using that $\hat{K}$, $\hat{K}^{-1}$ are group-like[7] and $\hat{E}$, $\hat{F}$ are twisted primitive with respect to $\hat{K}$, we obtain

$$
\begin{aligned}
\frac{\Delta(\Omega)}{(\sinh_q(1))^2} &= \frac{q^{-1}\hat{K}^2\otimes\hat{K}^2 + q\hat{K}^{-2}\otimes\hat{K}^{-2}}{(\sinh_q(1))^2} + \big(\hat{K}\otimes\hat{E} + \hat{E}\otimes\hat{K}^{-1}\big)\big(\hat{K}\otimes\hat{F} + \hat{F}\otimes\hat{K}^{-1}\big)\\
&= \frac{\hat{K}^2\otimes\Omega + \Omega\otimes\hat{K}^{-2} - \cosh_q(1)\hat{K}^2\otimes\hat{K}^{-2}}{(\sinh_q(1))^2} + \big(\hat{K}\hat{F}\otimes\hat{E}\hat{K}^{-1} + \hat{E}\hat{K}\otimes\hat{K}^{-1}\hat{F}\big).
\end{aligned}
$$

Note that all the factors that appear are specific versions of Y_K and Y_L. For example, $\hat{E}\hat{K}^{-1}$ is Y_L where $b_{\hat{F}} = b_t = 0$ and $b_{\hat{E}} = q^{1/2}$. Therefore, we can use (2.24) to obtain

$$
\begin{aligned}
f_{\mathcal{U}_q^{\otimes 2}}(1\otimes Y_K, \Delta(\Omega), 1\otimes Y_K) &= (\sinh_q(1))^2\hat{K}^2\otimes Y_K^2\Omega + \Omega\otimes f_{\mathcal{U}_q}\big(Y_K,\hat{K}^{-2},Y_K\big)\\
&\quad + (\sinh_q(1))^2\hat{K}\hat{F}\otimes f_{\mathcal{U}_q}\big(Y_K,\hat{E}\hat{K}^{-1},Y_K\big)\\
&\quad + (\sinh_q(1))^2\hat{E}\hat{K}\otimes f_{\mathcal{U}_q}\big(Y_K,\hat{K}^{-1}\hat{F},Y_K\big)\\
&\quad - \cosh_q(1)\hat{K}^2\otimes f_{\mathcal{U}_q}\big(Y_K,\hat{K}^{-2},Y_K\big).
\end{aligned}
\tag{A.2}
$$

Using (2.2) and (2.24), we get

$$
\begin{aligned}
&f_{\mathcal{U}_q}\big(Y_K,\hat{K}^{-2},Y_K\big) = (\sinh_q(1))^2 a_s Y_K + C_1\hat{K}^{-2} + \cosh_q(1) a_{\hat{E}} a_{\hat{F}}\Omega,\\
&f_{\mathcal{U}_q}\big(Y_K,\hat{E}\hat{K}^{-1},Y_K\big) = q^{1/2} a_{\hat{F}}\Omega Y_K + C_1\hat{E}\hat{K}^{-1} - q^{1/2}\cosh_q(1) a_{\hat{F}} a_s,\\
&f_{\mathcal{U}_q}\big(Y_K,\hat{K}^{-1}\hat{F},Y_K\big) = q^{1/2} a_{\hat{E}}\Omega Y_K + C_1\hat{K}^{-1}\hat{F} - q^{1/2}\cosh_q(1) a_{\hat{E}} a_s.
\end{aligned}
$$

Therefore, (A.2) becomes

$$
\begin{aligned}
f_{\mathcal{U}_q^{\otimes 2}}(1\otimes Y_K, \Delta(\Omega), 1\otimes Y_K) &= (\sinh_q(1))^2(1\otimes\Omega)\Delta(Y_K)(1\otimes Y_K) + C_1\Delta(\Omega)\\
&\quad + \cosh_q(1) a_{\hat{E}} a_{\hat{F}}\Omega\otimes\Omega + (\sinh_q(1))^2 a_s(\Omega\otimes 1)(1\otimes Y_K)\\
&\quad - \cosh_q(1)(\sinh_q(1))^2 a_s\Delta(Y_K).
\end{aligned}
$$

Computing $f_{\mathcal{U}_q^{\otimes 2}}(\Delta(\Omega), 1\otimes Y_K, \Delta(\Omega))$ is the most tedious, since we end up with 75 terms. We will not present the full calculation, as it is just a straightforward computation. One can simplify the calculation by observing that some terms vanish because

$$
\begin{aligned}
&f_A(a,b,c) = 0 \qquad \text{if } ab = q^2 ba \ \text{ and } \ cb = q^2 bc,\\
&f_A(a,b,c) + f_A(c,b,a) = 0 \qquad \text{if } ab = q^2 ba, ac = q^{-4}ca \ \text{ and } \ bc = cb.
\end{aligned}
$$

B Calculations for Remark 3.7

Let $S\subset\mathcal{U}_q\otimes\mathcal{U}_q$ be the subalgebra generated by

$$
Y_L\otimes 1,\quad \Delta(Y_L),\quad 1\otimes Y_K,\quad \Delta(Y_K),\quad \Omega\otimes 1,\quad 1\otimes\Omega,\quad \Delta(\Omega).
$$

[7] An element $Y\in\mathcal{U}_q$ is called group-like if $\Delta(Y) = Y\otimes Y$.

Then we have the following correspondence between S and AW(4):

$$\begin{aligned}
&\Lambda_{\{1\}} = b_t, \qquad \Lambda_{\{2\}} = \Omega\otimes 1, \qquad \Lambda_{\{3\}} = 1\otimes\Omega, \qquad \Lambda_{\{4\}} = a_s,\\
&\Lambda_{\{1,2\}} = Y_L\otimes 1, \qquad \Lambda_{\{2,3\}} = \Delta(\Omega), \qquad \Lambda_{\{3,4\}} = 1\otimes Y_K,\\
&\Lambda_{\{1,2,3\}} = \Delta(Y_L), \qquad \Lambda_{\{1,2,3,4\}} = -\frac{a_{\hat{E}}b_{\hat{F}} + a_{\hat{F}}b_{\hat{E}}}{(\sinh_q(1))^2}, \qquad \Lambda_{\{2,3,4\}} = \Delta(Y_K).
\end{aligned} \tag{B.1}$$

We can write the relation [8, equation (23)], when translating to the Askey–Wilson algebra setting, as

$$\Lambda_C = \alpha[\Lambda_A, \Lambda_B]_q + \beta\big(\Lambda_{A\cap B}\Lambda_{A\cup B} + \Lambda_{A\setminus B}\Lambda_{B\setminus A}\big),$$

with $\alpha = -(\sinh_q(2))^{-1}$ and $\beta = (\cosh_q(1))^{-1}$. We use this relation to define $\Lambda_{\{1,3\}}$ by

$$\Lambda_{\{1,3\}} = \alpha[\Lambda_{\{1,2\}}, \Lambda_{\{2,3\}}]_q + \beta\left(\Lambda_{\{2\}}\Lambda_{\{1,2,3\}} + \Lambda_{\{1\}}\Lambda_{\{3\}}\right).$$

Similarly, we define $\Lambda_{\{1,4\}}$ and $\Lambda_{\{1,3,4\}}$ by

$$\begin{aligned}
&\Lambda_{\{1,4\}} = \alpha\big[\Lambda_{\{1,2,3\}}, \Lambda_{\{2,3,4\}}\big]_q + \beta\big(\Lambda_{\{2,3\}}\Lambda_{\{1,2,3,4\}} + \Lambda_{\{1\}}\Lambda_{\{4\}}\big),\\
&\Lambda_{\{1,3,4\}} = \alpha\big[\Lambda_{\{1,2\}}, \Lambda_{\{2,3,4\}}\big]_q + \beta\big(\Lambda_{\{2\}}\Lambda_{\{1,2,3,4\}} + \Lambda_{\{1\}}\Lambda_{\{3,4\}}\big).
\end{aligned}$$

Let us now check that the relation

$$\Lambda_{\{1,4\}} = \alpha\big[\Lambda_{\{1,3\}}, \Lambda_{\{3,4\}}\big]_q + \beta\big(\Lambda_{\{3\}}\Lambda_{\{1,3,4\}} + \Lambda_{\{1\}}\Lambda_{\{4\}}\big) \tag{B.2}$$

also holds in S. We will substitute (B.1) in (B.2) and focus only on factors of the form $\hat{F}^2\otimes X$, with $X\in\mathcal{U}_q$. Substituting the definition of $\Lambda_{\{1,4\}}$, we obtain for the left-hand side,

$$\alpha[\Delta(Y_L), \Delta(Y_K)]_q - \beta\frac{(a_{\hat{E}}b_{\hat{F}} + a_{\hat{F}}b_{\hat{E}})\Delta(\Omega)}{(\sinh_q(1))^2} + \beta b_t a_s.$$

The only term that has a factor $\hat{F}^2\otimes X$ is $\alpha[\Delta(Y_L), \Delta(Y_K)]_q$, which is equal to

$$-a_{\hat{F}}b_{\hat{F}}\ \hat{F}^2\otimes\hat{K}^{-2}.$$

For the right-hand side of (B.2), we substitute the definitions for $\Lambda_{\{1,3\}}$ and $\Lambda_{\{1,3,4\}}$ to obtain

$$\begin{aligned}
&\alpha\big[\alpha[Y_L\otimes 1, \Delta(\Omega)]_q + \beta(\Omega\otimes 1)\Delta(Y_L) + \beta b_t(1\otimes\Omega), 1\otimes Y_K\big]_q\\
&\qquad +\beta(1\otimes\Omega)\left(\alpha[Y_L\otimes 1, \Delta(Y_K)]_q - \beta\frac{(a_{\hat{E}}b_{\hat{F}} + a_{\hat{F}}b_{\hat{E}})(\Omega\otimes 1)}{(\sinh_q(1))^2} + \beta b_t 1\otimes Y_K\right) + \beta b_t a_s.
\end{aligned}$$

Here, the only terms that have a factor $\hat{F}^2\otimes X$ is $\alpha^2[[Y_L\otimes 1, \Delta(\Omega)]_q, 1\otimes Y_K]_q$ and $\alpha\beta(1\otimes\Omega)[Y_L\otimes 1, \Delta(Y_K)]_q$. These are, respectively,

$$\alpha^2 a_{\hat{F}}b_{\hat{F}}\sinh_q(2)(\sinh_q(1))^2\hat{F}^2\otimes[E,F]_q \qquad \text{and} \qquad \alpha\beta a_{\hat{F}}b_{\hat{F}}\sinh_q(2)\hat{F}^2\otimes\Omega.$$

It follows from writing out Ω and using defining relations for $\mathcal{U}_q$ that the factor with $\hat{F}^2\otimes X$ on the right-hand side of relation (B.2) is equal to the the factor on the left-hand side. In the same way the other terms in (B.2) can be checked, showing that this relation is indeed valid in S.

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