ADE Bundles over Surfaces

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Abstract. This is a review paper about ADE bundles over surfaces. Based on the deep connections between the geometry of surfaces and ADE Lie theory, we construct the corresponding ADE bundles over surfaces and study some related problems.

Key words: ADE bundles; surfaces; Cox ring

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1   Introduction

Studies of curves on surfaces is one of the oldest problems in mathematics, starting from the intersection theory of configurations of lines on a plane. It is also an important tool in understanding the geometry of surfaces, see for example [1, 12]. Enumeration of curves on surfaces is an old subject which dates back to the 19th century. It has received much renewed interest and had become a very active research area, probably started from the ground breaking paper by Yau and Zaslow [56] which gave a beautiful formula for the number of curves on a K3 surface in terms of a modular form. Namely the number of rational curves on a K3 surface of a given degree is given by the coefficients of the generating function

$$q/\Delta(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-24},$$

where Δ is the well-known modular form of weight 12. Beauville [4] (see also [23]) gave a mathematical approach to this formula by interpreting the BPS count in terms of compactified Jacobians of curves. For primitive classes, the Yau–Zaslow formula was proven by Bryan and the second author [9, 10, 11, 38, 39] and the full conjecture was proven later by Klemm, Maulik, Pandharipande and Scheidegger [33] (see also [48]) via mirror symmetry for a Calabi–Yau threefold with a K3 fibration.

In [26], Göttsche gave an intriguing generalization of the Yau–Zaslow formula which applies to any surface X and any genus g as long as the curve class C is sufficiently ample. Concretely the number of genus g curves in $|C|$ passing through an appropriate number r of points is given as the coefficient of $q^{C \cdot (C-K)/2}$ in the following power series in q

$$B_1^{K^2} B_2^{C \cdot K} (DG_2)^r (D^2 G_2) / (\Delta(D^2 G_2))^{\chi(O_X)/2},$$

where $D = q\frac{\mathrm{d}}{\mathrm{d}q}$, $G_2(q) = -\frac{1}{24} + \sum_{k>0}\left(\sum_{d|k}d\right)q^k$ is the Eisenstein series and $B_1(q)$ and $B_2(q)$ are universal power series in q. The universality of this amazing conjecture has been solved by Kool–Shende–Thomas [37] and Tzeng [55] independently in 2011 and 2012.

There is also a refined curve counting defined by Block and Göttsche which is related to tropical geometry [5] and real algebraic geometry [27] as well.

In this article, we study very different aspects of the geometry of curves on surfaces, namely the intricate relationships between configurations of low degree curves, for instance lines, and representation theory of simple Lie algebras. The most famous example is probably the 27 lines on a cubic surface discovered by Cayley and Salmon in 1849 and their relationships with the exceptional Lie algebra E_6 (see, e.g., [19, 46]).

The organization of this paper is as follows. In Sections 2, 3, 4 and 5, we introduce the famous examples that reflect the deep connections between the geometry of surfaces and Lie theory. Specifically, we study ADE singularities in Section 2, cubic surfaces in Section 3, del Pezzo surfaces in Section 4 and ADE surfaces in Section 5. Based on Section 5 (ADE surfaces), Sections 6 and 7 are related to F/string theory duality. In Sections 8, 9, 10 and 11, we introduce some results related to deformation of ADE bundles over surfaces with ADE singularities, based on Section 2 (ADE singularities). We study Cox rings of ADE surfaces in Section 12. And the final Section 13 is a summary.

2   ADE singularities vs ADE Lie theory

In this section, we will recall ADE surface singularities, Lie algebras of ADE types and their fundamental represenations. Another intricate relationship between geometry of surfaces and Lie theory is the McKay correspondence [50]. The simplest type of surface singularity $p \in X$ is called a simple singularity (also called a rational double point, canonical singularity, Du Val singularity or ADE singularity) [1]. Locally it is given by the quotient $\mathbb{C}^2/\Gamma$ of $\mathbb{C}^2$ by a finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. The exceptional curve $C = \bigcup_{i=1}^r C_i$ of its minimal resolution $\tilde{X} \to X$ is a union of smooth rational curves C_i's satisfying $C_i \cdot C_i = -2$, i.e., (-2)-curves, and the configuration can be described by its dual graph which is a simply-laced Dynkin diagram, i.e., of ADE type.

Recall that a simple Lie algebra $\mathfrak{g}$ is called simply-laced if all roots have the same length and they are exactly those of ADE types in the classification of simple Lie algebras [29], see Figure 1. Nodes in a Dynkin diagram label fundamental representations of $\mathfrak{g}$.

The Lie algebra $A_n = \mathfrak{sl}(n+1)$ is the algebra of symmetries of a volume form on $V \simeq \mathbb{C}^{n+1}$. The fundamental representations of $\mathfrak{sl}(n+1)$ consist of the standard representation $V \simeq \mathbb{C}^{n+1}$ together with its exterior powers $\Lambda^k V$ with $k = 2, 3, \ldots, n$.

The Lie algebra $D_n = \mathfrak{o}(2n)$ is the algebra of infinitesimal symmetries of a nondegenerate quadratic form $q \in S^2 V^*$ on $V \simeq \mathbb{C}^{2n}$. The fundamental representations of $\mathfrak{o}(2n)$ consist of the standard representation V together with its exterior powers $\Lambda^k V$ with $k = 2, 3, \ldots, n-2$ and also the two spinor representations S^+ and S^-.

Furthermore, E_6 is the algebra of infinitesimal symmetries of a specific cubic form $c \in S^3 V^*$ on $V \simeq \mathbb{C}^{27}$ and E_7 is the algebra of infinitesimal symmetries of a specific quartic form $t \in S^4 V^*$ on $V \simeq \mathbb{C}^{56}$. The explicit forms of these cubic form and quartic form can be described in terms of the geometry of del Pezzo surfaces of degree 3 and 2 respectively. The situation for E_8 is more complicated as its smallest representation V is not a miniscule representation, instead it is the adjoint representation of E_8. We will call the above V's as the standard representations of $\mathfrak{g}$, see Figure 2.
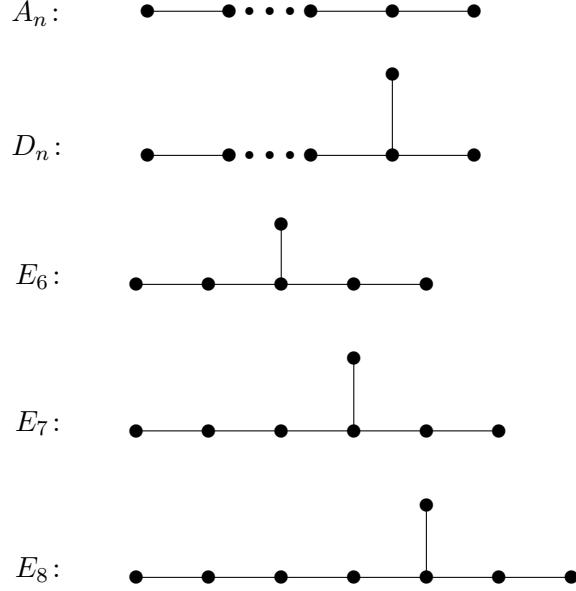
Figure 1. Dynkin diagrams of ADE types.

3   Cubic surfaces vs E_6 Lie theory

Let us come back to explain the relationships between cubic surfaces, or more generally del Pezzo surfaces, with the representation theory of exceptional Lie algebras of type E.

In this section, we will recall various ways to realize the famous 27 lines on cubic surfaces in $\mathbb{P}^3$. Each such geometric setting corresponds to the branching rule for the 27-dimensional standard representation of E_6 to a Lie subalgebra.

Geometric settings	Lie subalgebra of E_6
Degenerate to 3 planes	$\mathfrak{sl}(3) \times \mathfrak{sl}(3) \times \mathfrak{sl}(3)$
Degenerate to plane + quadric surface	$\mathfrak{sl}(6) \times \mathfrak{sl}(2)$
Blowing down a line	$\mathfrak{o}(10)$

We will also explain how to realize the E_6 structure from the configuration of these 27 lines.

It is a classical result that there are exactly 27 lines on a smooth cubic surfaces $X = \{f(z) = 0\} \subset \mathbb{P}^3$ and the symmetry of their intersection pattern is the Weyl group W_{E_6} of E_6 (see, e.g., [46, 49]). Could we recover the Lie algebra E_6 itself, rather than just its Weyl group? Indeed this can be done from the geometry of these 27 lines.

One way to locate these lines is to consider a family of cubic surfaces of the form [51]

$$X(t) = \{z_1 z_2 z_3 + t f(z_0, z_1, z_2, z_3) = 0\} \subset \mathbb{P}^3$$

for a homogeneous polynomial f of degree 3. Suppose f is generic, then $X(t)$ is a smooth cubic surface for any generic $t \neq 0$. When $t = 0$, $X(0)$ is the union of 3 coordinate planes, which certainly contain infinitely many lines. But only a few of them could survive for nearby t. To explain this, we note that the singular set of $X(0)$ is the union of 3 coordinate axes. In the smoothing $X (t \neq 0)$, each such singular point p becomes a *vanishing* S^1 as in the standard model of smoothing from $\{xy = 0\}$ to $\{xy = t\}$, with the exception when f vanishes at p. As $\deg(f) = 3$, f vanishes at 3 points on each coordinate axis. We call these *unstable points* as without them the total family of $X(t)$'s would be a semi-stable degeneration for small t. A line on a coordinate
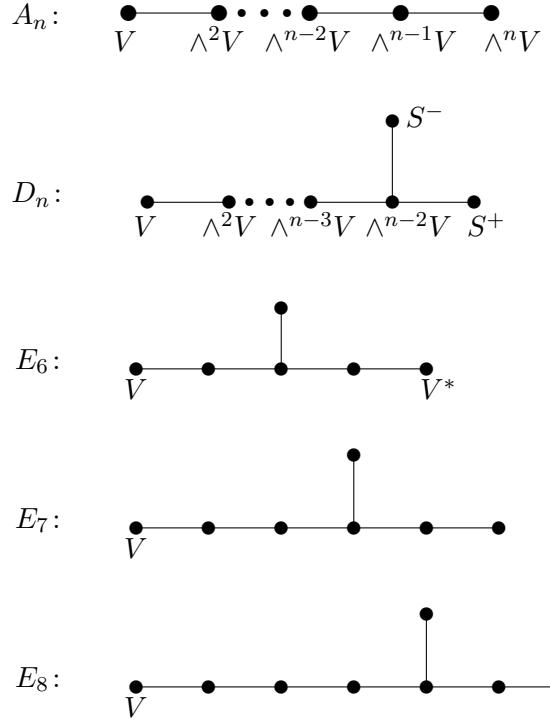
Figure 2. Dynkin diagrams with fundamental representations labelled.

plane, say $\{z_1 = 0\} \subset X(0)$, survives on nearby $X(t)$'s if and only if it meets one of these 3 points on the z_2-axis, as well as on z_3-axis. Therefore the total number of lines on $X(0)$ which survive on nearby $X(t)$'s is $3 \times 3 + 3 \times 3 + 3 \times 3 = 27$.

We will see that these 27 lines can be used to construct the fundamental representation $L = \underline{27}$ of E_6. Here $\underline{27}$ means a particular representation of E_6 of dimension 27. In fact, the above description of these lines corresponds to the branching rule from E_6 to its subalgebra $\mathfrak{su}(3)_1 \times \mathfrak{su}(3)_2 \times \mathfrak{su}(3)_3$:

$$\underline{27} = \underline{3_1} \times \underline{3_2^*} + \underline{3_2} \times \underline{3_3^*} + \underline{3_3} \times \underline{3_1^*}.$$

Here $\underline{3_i}$ refers to the standard representation of the ith-factor $\mathfrak{su}(3)_i$ in $\mathfrak{su}(3)_1 \times \mathfrak{su}(3)_2 \times \mathfrak{su}(3)_3$ and $\underline{3_i^*}$ is its dual representation.

Similarly, we could degenerate $X(t)$ to a union of a plane H and a smooth quadric surface Q in $\mathbb{P}^3$, i.e., $X(0) = H \cup Q$ [42]. Then there are 6 points on the curve $C = H \cap Q$ which stay on $X(t)$ for t infinitesimally close to 0. They play the same role as the 3 points on each coordinate axis in the previous example, namely they are the unstable points for the family $X(t)$'s. A line on the plane H joining any 2 of these 6 points will deform to a line on nearby $X(t)$'s. The total number of such lines is $\binom{6}{2} = 15$. On the other hand, the quadric Q has 2 rulings, each is a $\mathbb{P}^1$-family of lines. A line in Q passing through one the these 6 points in C will also deform to one in nearby $X(t)$, and the total number of such lines is $6 \times 2 = 12$. All together, we have $\binom{6}{2} + 6 \times 2 = 27$ lines on $X(t)$. This corresponds to the branching rule for the fundamental representation $L_{E_6} = \underline{27}$ of E_6 to its subalgebra $\mathfrak{sl}(6) \times \mathfrak{sl}(2) \subset E_6$ as follows:

$$L_{E_6} \simeq \Lambda^2 V_6 + V_6 \otimes V_2,$$

where $V_6 \simeq \mathbb{C}^6$ and $V_2 \simeq \mathbb{C}^2$ are the standard representations of $\mathfrak{sl}(6)$ and $\mathfrak{sl}(2)$ respectively.

There is another way to see these lines once a particular line $l \subset X$ is given. Any hyperplane section containing l must be of the form $l + C$ for some conic curve $C \subset X$ [46, 49]. The pencil

of such hyperplane sections degenerates into sum of three lines $l + l_i' + l_i''$ five times $i = 1, \ldots, 5$. These 10 divisors l_i''s and l_i'''s determine the standard representation of $O(5,5) = D_5$ (see D_n-surfaces). The remaining 16 lines, which do not intersect l, form a spinor representation of D_5 and

$$\underline{27} = \underline{1} + \underline{16} + \underline{10}$$

is the branching rule from E_6 to $E_5 = D_5$. From the surface perspective, these three types of lines correspond to lines with intersection numbers -1, 0 and 1 with the given line l.

We could realize the E_6 structure more concretely. We recall that the standard representation $\underline{27}$ of E_6 admits a cubic form

$$c\colon \underline{27} \otimes \underline{27} \otimes \underline{27} \to \mathbb{C}$$

such that $E_6 \simeq \mathrm{Aut}(\underline{27}, c)$, similar to $D_n \simeq \mathrm{Aut}(\underline{2n}, q)$ as the symmetry group of a quadratic form on its standard representation. We consider the direct sum of line bundles from lines on X:

$$\mathcal{L}^{E_6} = \bigoplus_{i=1}^{27} O(l_i).$$

If 3 of them l_1, l_2, l_3 form a $triangle$, i.e., $l_i \cdot l_j = 1$ whenever $i \neq j$, then $l_1 + l_2 + l_3$ is a hyperplane section. Therefore

$$O(l_1) \otimes O(l_2) \otimes O(l_3) \simeq O(1) \simeq K^{-1},$$

where K is the canonical line bundle of X. With suitable choices of these isomorphisms [42] and using all triangles in X, we obtain

$$c_{\mathcal{L}}\colon \mathcal{L}^{E_6} \otimes \mathcal{L}^{E_6} \otimes \mathcal{L}^{E_6} \to K^{-1}$$

so that the fiberwise cubic structures give a conformal E_6-bundle $\mathcal{E}^{E_6}$ over the cubic surface X.

The root lattice Λ_{E_6} of E_6 can be identified as the orthogonal complement $\langle K \rangle^\perp$ of K in $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{1,6}$. Here $\mathbb{Z}^{1,6}$ denotes the lattice $\mathbb{Z}^7$ with the quadratic form $(1) \oplus (-1)^{\oplus 6}$, which is isomorphic to $H^2(X, \mathbb{Z})$ equipped with the intersection form. This is because any smooth cubic surface is a blowup of $\mathbb{P}^2$ at 6 points. Note that $\alpha \in H^2(X, \mathbb{Z})$ is a root in Λ_{E_6} if and only if $\alpha \cdot \alpha = -2$ and $\alpha \cdot K = 0$, i.e., $\alpha \in \langle K \rangle^\perp \simeq \Lambda_{E_6}$. We denote the collection of all roots as Φ. If $\alpha \in \Phi$ can be represented by an effective divisor C, then $C \simeq \mathbb{P}^1$ is called an (-2)-curve. Furthermore, (1) $l \in \mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ satisfying $l \cdot l = -1$ and $l \cdot K = -1$ is always represented by a unique line on X; (2) $f \in \mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ satisfying $f \cdot f = 0$ and $f \cdot K = -2$ will be called a $ruling$, or conic bundle, as it defines a $\mathbb{P}^1$-bundle $\Phi_f\colon X \to |f| \simeq \mathbb{P}^1$ on X with fiber degree 2; (3) $h \in \mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ satisfying $h \cdot h = 1$ and $h \cdot K = -3$ gives $\Phi_h\colon X \to |h| \simeq \mathbb{P}^2$ which realizes X as a blowup of $\mathbb{P}^2$ at 6 points. Similar structures hold for all del Pezzo surfaces and they are closely related to the fundamental representations L, R and H of E_n corresponding the left node, the right node and the top node in the Dynkin diagram of E_n's, see Figure 3.
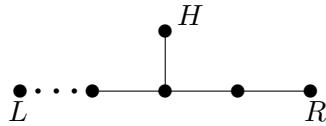

Figure 3. Dynkin diagram of E_n with L, R and H labelled.

In fact, the Lie algebra E_6-bundle of fiberwise infinitesimal automorphisms of $\left(\mathcal{L}^{E_6}, c_{\mathcal{L}}\right)$ is naturally

$$\mathcal{E}^{E_6} = O_X^{\oplus 6} \bigoplus_{\substack{\alpha \cdot K = 0 \\ \alpha \cdot \alpha = -2}} O_X(\alpha),$$

similar to the root space decomposition of Lie algebra $\mathrm{Lie}(E_6) = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

Given any smooth anti-canonical curve $\Sigma \in |-K|$, then it is always an elliptic curve by the adjunction formula. For any $\alpha \in \Phi$, $O(\alpha)|_\Sigma$ has degree 0 and therefore defines a flat line bundle over Σ. Indeed $\mathcal{E}^{E_6}|_\Sigma$ is a flat E_6-bundle over Σ and every flat E_6-bundle over Σ always arises this way for an embedding of Σ into some cubic surface. Such a correspondence works for any E_n-bundle, or even any ADE bundle, as we will describe next.

4 Del Pezzo surfaces vs E_n Lie theory

Cubic surfaces are degree three del Pezzo surfaces. In this section, we explain the analogous results between the configuration of lines in degree $9 - n$ del Pezzo surfaces and representation theory for the Lie algebra E_n for any n between 1 and 8, as given in [42].

A smooth surface X is called a del Pezzo surface if K^{-1} is ample. Its degree $\deg X = K \cdot K$ is always between 1 and 9. A del Pezzo surface of degree d is always a blowup of $\mathbb{P}^2$ at $n = 9 - d$ points in general position, with the exception of $\mathbb{P}^1 \times \mathbb{P}^1$. We will denote them as X_n and $X_1' = \mathbb{P}^1 \times \mathbb{P}^1$ respectively. For example $X_0 = \mathbb{P}^2$, X_5 is the complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$, X_6 is a cubic surface in $\mathbb{P}^3$, X_7 is a branched cover of $\mathbb{P}^2$ branched along a quartic curve and X_8 has a pencil of elliptic curves, i.e., $|-K| \simeq \mathbb{P}^1$. If we blowup the unique base point of this pencil, we obtain a surface X_9 which is a rational elliptic surface. We will see in later sections that the geometry of this surface is related to affine Kac–Moody Lie algebras of type $\hat{E}_8$'s.

In fact, the X_n's and X_1' correspond to simple Lie algebras of type E_n, $1 \le n \le 9$ and $\mathbb{C}$ if the Abelian Lie algebra $\mathbb{C}$ is also included as a rank one simple Lie algebra. We recall that E_n's with $n \le 5$ coincide with certain classical Lie algebras, as can be seen directly from Dynkin diagrams. Concretely, we have $E_5 \simeq D_5 = \mathfrak{o}(10)$, $E_4 \simeq A_4 = \mathfrak{sl}(5)$, $E_3 \simeq A_2 \times A_1 = \mathfrak{sl}(3) \times \mathfrak{sl}(2)$, $E_2 \simeq \mathfrak{sl}(2) \times \mathbb{C}$, $E_1 \simeq \mathbb{C}$, $E_1' \simeq \mathfrak{sl}(2)$ and $E_0 = 0$. We remark that there is a physical explanation of having two E_n's when $n = 1$ [31] via supergravity in eleven dimensions where E_n arises as one compactifies the 11-dimensional space-time along a n-dimensional torus.

The first relationship between the geometry of a del Pezzo surface X_n and the Lie algebra E_n is the following: the orthogonal complement $\langle K \rangle^\perp \subset H^2(X_n, \mathbb{Z}) \simeq \mathbb{Z}^{n,1}$ is a root system Λ_{E_n} for E_n.

Similar to the cubic surface case, a line $l \subset X_n$ means a curve in X_n of degree one with respect to the anti-canonical class, which is equivalent to $l \in H^2(X_n, \mathbb{Z})$ satisfying $l \cdot l = -1$ and $l \cdot K = -1$. It is because each such cohomology class can be represented as the Poincare dual of a unique line in X_n. The number of lines in X_n is equal to the dimension of the standard representation L of E_n, unless $n = 8$ in which case it is $\dim L - 8 = 240$ because L is no longer a minuscule representation. Explicitly these numbers are 1, 3, 6, 10, 16, 27, 56 and 240. Similarly $f \in \mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ satisfying $f \cdot f = 0$ and $f \cdot K = -2$ will be called a *ruling*, or conic bundle, as it defines a $\mathbb{P}^1$-bundle $\Phi_f \colon X \to |f| \simeq \mathbb{P}^1$ on X with fiber degree 2. The number of rulings in X_n is equal to the dimension of the fundamental representation R of E_n for $n < 7$.

We define

$$\mathcal{E}^{E_n} = O^{\oplus n} \bigoplus_{\substack{\alpha \cdot K = 0 \\ \alpha \cdot \alpha = -2}} O(\alpha), \qquad \mathcal{L}^{E_n} = \bigoplus_{\substack{l \cdot K = -1 \\ l \cdot l = -1}} O(l), \qquad \mathcal{R}^{E_n} = \bigoplus_{\substack{f \cdot K = -2 \\ f \cdot f = 0}} O(f).$$

Then (1) $\mathcal{E}^{E_n}$ is always an E_n-bundle over X_n; (2) $\mathcal{L}^{E_n}$ is an $\mathcal{E}^{E_n}$-representation bundle over X_n corresponding to the E_n representation L, for $n < 8$ and (3) $\mathcal{R}^{E_n}$ is an $\mathcal{E}^{E_n}$-representation bundle over X_n corresponding to the E_n-representation R, for $n < 7$. The restriction on n is related to the fact that L for E_8 and R for E_7 are the adjoint representations.

If l_1 and l_2 are two lines in X_n satisfying $l_1 \cdot l_2 = 1$, then $l_1 + l_2$ is a ruling. This is reflected by the fact that R is an irreducible component of the tensor product E_n-representation $L \otimes L$ and in fact we have a natural bundle homomorphism

$$\mathcal{L}^{E_n} \otimes \mathcal{L}^{E_n} \to \mathcal{R}^{E_n}$$

over any X_n.

For example X_1 is the blowup of $\mathbb{P}^2$ and the exceptional curve is its unique line. The two exceptional curves in X_2 together with the strict transform of the line joining two blow up points in $\mathbb{P}^2$ give the 3 lines in X_2. Having two different types of lines in X_2 reflects that L_{E_2} is a reducible representation of E_2 and it is also responsible for the fact that there are two different degree 8 del Pezzo surfaces, namely X_1 and $X_1' = \mathbb{P}^1 \times \mathbb{P}^1$. The $6 = 3 \times 2$ lines in X_3 is reflecting the fact that X_3 is the blow up of $\mathbb{P}^2$ in 2 different ways, which in turns is the origin of the Cremona transformation. In terms of the representation L_{E_3} of $E_3 \simeq \mathfrak{sl}(3) \times \mathfrak{sl}(2)$ we have

$$L_{E_3} \simeq V_3 \otimes V_2,$$

where V_3 and V_2 are the standard representations of $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2)$ respectively. On X_4, there are 10 lines and 5 rulings. Under the identification of E_4 with $\mathfrak{sl}(5)$, we have $R_{E_4} \simeq V_5$ the standard representation of $\mathfrak{sl}(5)$ and $L_{E_4} \simeq \Lambda^3 V_5$. We could also see the relationship $L_{E_4} \simeq \Lambda^3 R_{E_4}$ from the fact that every 3 distinct rulings on X_4 determines a unique line in X_4 which is a bisection to each of these 3 rulings. In fact, we have a natural bundle isomorphism

$$\mathcal{L}^{E_4} \simeq \Lambda^3 \mathcal{R}^{E_4}(K).$$

On X_5 there are 16 lines and 10 rulings, which are related to a spinor representation S_{10}^+ and the standard representation V_{10} of $E_5 \simeq D_5 = \mathfrak{o}(10)$. The defining quadratic form of $\mathfrak{o}(10)$ on $V_{10} \simeq \mathbb{C}^{10}$ is reflected by the geometric fact that given any two rulings f_1 and f_2, we have $f_1 \cdot f_2 \leq 2$ and the equality sign holds if and only if $f_1 + f_2 = -K$. In fact, we have a fiberwise quadratic form q over X_5,

$$q\colon \mathcal{R}^{E_5} \otimes \mathcal{R}^{E_5} \to O(-K),$$

so that the Lie algebra E_5-bundle over X_5 is the bundle of infinitesimal symmetries of q on $\mathcal{R}^{E_5}$ [40]. A smooth X_5 is the complete intersection of two quadric hypersurfaces Q_0 and Q in $\mathbb{P}^4$. If Q_0 varies in a one parameter family $Q_0(t)$ and degenerates into a union of two hyperplanes $Q_0(0) = H' \cup H''$, thus $X_5(0) = X_5' \cup X_5''$, then its geometry is governed by the reduction of the Lie algebra $E_5 \simeq D_5 = \mathfrak{o}(10)$ to $A_3 \times A_1 \times A_1 \simeq D_3 \times D_2 = \mathfrak{o}(6) \times \mathfrak{o}(4)$. For instance the branching rule

$$S_{10}^+ \simeq S_6^+ \otimes S_4^+ + S_6^- \otimes S_4^-$$

is reflected by the following geometric statements: X_5' and X_5'' are quadric surfaces and therefore each admits two rulings. Lines on each ruling passing through one of the four points in the curve $Q \cap H' \cap H''$, which are the unstable points for the total family $X_5(t)$, are exactly those lines on $X_5(0)$ which will survive for nearby $X_5(t)$'s. Thus the $16 = 4 \times 2 + 4 \times 2$ lines on $X_5(t)$ is reflecting the above branching rule of S_{10}^+. The relationship between the branching rule $V_{10} \simeq V_6 + V_4$ and the geometry of rulings on $X_5(t)$ can be described in a similar fashion.

The relationship between the geometry of cubic surfaces X_6 and representation theory of E_6 has been discussed in the previous section. Recall that a smooth degree 2 del Pezzo surface X_7 is a double cover of $\mathbb{P}^2$, branched along a quartic curve $Q \subset \mathbb{P}^2$. A bitangent line to Q in $\mathbb{P}^2$ determines a pair of lines in X_7 as its double cover, thus the 28 bitangents to Q gives 56 lines in X_7. If l and l' is any such pair of lines in X_7, then $l + l' = -K$ and we have a fiberwise non-degenerate quadratic form q on $\mathcal{L}^{E_7}$,

$$q\colon \ \mathcal{L}^{E_7} \otimes \mathcal{L}^{E_7} \to O(-K).$$

Furthermore, there is a fiberwise quartic form f on $\mathcal{L}^{E_7}$,

$$f\colon \ \mathcal{L}^{E_7} \otimes \mathcal{L}^{E_7} \otimes \mathcal{L}^{E_7} \otimes \mathcal{L}^{E_7} \to O(-2K),$$

so that its bundle of infinitesimal symmetries is the Lie algebra E_7-bundle $\mathcal{E}^{E_7}$ over X_7. We also have

$$\mathcal{E}^{E_7} \simeq \left(O^{\oplus 7} + \mathcal{R}^{E_7}\right)(K).$$

If we degenerate the smooth quartic branch curve $Q \subset \mathbb{P}^2$ to a double conic $2C$, then we obtain a family of smooth del Pezzo surface $X_7(t)$ degenerating to a nonnormal surface $X_7(0) = X_7' \cup X_7''$ with $X_7' \simeq X_7'' \simeq \mathbb{P}^2 \supset C$. There are 8 unstable points on $C \subset X_7(0)$. Each line in X_7' or X_7'' passing through 2 of these 8 unstable points will survive on nearby $X_7(t)$'s, thus giving $\binom{8}{2} + \binom{8}{2} = 56$ lines on X_7. This is reflected by the branching rule of L_{E_7} of E_7 to its subalgebra $A_7 = \mathfrak{sl}(8)$,

$$L_{E_7} \simeq \Lambda^2 V_8 + \Lambda^2 V_8^*,$$

where $V_8 \simeq \mathbb{C}^8$ is the standard representation of $\mathfrak{sl}(8)$. For degree 1 del Pezzo surfaces X_8, the relationships between their geometry with E_8 are discussed in [40].

5 ADE surfaces vs ADE Lie theory

In this section, the above relationships between the representation theory of E_n and the geometry of del Pezzo surfaces will be generalized to other simply-laced Lie algebras, i.e., Lie algebras of type ADE [42]. We call the corresponding surfaces ADE surfaces. In this unified description, an E_n-surface is an X_{n+1} together with a choice of a line, or equivalently X_n with a point in it.

First we describe D_n-surfaces. Given any ruling f on X_{n+1}, its linear system defines a $\mathbb{P}^1$-bundle on X_{n+1}

$$\Phi_f\colon \ X_{n+1} \to |f| \simeq \mathbb{P}^1.$$

For a generic X_{n+1}, there are n singular fibers and each is a union of two lines intersecting at a point. To see this, we note the Euler characteristic of X_{n+1} equals

$$\chi(X_{n+1}) = \chi\left(\mathbb{P}^2\right) + n + 1 = n + 4.$$

Smooth (resp. singular) fibers have Euler characteristic 2 (resp. 3). We have

$$\chi(X_{n+1}) = \chi\left(\mathbb{P}^1 \times \mathbb{P}^1\right) + \# \text{ (singular fibers)}$$

and therefore there are n singular fibers [3]. Using these fiberwise lines, i.e., $l \cdot f = 0$, we construct a rank $2n$ vector bundle $\mathcal{L}^{D_n}$ over X_{n+1},

$$\mathcal{L}^{D_n} = \bigoplus_{\substack{l \cdot K = -1 \\ l \cdot l = -1 \\ l \cdot f = 0}} O(l).$$

Notice that any two such lines l and l' intersect if and only if $l + l'$ is a singular fiber of Φ_f. When this happens, we have

$$O(l) \otimes O(l') \simeq O(f).$$

Putting these isomorphisms together, we obtain a fiberwise quadratic form $q_{\mathcal{L}}$ on $\mathcal{L}^{D_n}$ over X_{n+1},

$$q_{\mathcal{L}} \colon \ \mathcal{L}^{D_n} \otimes \mathcal{L}^{D_n} \to O(f).$$

The bundle of infinitesimal symmetries of $\left(\mathcal{L}^{D_n}, q_{\mathcal{L}} \right)$ is naturally

$$\mathcal{E}^{D_n} = O^{\oplus n} \bigoplus_{\substack{\alpha \cdot K = 0 \\ \alpha \cdot \alpha = -2 \\ \alpha \cdot f = 0}} O(\alpha),$$

which is a D_n-Lie algebra bundle over X_{n+1} so that $\mathcal{L}^{D_n}$ is its representation bundle corresponding to the standard representation of $D_n = \mathfrak{o}(2n)$. We call (X_{n+1}, f) a D_n-surface, denoted as X_{D_n}.

In fact, the orthogonal complement of K and f in $H^2(X_{n+1}, \mathbb{Z})$ is the root lattice of type D_n

$$\langle K, f \rangle^{\perp} \simeq \Lambda_{D_n}.$$

Furthermore, this holds true for any n without the restriction $n \leq 8$.

In a similar fashion, an A_n-surface X_{A_n} is a pair (X_{n+1}, h) with $h \cdot K = -3$ and $h \cdot h = 1$. The rank $n + 1$ vector bundle

$$\mathcal{L}^{A_n} = \bigoplus_{\substack{l \cdot K = -1 \\ l \cdot l = -1 \\ l \cdot h = 0}} O(l)$$

over X_{A_n} admits a fiberwise determinant morphism

$$\det \colon \ \Lambda^{n+1} \mathcal{L}^{A_n} \xrightarrow{\sim} O(K + 3h).$$

This is because

$$\Phi_h \colon \ X_{n+1} \to |h| \simeq \mathbb{P}^2,$$

which realizes X_{n+1} as a blowup of $\mathbb{P}^2$ at $n+1$ points and exceptional curves for Φ_h are precisely those lines used to construct $\mathcal{L}^{A_n}$. The bundle of infinitesimal symmetries of $\left(\mathcal{L}^{A_n}, \det \right)$ is naturally

$$\mathcal{E}^{A_n} = O^{\oplus n} \bigoplus_{\substack{\alpha \cdot K = 0 \\ \alpha \cdot \alpha = -2 \\ \alpha \cdot h = 0}} O(\alpha),$$

which is an A_n-Lie algebra bundle over X_{A_n} so that $\mathcal{L}^{A_n}$ is its representation bundle corresponding to the standard representation of $A_n = \mathfrak{sl}(n+1)$. Again the orthogonal complement of K and h in $H^2(X_{n+1}, \mathbb{Z})$ is the root lattice of type A_n

$$\langle K, h \rangle^{\perp} \simeq \Lambda_{A_n}.$$

We remark that an E_n-surface can be interpreted as (X_{n+1}, l), as in the D_n and A_n cases. But since l is an exceptional curve and therefore it can be blown down to X_n. We also remark that a choice of h, f or l in X_{n+1} defines an A_n-surface, D_n-surface or E_n-surface accordingly. From the Lie theoretical perspective, they correspond to the fundamental representations H, R and L for E_{n+1}. By removing the corresponding nodes in the E_{n+1} Dynkin diagram, we also obtain Dynkin diagrams of type A, D and E respectively.

An ADE surface is a surface X_{n+1}, $\mathbb{P}^2$ blown up at $n+1$ points in general position, together with a divisor h, r or l as below:

A_n-surface	(X_{n+1}, h) with $h \cdot K = -3$ and $h \cdot h = 1$
D_n-surface	(X_{n+1}, r) with $r \cdot K = -2$ and $r \cdot r = 0$
E_n-surface	(X_{n+1}, l) with $l \cdot K = -1$ and $l \cdot l = -1$

6 F/String theory duality

In this section, we recall a physical motivation for the construction of E_n-bundles over del Pezzo surfaces of degree $9-n$ given in [17, 20, 21, 24]. Physically, if G is a simple subgroup of $E_8 \times E_8$, then G-bundles are related to the duality between F-theory and heterotic string theory. Among other things, this duality predicts that the moduli of flat E_n-bundles over a fixed elliptic curve Σ can be identified with the moduli of del Pezzo surfaces with a fixed anti-canonical curve Σ.

Given any smooth elliptic curve Σ and any ADE Lie group G of adjoint type, if Σ is an anti-canonical curve in a G-surface X, then the natural $\mathfrak{g}$ Lie algebra bundle $\mathcal{E}^{\mathfrak{g}}$ restricts to a flat G-bundle over Σ. In fact, every flat G-bundle on Σ arises this way. To state the result, we denote $\mathcal{M}_\Sigma^G$ to be the moduli space of flat G-bundles over Σ and $\mathcal{S}_\Sigma^G$ to be the moduli space of pairs $(X, \Sigma \in |-K_X|)$ with X being an ADE surface of type G and Σ is an anti-canonical curve in X.

Theorem 6.1 ([42]). *Given Σ and G as above, there is an open dense embedding*

$$\Phi \colon \ \mathcal{S}_\Sigma^G \to \mathcal{M}_\Sigma^G$$

given by the restriction of the natural $\mathfrak{g}$-bundle $\mathcal{E}^{\mathfrak{g}}$ over X to Σ. Furthermore, there is a natural compactification $\bar{\mathcal{S}}_\Sigma^G$ of $\mathcal{S}_\Sigma^G$ given by those surfaces X equipped with G-configurations and Φ extends to an isomorphism $\Phi \colon \bar{\mathcal{S}}_\Sigma^G \xrightarrow{\sim} \mathcal{M}_\Sigma^G$.

This particular form is given in [42] and its generalization for non-simply laced G is given in [43, 44]. Such a correspondence was originally motivated from the duality between F-theory and heterotic string theory in physics by the work of Friedman–Morgan–Witten [24] and Donagi [20] where different proofs of this correspondence are also given.

Let us very briefly describe how such a correspondence arises from the physical duality between F-theory and heterotic string theory. The space-time in F-theory is a Calabi–Yau fourfold Z equipped with an elliptic K3 fibration over a complex surface B and the space-time in heterotic string theory is a Calabi–Yau threefold Y equipped with an elliptic fibration over the same complex surface B and coupled with an $E_8 \times E_8$ Hermitian Yang–Mills bundle over Y. When the two theories are dual to each other, in a certain adiabatic limit, the duality becomes a fiberwise duality. Namely an elliptic K3 fiber X over $b \in B$ in Z is dual to an elliptic curve fiber Σ over $b \in B$ in Y coupled with a flat $E_8 \times E_8$-bundle over Σ. To obtain a geometric correspondence, the dilaton field in the string theory should vanish, which corresponds to a type II degeneration of the elliptic K3 surface, that is X is a fiber sum $X_1 \#_\Sigma X_2$, each X_i is a rational elliptic surface with section with Σ being a fiber. In particular, X_i is a blowup of $\mathbb{P}^2$ at 9 points with one exceptional curve identified with the section, namely X_i is an E_8-surface as defined previously. Hence each (X_i, Σ) is an E_8-surface and therefore gives an E_8-bundle over Σ by our above discussions. Together we obtain the flat $E_8 \times E_8$-bundle over Σ.

7 Generalization to Kac–Moody cases

In this section, we generalize the above results for E_n-bundles in Theorem 6.1 to Kac–Moody $\hat{E}_n$-bundles in [41, Theorems 2 and 3]. If we blowup $\mathbb{P}^2$ at 9 points, then the resulting surface X_9 is no longer a del Pezzo surface. As $K \cdot K = 0$, $\langle K_X \rangle^\perp \subset H^2(X_9, \mathbb{Z})$ is only non-positive definite and there are infinitely many roots. The latter is because for any root α, namely $\alpha \cdot \alpha = -2$ and $\alpha \cdot K = 0$, $\alpha + nK$ is also a root for any integer n [15, 46].

In fact, $\Lambda_{E_9} \triangleq \langle K_X \rangle^\perp$ is the root system of the affine Kac–Moody Lie algebra $\hat{E}_8$, or the loop algebra LE_8 of E_8, up to central extension. The following is the list of Dynkin diagrams of simply-laced affine Kac–Moody Lie algebras, which coincides with extended Dynkin diagrams of simply-laced simple Lie algebras, namely ADE types (see [32] for details):
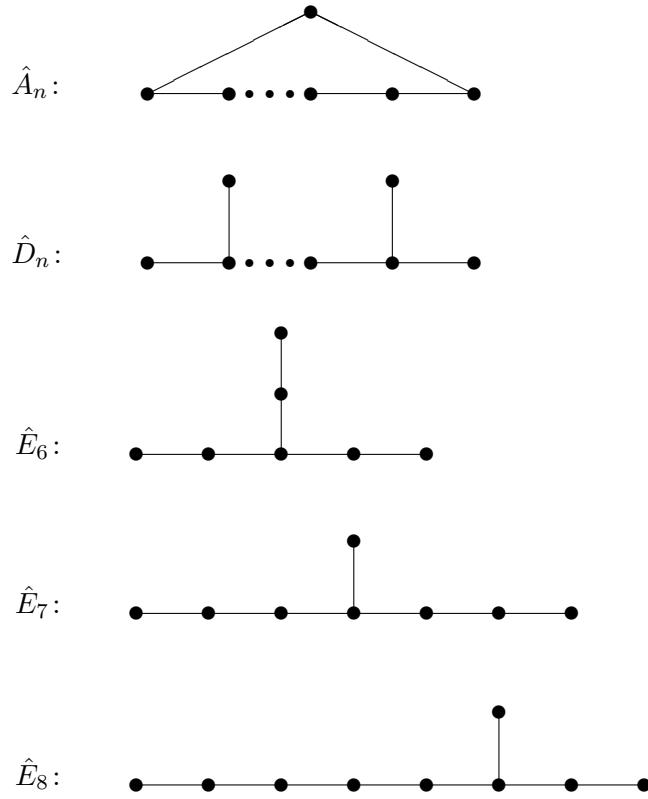

Figure 4. Dynkin diagrams of affine ADE types.

We could also realize other $\hat{E}_n$'s with $n \leq 8$ from rational surfaces as follows: Let $C_1, \ldots, C_{8-n}$ be an A_{8-n}-chain of (-2)-curves in X_9, i.e., $C_i \simeq \mathbb{P}^1$, $C_i \cdot C_i = -2$, $C_i \cdot C_{i+1} = 1$ and other $C_i \cdot C_j$'s are zero. Then we could blow down this chain of (-2)-curves in X_9 to obtain a rational surface, denoted $X_{\hat{E}_n}$, with a rational singular point of type A_{8-n}. Equivalently $X_{\hat{E}_n}$ is a singular del Pezzo surface of degree $8 - n$ with a canonical singularity of type A_{8-n}, for $n \leq 7$. When $n = 8$, further care is needed as the representation L_n is no longer a miniscule representation in this case. We call these $X_{\hat{E}_n}$'s $\hat{E}_n$-surfaces.

Then we have the following results [41] generalizing the construction of E_n-bundles over del Pezzo surfaces and their relationships with flat E_n-bundles over elliptic curves.

Theorem 7.1 ([41])**.** *For any $\hat{E}_n$-surface $X_{\hat{E}_n}$, the orthogonal complement of K in $H^2(X_{\hat{E}_n}, \mathbb{Z})$ is isomorphic to the root system of the affine Kac–Moody Lie algebra $\hat{E}_n$, i.e.,*

$$\langle K \rangle^\perp \simeq \Lambda_{\hat{E}_n}.$$

Furthermore, the real root system of $\hat{E}_n$ is $\Delta^{\mathrm{re}}(\hat{E}_n) \simeq \langle K \rangle^\perp \cap \{\alpha \cdot \alpha = -2\}$, the imaginary root system of $\hat{E}_n$ is $\Delta^{\mathrm{im}}(\hat{E}_n) \simeq Z_{\neq 0}\langle K \rangle$ and the null root is $-K$.

There is also a canonically defined $\hat{E}_n$-bundle $\mathcal{E}^{\hat{E}_n}$ over $X_{\hat{E}_n}$,

$$\mathcal{E}^{\hat{E}_n} = \mathcal{O}^{\oplus n+1} \oplus \bigoplus_{\alpha \in \Delta^{\mathrm{re}}(\hat{E}_n)} O(\alpha) \bigoplus_{\beta \in \Delta^{\mathrm{im}}(\hat{E}_n)} O(\beta)$$

$$\simeq \mathcal{E}^{E_n} \otimes \left(\bigoplus_{n \in \mathbb{Z}} O(nK) \right) \oplus O.$$

Note that the last isomorphism is a realization of the Kac–Moody bundle as a central extension of a loop algebra bundle over X, which depends on a choice of a fixed line in X.

Given any fixed smooth elliptic curve Σ, let $\mathcal{S}_\Sigma^{\hat{E}_n}$ be the moduli space of $\hat{E}_n$-surfaces $X_{\hat{E}_n}$ containing Σ as an anti-canonical curve and $\mathcal{M}_\Sigma^{\hat{E}_n}$ be the moduli space of holomorphic $\hat{E}_n$-bundles over Σ, in particular

$$\mathcal{M}_\Sigma^{\hat{E}_n} \simeq \mathrm{Hom}(\Lambda_{\hat{E}_n}, \Sigma)/W_{\hat{E}_n \times \mathbb{Z}_2}.$$

Theorem 7.2 ([41]). *Given any smooth elliptic curve Σ, there is an open embedding*

$$\Phi \colon \mathcal{S}_\Sigma^{\hat{E}_n} \to \mathcal{M}_\Sigma^{\hat{E}_n}$$

given by the restriction of the canonically defined $\hat{E}_n$-bundle $\mathcal{E}^{\hat{E}_n}$ over $X_{\hat{E}_n}$ to $\Sigma \subset X_{\hat{E}_n}$. Furthermore, there is a natural extension $\Phi \colon \bar{\mathcal{S}}_\Sigma^{\hat{E}_n} \xrightarrow{\sim} \mathcal{M}_\Sigma^{\hat{E}_n}$.

In above discussions, we contract an A_d-chain of (-2)-curves on X_9 to realize affine E_{9-d} structures on rational surfaces and elliptic curves. However, in Section 4, E_n structures on rational surfaces and elliptic curves also appear for considerations of $\mathbb{P}^2$ blown up at n points. In general we could consider A_d-chains of (-2)-curves on $\mathbb{P}^2$ blown up at $n+1$ points. Corresponding Lie algebras are listed in the following magic triangle of Julia [31], see Table 1.

Table 1. Julia's magic triangle.

	$n = 8$	$n = 7$	$n = 6$	$n = 5$	$n = 4$	$n = 3$	$n = 2$	$n = 1$	$n = 0$
$d = 11$	$+$								
$d = 10$	$\mathbb{R}$ or A_1	$+$							
$d = 9$	$\mathbb{R} \times A_1$	$\mathbb{R}$							
$d = 8$	$A_1 \times A_2$	$\mathbb{R} \times A_1$ or A_2	A_1						
$d = 7$	E_4	$\mathbb{R} \times A_2$	$\mathbb{R} \times A_1$	$\mathbb{R}$	$+$				
$d = 6$	E_5	$A_1 \times A_3$	$\mathbb{R} \times A_1^2$	$\mathbb{R}^2$ or A_1^2	$\mathbb{R}$				
$d = 5$	E_6	A_5	A_2^2	$\mathbb{R} \times A_1^2$	$\mathbb{R} \times A_1$	A_1			
$d = 4$	E_7	D_6	A_5	$A_1 \times A_3$	$\mathbb{R} \times A_2$	$\mathbb{R} \times A_1$ or A_2	$\mathbb{R}$	$+$	
$d = 3$	E_8	E_7	E_6	E_5	E_4	$A_1 \times A_2$	$\mathbb{R} \times A_1$	$\mathbb{R}$ or A_1	$+$

Notice that there is a symmetry between d and n in this magic triangle, extending our earlier descriptions on E_n bundles and affine E_n bundles. The motivation of Julia is from the studies of 11-dimensional supergravity in which the bosonic fields are pairs (g, C), where g is a metric tensor on $\mathbb{R}^{1,10}$ and C is a three form field on $\mathbb{R}^{1,10}$, called the C-field. Consider a physical toroidal compactification, namely one replaces $\mathbb{R}^{1,10}$ by $\mathbb{R}^{1,10-n} \times T^n$ and requires the volume of T^n to shrink to zero size. By viewing g as a family of metrics on T^n parametrized by $\mathbb{R}^{1,10-n}$, we might expect to obtain an effective theory which is a sigma model for maps from $\mathbb{R}^{1,10-n}$ to

$SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R}) / SO(n)$, as the latter space parametrizes Einstein metrics on T^n. However, the C-field will also decompose and regroup with components of g to enhance the sigma model to $E_{n,\mathbb{Z}} \backslash E_n^{\text{split}} / K_n$ where K_n is the maximal compact subgroup of the split Lie group E_n^{split} of type E_n. Various structures that we mentioned on ADE structures would have their counterparts in this supergravity theory, which is a fascinating connection between algebraic geometry and physics.

8 ADE bundles over surfaces with ADE singularities

In this section, we explain that ADE singularities on a surface X with $q = p_g = 0$ leads to ADE bundles over X, as stated in Theorem 8.1. Suppose X' is a singular surface with $q(X') = 0$ and with a simple singularity p and X is its minimal resolution with exceptional divisor $C = \bigcup_{i=1}^{r} C_i$. As explained in Section 2, each irreducible component C_i is an (-2)-curve, i.e., $C_i \simeq \mathbb{P}^1$ with $C_i \cdot C_i = -2$. In particular, $C_i \cdot K = 0$ by the adjunction formula. The dual graph for the configuration of these C_i's is a Dynkin diagram of ADE type, thus there is a corresponding simple Lie algebra $\mathfrak{g}$ of ADE type, and we also call the corresponding singularity p an ADE singularity. The $\mathbb{Z}$-span of the C_i's is a root lattice $\Lambda_{\mathfrak{g}}$ of type $\mathfrak{g}$ inside $H^2(X, \mathbb{Z})$,

$$\mathbb{Z}\langle C_1, \ldots, C_r \rangle = \Lambda_{\mathfrak{g}} \subset H^2(X, \mathbb{Z}).$$

We denote the set of roots in $\Lambda_{\mathfrak{g}}$ as Φ, i.e., $\alpha \in \Lambda_{\mathfrak{g}}$ lies in Φ if $\alpha \cdot \alpha = -2$. We write $\Phi = \Phi_+ \amalg \Phi_-$, where $\alpha \in \Phi_+$ if $\alpha = \sum n_i C_i$ with $n_i \leq 0$ and $\Phi_- = -\Phi_+$. Since $q(X) = 0$, every class in $H^2(X, \mathbb{Z}) \simeq \text{Pic}(X)$ corresponds to a unique line bundle over X up to isomorphisms. Similar to our earlier constructions, we could construct a Lie algebra bundle of ADE type $\mathfrak{g}$ over X as follows:

$$\mathcal{E}^{\mathfrak{g}} = O_X^{\oplus r} \bigoplus_{\alpha \in \Phi} O_X(\alpha).$$

It is natural to ask whether this ADE bundle $\mathcal{E}^{\mathfrak{g}}$ over X can be descended to the original surface X' which admits the ADE singularity p. Namely surfaces with an ADE singularity has a natural ADE bundle over it. In [25], Friedman and Morgan showed that it is possible for del Pezzo surfaces after small deformations of $\mathcal{E}^{\mathfrak{g}}$. In [13], we gave a direct construction of these deformations, which also works for general surfaces X' satisfying $q = p_g = 0$.

In the simplest case where p is an A_1 singularity, namely locally $\mathbb{C}^2 / \{\pm 1\}$, there is only one (-2)-curve C_1 in the exceptional locus and we have $\mathcal{E}^{A_1} = O_X \oplus O_X(-C_1) \oplus O_X(C_1) = \text{End}_0(O_X \oplus O_X(C_1))$, the bundle of traceless endomorphisms of $O_X \oplus O_X(C_1)$. Restricting to $C_1 \subset X$ we have

$$(O_X \oplus O_X(C_1))|_{C_1} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2),$$

which admits a nontrivial deformation as an extension

$$0 \to O_{\mathbb{P}^1}(-2) \to O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1) \to O_{\mathbb{P}^1} \to 0.$$

Using $p_g(X) = 0$, we could lift this extension from C_1 to X. The corresponding deformations of $\mathcal{E}^{A_1}$ would then be trivial along C_1 and therefore can be descended to X' as a Lie algebra bundle.

In the general case, we have the following result about these Lie algebra bundles $\mathcal{E}^{\mathfrak{g}}$ through studying their minuscule representation bundles in terms of (-1)-curves in X.

Theorem 8.1 ([13]). *Let p be an ADE singularity of a surface X' with $q(X') = p_g(X') = 0$ and X be its minimal resolution with exceptional curve $C = \bigcup_{i=1}^{r} C_i$. Then*

(i) given any $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}\big(X, \bigoplus_{i=1}^n O(C_i)\big)$ with $\overline{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^-} \in \Omega^{0,1}\big(X, \bigoplus_{\alpha \in \Phi^-} O(\alpha)\big)$ such that $\overline{\partial}_\varphi := \overline{\partial} + \mathrm{ad}(\varphi)$ is a holomorphic structure on $\mathcal{E}^{\mathfrak{g}}$. We denote this new holomorphic bundle as $\mathcal{E}^{\mathfrak{g}}_\varphi$;

(ii) such a $\overline{\partial}_\varphi$ is compatible with the Lie algebra structure;

(iii) $\mathcal{E}^{\mathfrak{g}}_\varphi$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$;

(iv) there exists $[\varphi_{C_i}] \in H^1(X, O(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$;

(v) such a $\mathcal{E}^{\mathfrak{g}}_\varphi$ can descend to X' if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$ for every i.

9   Relation to flag varieties of ADE type

In this section, we explain a relationship between our G-bundles over a surface X with a tautological G-bundle over the flag variety G/B and its cotangent bundle $G \times \mathfrak{n}/B$, as given in [16, Theorems 5, 6 and 7]. An ADE singularity of type $\mathfrak{g}$ is locally given by the intersection of the transversal slice S_x of a subregular nilpotent element x and the nilpotent variety $N(\mathfrak{g})$ of the complex Lie algebra $\mathfrak{g}$. Recall that $N(\mathfrak{g})$ is the fiber over zero of the adjoint quotient $\mathfrak{g} \to \mathfrak{g}/G \simeq \mathfrak{t}/W$, where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{g}$ and W is the corresponding Weyl group. Furthermore, the restriction of the adjoint quotient $\mathfrak{g} \to \mathfrak{t}/W$ to the transversal slice S_x is a semiuniversal deformation of the corresponding ADE singularity. This result is conjectured by Grothendieck and proved by Brieskorn in 1970 [6]. After that, Grothendieck defined a morphism $G \times \mathfrak{b}/B \to \mathfrak{t}$ and gave a simultaneous resolution of the adjoint quotient $\mathfrak{g} \to \mathfrak{t}/W$ using it. The restriction of the Grothendieck resolution to the above transversal slice S_x is also a simultaneous resolution [54]. In 1969, Springer gave a resolution of singularities for the nilpotent variety $N(\mathfrak{g})$ through $G \times \mathfrak{n}/B \to N(\mathfrak{g})$. Note that $G \times \mathfrak{n}/B \cong T^*(G/B)$ is the cotangent bundle of the flag variety G/B. The connection among these resolutions can be shown in the following Brieskorn–Slodowy–Grothendieck diagram (here $\widetilde{S}$ is the minimal resolution of S and $C = \bigcup_{i=1}^r C_i$ is the exceptional locus with each C_i irreducible component):

$$
\begin{array}{ccccc}
C = \bigcup_{i=1}^r C_i & \subset & \widetilde{S} & \longrightarrow & S = N(\mathfrak{g}) \cap S_x \\
\cap & & \cap & & \cap \\
G/B & \subset & G \times \mathfrak{n}/B & \longrightarrow & N(\mathfrak{g}) \\
& & \cap & & \cap \\
& & G \times \mathfrak{b}/B & \longrightarrow & \mathfrak{g} \\
& & \downarrow & & \downarrow \\
& & \mathfrak{t} & \longrightarrow & \mathfrak{t}/W.
\end{array}
$$

Under the above background, we consider the associated Lie algebra bundles $G \times \mathfrak{g}/B$ over G/B and $G \times \mathfrak{n} \times \mathfrak{g}/B$ over $G \times \mathfrak{n}/B$ respectively. It is obvious that these bundles are trivial as the action of B on $\mathfrak{g}$ can extend to the whole G. We [16] describe natural holomorphic filtration structures on these bundles explicitly.

As B is a solvable Lie group, the associated representation bundle $G \times \mathfrak{g}/B$ over G/B is an iterated extension of holomorphic line bundles. Also for the full flag variety G/B, we have $\mathrm{Pic}(G/B) = \Lambda$, where Λ is the weight lattice of the Lie algebra $\mathfrak{g}$. Hence for every $\lambda \in \Lambda$, we can associate a line bundle L_λ over G/B. From the Borel–Weil–Bott theorem, we can compute some particular cases of cohomology of line bundles over G/B easily. Note that our choice of B gives rise to Φ_+, the set of positive roots, such that $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$. We let $\{\alpha_1, \ldots, \alpha_r\}$ be the simple roots, then for any root $\alpha \in \Phi$, we have

(i) $H^i(G/B, L_\alpha) = 0$ for any $i \geq 2$;

(ii) $H^1(G/B, L_\alpha) \cong \mathbb{C}$ if $\alpha = -\alpha_i$ for some simple root α_i and $H^1(G/B, L_\alpha) = 0$ otherwise;

(iii) the restriction map $H^1(G/B, L_{-\alpha_i}) \to H^1(C_i, L_{-\alpha_i}|_{C_i}) \cong \mathbb{C}$ is an isomorphism for every simple root α_i. Hence $[\varphi_{-\alpha_i}|_{C_i}] \neq 0$ if and only if $[\varphi_{-\alpha_i}] \neq 0 \in H^1(G/B, L_{-\alpha_i})$.

Now we try to write the holomorphic structures on $G \times \mathfrak{g}/B$ explicitly. The filtration of the representation $\mathfrak{g}$ is given by the Chevalley order of its weights, hence not unique and we will choose an arbitrary one. Then the holomorphic structure $\overline{\partial}_\varphi$ on $G \times \mathfrak{g}/B$ can be written in an upper-triangular form with respect to the holomorphic structure $\overline{\partial}$ on the graded vector bundle associated to this filtration. Note that for a homogenous space G/P, a vector bundle V on G/P is trivial if and only if the restriction of V to every Schubert line is trivial [47]. Back to our cases, the Schubert lines in G/B are given by $C_i = P_{\alpha_i}/B$, where α_i's run through all the simple roots, and P_{α_i} is the parabolic subgroup of G corresponding to α_i. The main result is as follows:

Theorem 9.1 ([16])**.** *For the Lie algebra bundle* $(G \times \mathfrak{g}/B, [\,,\,])$ *over* G/B *with holomorphic structure* $\overline{\partial}_\varphi$ *as above, we have:*

(i) $\overline{\partial}_\varphi[\,,\,] = 0$ *if and only if* $\overline{\partial}_\varphi = \overline{\partial} + \sum_{\alpha \in \Phi^-} \mathrm{ad}(\varphi_\alpha)$ *with* $\varphi_\alpha \in \Omega^{0,1}(G/B, L_\alpha)$ *for some* $\alpha \in \Phi^-$.

(ii) *The bundle* $\left(G \times \mathfrak{g}/B, \overline{\partial}_\varphi = \overline{\partial} + \sum_{\alpha \in \Phi^-} \mathrm{ad}(\varphi_\alpha)\right)$ *is holomorphically trivial if and only if* $[\varphi_{-\alpha_i}|_{C_i}] \neq 0$ *for every simple root* α_i.

(iii) *The holomorphic structure of* $(G \times \mathfrak{g}/B, [\,,\,])$ *over* G/B *is* $\overline{\partial}_\varphi = \overline{\partial}_0 + \sum_{\alpha \in \Phi^-} \mathrm{ad}(\varphi_\alpha)$ *with* $[\varphi_{-\alpha_i}] \neq 0$ *for every simple root* α_i.

Consider the holomorphic structure of $G \times \mathfrak{n} \times \mathfrak{g}/B$ over $G \times \mathfrak{n}/B \cong T^*(G/B)$ when $\mathfrak{g}$ is of *ADE* type. First, we know that $G \times \mathfrak{n} \times \mathfrak{g}/B$ is an iterated extension of line bundles over $G \times \mathfrak{n}/B$ as B is solvable. And any line bundle over $G \times \mathfrak{n}/B$ is the pull back of a line bundle over G/B through the projection map $\pi \colon T^*(G/B) \cong G \times \mathfrak{n}/B \to G/B$. Denote $\mathfrak{L}_\lambda := \pi^* L_\lambda$ to be the corresponding line bundle over $G \times \mathfrak{n}/B$ for any weight $\lambda \in \Lambda$. Denote $H^i(\lambda) := H^i(G \times \mathfrak{n}/B, \mathfrak{L}_\lambda)$ for convenience. Using cohomology of line bundles on the cotangent bundle of the flag variety [7, 8, 28], we have:

(i) for any positive root $\alpha \in \Phi^+$, $H^i(\alpha) = 0$ for all $i \geq 1$;

(ii) for any negative root $\alpha \in \Phi^-$, $H^1(\alpha) \neq 0$, $H^2(\alpha) = 0$;

(iii) the restriction map $H^1(G \times \mathfrak{n}/B, \mathfrak{L}_{-\alpha_i}) \to H^1(G/B, L_{-\alpha_i})$ is surjective for every simple root α_i.

Theorem 9.2 ([16])**.** *The holomorphic structure of* $(G \times \mathfrak{n} \times \mathfrak{g}/B, [\,,\,])$ *over* $G \times \mathfrak{n}/B$ *is* $\overline{\partial}_\varphi = \overline{\partial} + \sum_{\alpha \in \Phi^-} \mathrm{ad}(\varphi_\alpha)$ *with* $[\varphi_{-\alpha_i}|_{G/B}] \neq 0 \in H^1(G/B, L_{-\alpha_i})$ *for every simple root* α_i.

Since the minimal resolution $\widetilde{S}$ of the *ADE* singular surface S is contained in $G \times \mathfrak{n}/B$, we also consider the restriction of the $\mathfrak{g}$-bundle $G \times \mathfrak{n} \times \mathfrak{g}/B$ from $G \times \mathfrak{n}/B$ to $\widetilde{S}$. Note that $\widetilde{S}$ is the minimal resolution of the *ADE* singular surface $S = \mathbb{C}^2/\Gamma$. It is obvious that this $\mathfrak{g}$-bundle over $\widetilde{S}$ is also an iterated extension of line bundles. The Picard group of $\widetilde{S}$ is a free abelian group generated by divisors dual to the irreducible curves C_i [22], i.e., $\mathrm{Pic}(\widetilde{S}) = \mathbb{Z}\langle D_i \rangle$ with each D_i dual to C_i.

As before, we know that the irreducible curves $C_i = P_{\alpha_i}/B$ are Schubert lines in G/B, where α_i's run through all the simple roots. Now for any weight λ, we consider the restriction of the line bundle L_λ from G/B to C_i and it is easy to see that $L_\lambda|_{C_i} \cong \mathcal{O}_{\mathbb{P}^1}(\langle \lambda, \alpha_i \rangle)$. For the restriction of the line bundle $\mathfrak{L}_\lambda = \pi^* L_\lambda$ from $G \times \mathfrak{n}/B$ to $\widetilde{S}$, we know that for any root $\alpha = \sum n_i \alpha_i$, $\mathfrak{L}_\alpha|_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{S}}\left(\sum -n_i C_i\right)$.

Theorem 9.3 ([16]). *The restriction of the $\mathfrak{g}$-bundle $G \times \mathfrak{n} \times \mathfrak{g}/B$ from $G \times \mathfrak{n}/B$ to $\widetilde{S}$ is*

$$\left(\mathcal{O}^{\oplus r} \oplus \bigoplus_{(\sum n_i C_i)^2 = -2} \mathcal{O}\Big(\sum n_i C_i\Big), \bar{\partial}_\varphi = \bar{\partial} + \sum_{\alpha \in \Phi^-} \mathrm{ad}(\varphi_\alpha) \right)$$

with $[\varphi_{-\alpha_i}] \neq 0 \in H^1\big(\widetilde{S}, \mathcal{O}(C_i)\big) \cong \mathbb{C}$ for every simple root α_i.

We note that the holomorphic structures described here have the same form as the holomorphic structures constructed in Section 8.

10   Generalization to affine ADE bundles

In this section, we will generalize the results for ADE bundles in Section 8 to affine ADE bundles and obtain [15, Theorem 8]. For convenience, we will call a curve $C = \cup C_i$ in a surface X an ADE (resp. affine ADE) curve of type $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) if each C_i is a smooth (-2)-curve in X and the dual graph of C is a Dynkin diagram of the corresponding type. By Kodaira's classification [34, 35, 36] of fibers of relative minimal elliptic surfaces, every singular fiber is a affine ADE curve unless it is rational with a cusp, tacnode or triplepoint (corresponding to type II or $III\big(\widehat{A}_1\big)$ or $VI\big(\widehat{A}_2\big)$ in Kodaira's notations), which can also be regarded as a degenerated affine ADE curve of type $\widehat{A}_0$, $\widehat{A}_1$ or $\widehat{A}_2$ respectively. We will not distinguish affine ADE curves from their degenerated forms since they have the same intersection matrices. We also call the affine ADE curves as Kodaira curves.

Suppose $C = \cup_{i=0}^r C_i$ is a affine ADE curve of type $\widehat{\mathfrak{g}}$ in X with C_0 corresponding to the extended root, then $\cup_{i=1}^r C_i$ will be the corresponding ADE curve of type $\mathfrak{g}$ and

$$\Phi := \left\{ \alpha = \Big[\sum_{i \neq 0} a_i C_i\Big] \in H^2(X, \mathbb{Z}) \,|\, \alpha^2 = -2 \right\}$$

is the root system of $\mathfrak{g}$. As before, we have a $\mathfrak{g}$-bundle

$$\mathcal{E}^{\mathfrak{g}} = O^{\oplus r} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha).$$

Also, there exists unique n_i's up to overall scalings such that $F := \sum n_i C_i$ satisfies $F \cdot F = 0$. We know

$$\Phi_{\widehat{\mathfrak{g}}} := \{\alpha + nF \,|\, \alpha \in \Phi, n \in \mathbb{Z}\} \cup \{nF \,|\, n \in \mathbb{Z}, n \neq 0\}$$

is a affine root system and it decomposes into union of positive and negative roots, i.e., $\Phi_{\widehat{\mathfrak{g}}} = \Phi_{\widehat{\mathfrak{g}}}^+ \cup \Phi_{\widehat{\mathfrak{g}}}^-$, where

$$\begin{aligned} \Phi_{\widehat{\mathfrak{g}}}^- &= \Big\{ \sum a_i C_i \in \Phi_{\widehat{\mathfrak{g}}} \,|\, a_i \geq 0 \text{ for all } i \Big\} \\ &= \{\alpha + nF \,|\, \alpha \in \Phi^+, \, n \in \mathbb{Z}_{\geq 1}\} \cup \{\alpha + nF \,|\, \alpha \in \Phi^-, \, n \in \mathbb{Z}_{\geq 0}\} \cup \{nF \,|\, n \in \mathbb{Z}_{\geq 1}\} \end{aligned}$$

and $\Phi_{\widehat{\mathfrak{g}}}^+ = -\Phi_{\widehat{\mathfrak{g}}}^-$. Then there is a canonically defined $\widehat{\mathfrak{g}}$-bundle $\mathcal{E}^{\widehat{\mathfrak{g}}}$ over X,

$$\mathcal{E}^{\widehat{\mathfrak{g}}} = \mathcal{E}^{\mathfrak{g}} \otimes \Big(\bigoplus_{n \in \mathbb{Z}} O(nF) \Big) \oplus O.$$

We remark that if we remove the central extension part O in $\mathcal{E}^{\widehat{\mathfrak{g}}}$, then this is a loop algebra $L\mathfrak{g}$ bundle over X and the Lie algebra bundle structure is independent of the choice of the affine root C_0 in F. Similar to Section 8, we have the following results:

Theorem 10.1 ([15])**.** *Given any complex surface X with $p_g = 0$. If X has a Kodaira curve $C = \cup_{i=0}^{r} C_i$ of type $\widehat{\mathfrak{g}}$, then*

(i) *given any $(\varphi_{C_i})_{i=0}^{r} \in \Omega^{0,1}\big(X, \bigoplus_{i=0}^{r} O(C_i)\big)$ with $\overline{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi_{\widehat{\mathfrak{g}}}^{-}} \in \Omega^{0,1}\big(X, \bigoplus_{\alpha \in \Phi_{\widehat{\mathfrak{g}}}^{-}} O(\alpha)\big)$ such that $\overline{\partial}_\varphi := \overline{\partial} + \mathrm{ad}(\varphi)$ is a holomorphic structure on $\mathcal{E}^{\widehat{\mathfrak{g}}}$. We denote the new bundle as $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$;*

(ii) *$\overline{\partial}_\varphi$ is compatible with the Lie algebra structure on $\mathcal{E}^{\widehat{\mathfrak{g}}}$;*

(iii) *$\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$;*

(iv) *there exists $[\varphi_{C_i}] \in H^1(X, O(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$.*

11 Deformability of Lie algebra bundles and geometry of rational surfaces

In this section, we discuss the hidden geometry underlying the deformability of the affine E_8 bundle over X_9, as stated in [14, Theorems 9 and 10]. For an ADE curve C in a surface X with $p_g = 0$, the corresponding Lie algebra bundle $\mathcal{E}^{\mathfrak{g}}$ over X admits a deformation which can be descended to the surface obtained by blowing down C in X. On the other hand, an affine ADE curve can never be blown down. Nevertheless, we could explain the geometric meaning of such deformabilities as below.

Recall that we have an E_n Lie algebra bundle $\mathcal{E}^{E_n}$ over X_n for $n \leq 8$. When $n = 9$, E_9 is the affine Kac–Moody algebra of E_8, i.e., $E_9 = \hat{E}_8$. When $H^2(X, \mathbb{Z})$ has a sublattice $\Lambda_{\mathfrak{g}}$ isomorphic to the root lattice of a simple Lie algebra $\mathfrak{g}$, then our construction also gives a $\mathfrak{g}$ Lie algebra bundle $\mathcal{E}^{\mathfrak{g}}$ over X. Infinitesimal deformations of $\mathcal{E}^{\mathfrak{g}}$ as a $\mathfrak{g}$-bundle are parametrized by $H^1(X, \mathrm{ad}(\mathcal{E}^{\mathfrak{g}})) \simeq H^1(X, \mathcal{E}^{\mathfrak{g}}) \subset H^1(X, \mathrm{End}(\mathcal{E}^{\mathfrak{g}}))$. We say $\mathcal{E}$ is (i) fully deformable if there is a base $\Delta \subset \Phi$ of the root system Λ_g such that $H^1(X, O(\alpha)) \neq 0$ for every $\alpha \in \Delta$, (ii) $\mathfrak{h}$-deformable with $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$ if there exists a strict $\mathfrak{h}$-subbundle of $\mathcal{E}$ which is fully deformable, (iii) totally non-deformable if $H^1(X, O(\alpha)) = 0$ for every $\alpha \in \Delta$, (iv) deformable in α-direction for $\alpha \in \Phi$ if $H^1(X, O(\alpha)) \neq 0$.

We proved the following results.

Theorem 11.1 ([14])**.** *On X_9, a blowup of $\mathbb{P}^2$ at 9 points, these points are in general position in $\mathbb{P}^2$ if and only if $\mathcal{E}^{E_9}$ is totally non-deformable.*

Theorem 11.2 ([14])**.** *If $-K$ is nef on X_9, then*

(i) *there exists an ADE curve $C \subset X_9$ of type $\mathfrak{g}$ if and only if $\mathcal{E}^{E_9}$ is $\mathfrak{g}$-deformable;*

(ii) *there exists a affine ADE curve $C \subset X_9$ of type $\widehat{\mathfrak{g}}$ if and only if $\mathcal{E}^{E_9}$ is $\widehat{\mathfrak{g}}$-deformable;*

(iii) *X_9 admits an elliptic fibration structure with a multiple fiber of multiplicity m if and only if $\mathcal{E}^{E_9}$ is deformable in the $(-mK)$-direction, but not in $(-m+1)K$-direction.*

12 Cox rings of ADE surfaces

In this section, we explain another mysterious relationship between del Pezzo surfaces X_n and the Lie group G of type E_n discovered by Batyrev and Popov [2] relating the Cox ring of X_n and the flag variety $G/P_L \subset \mathbb{P}(L)$. We also establish the the corresponding results for any ADE surface in [45, Theorem 11]. Very loosely speaking, the Cox ring of a variety X with $q(X) = 0$ and torsion free $H^2(X, \mathbb{Z})$ is the sum of spaces of sections of all line bundles on X,

$$\mathrm{Cox}(X) \sim \bigoplus_{[L] \in \mathrm{Pic}(X)} H^0(X, L),$$

with the ring structure given by the tensor products of sections. As elements in $\text{Pic}(X)$ are only isomorphism classes of line bundles, in order to define $\text{Cox}(X)$ properly, one needs to fix a collection of line bundles L_i's with $i = 1, \ldots, b$ (with $b = b_2(X)$) whose first Chern classes represent a basis of $H^2(X, \mathbb{Z})$, then the correct definition of the Cox ring of X with respect to this choice is

$$\text{Cox}(X) = \bigoplus_{n_i \in \mathbb{Z}} H^0\big(X, L_1^{\otimes n_1} \otimes L_2^{\otimes n_2} \otimes \cdots \otimes L_b^{\otimes n_b}\big).$$

Then the ring structure is simply given by the tensor products of holomorphic sections.

When X is a del Pezzo surface, the ample line bundle K^{-1} defines a grading on $\text{Cox}(X)$. We have seen before that the geometry of a degree d del Pezzo surface is closely related to the Lie algebra E_n with $n = 9 - d$. For example the number of lines in X is equal to the dimension of the fundamental representation L for $n < 8$. It turns out that the flag variety G/P_L corresponding to L, namely the unique closed G-orbit in $\mathbb{P}(L)$, is closely related to the Cox ring of X as follows: When the degree d of X satisfies $d \leq 5$, Batyrev–Popov [2], Derenthal [18], Serganova–Skorobogatov [52, 53] showed that there is a natural embedding of the projective spectrum of the graded ring $\text{Cox}(X)$ into G/P_L:

$$\text{Proj}(\text{Cox}(X)) \hookrightarrow G/P_L.$$

It was observed in [45] that this relationship can be easily generalized to all ADE surfaces X_G. Recall that an ADE surfac X_G of rank n is a blowup X_{n+1} of $\mathbb{P}^2$ at $n+1$ distinct points, together with a rational curve C in X_{n+1} whose class in $H^2(X_{n+1}, \mathbb{Z})$ is (i) h satisfying $h \cdot h = 1$ for type A, (ii) f satisfying $f \cdot f = 0$ for type D and (iii) l satisfying $l \cdot l = -1$ for type E. We have $\langle K, C \rangle^{\perp} \subset H^2(X_{n+1}, \mathbb{Z})$ is a root lattice Λ_G of corresponding ADE type. In particular, the blow down of the (-1)-curve l in an E_n-surface is a del Pezzo surface X of degree $d = 9 - n$ and $\text{Cox}(X) = \bigoplus_{L \in \langle l \rangle^{\perp}} H^0(X_{En}, L)$ defined loosely as before. Notice that $H^2(X, \mathbb{Z}) \simeq \langle l \rangle^{\perp} \subset H^2(X_{n+1}, \mathbb{Z})$.

Similarly we define a generalization of the Cox ring for an ADE surface $X_G = (X_{n+1}, C)$ of type G as follows:

$$\text{Cox}^G(X_G) = \bigoplus_{L \in \langle C \rangle^{\perp}} H^0(X_{n+1}, L).$$

We also define the flag variety of G corresponding to the fundamental representation L as G/P_L, i.e., G/P_L is the unique closed G-orbit in $\mathbb{P}(L)$. Then we have the following statement:

Theorem 12.1 ([45])**.** *For any ADE surface $X_G = (X_{n+1}, C)$ of type G, we have*

$$\text{Proj}\big(\text{Cox}^G(X_G)\big) \hookrightarrow G/P_L,$$

with $n \geq 4$ for type E_n cases.

13   Conclusions

We have discussed several intriguing relationships between the geometry of surfaces and Lie theory. A couple of these are related to physics, namely the duality between F-theory and heterotic string theory and supergravity theory in eleven dimensions. There is also a mysterious duality between the geometry of del Pezzo surfaces and toroidal compactification of M-theory in physics, as proposed by Iqbal, Neitzke and Vafa in [30]. We expect that more surprising connections will be uncovered in the future.

Acknowledgements

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