Symmetry, Integrability and Geometry: Methods and Applications SIGMA 18 (2022), 076, 15 pages

Universality of Descendent Integrals over Moduli Spaces of Stable Sheaves on $K3$ Surfaces

Georg OBERDIECK

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
E-mail: *georgo@math.uni-bonn.de*

Received January 23, 2022, in final form October 06, 2022; Published online October 13, 2022
https://doi.org/10.3842/SIGMA.2022.076

Abstract. We interprete results of Markman on monodromy operators as a universality statement for descendent integrals over moduli spaces of stable sheaves on $K3$ surfaces. This yields effective methods to reduce these descendent integrals to integrals over the punctual Hilbert scheme of the $K3$ surface. As an application we establish the higher rank Segre–Verlinde correspondence for $K3$ surfaces as conjectured by Göttsche and Kool.

Key words: moduli spaces of sheaves; $K3$ surfaces; descendent integrals

2020 Mathematics Subject Classification: 14D20; 14J28; 14J80; 14J60

1 Introduction

1.1 Descendent integrals

Let M be a proper and fine[1] moduli space of Gieseker stable sheaves F on a $K3$ surface S with Mukai vector

$$v(F) := \mathrm{ch}(F)\sqrt{\mathrm{td}_S} = v \in H^*(S,\mathbb{Z}).$$

Let π_M, π_S be the projections of $M\times S$ to the factors and let $\mathcal{F}\in \mathrm{Coh}(M\times S)$ be a universal family. We define the k-th descendent of a class $\gamma\in H^*(S,\mathbb{Q})$ by

$$\tau_k(\gamma) = \pi_{M*}(\pi_S^*(\gamma)\mathrm{ch}_k(\mathcal{F})) \in H^*(M). \tag{1.1}$$

Let $P(c_1,c_2,c_3,\dots)$ be a polynomial and consider an arbitrary integral of descendents and Chern classes of the tangent bundle over the moduli space:

$$\int_M \tau_{k_1}(\gamma_1)\cdots\tau_{k_\ell}(\gamma_\ell)P(c_r(T_M)). \tag{1.2}$$

The goal of this paper is to explain the following application of Markman's work [13] on monodromy operators:

Theorem 1.1. *Any integral of the form* (1.2) *can be effectively reconstructed from the set of all integrals* (1.2)*, where M is replaced by the Hilbert scheme of n points of a $K3$ surface, with $n=\dim M/2$.*

We refer to Section 2 for the precise form the reconstruction of the theorem takes. In particular, Theorem 2.9 is a universality statement for descendent integrals over M, that immediately implies Theorem 1.1.

This paper is a contribution to the Special Issue on Enumerative and Gauge-Theoretic Invariants in honor of Lothar Göttsche on the occasion of his 60th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Gottsche.html

[1]See Remark 2.1 for the extension to the case where only a quasi-universal family exists.

1.2 Segre numbers

As a concrete application of Theorem 1.1 we prove a conjecture of Göttsche and Kool which was made in [3, Conjecture 5.1]: Consider the decomposition of $v \in H^*(S,\mathbb{Z})$ according to degree

$$v = (\mathsf{rk}(v), c_1(v), v_2) \in H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}),$$

and assume that

$$\mathsf{rk}(v) > 0.$$

For any topological K-theory class $\alpha \in K(S)$ define

$$\alpha_M = \mathrm{ch}\big(-\pi_{M*}\big(\pi_S^*(\alpha) \otimes \mathcal{F} \otimes \det(\mathcal{F})^{-1/\mathsf{rk}(v)}\big)\big)$$

whenever a $\mathsf{rk}(v)$-th root of $\det(\mathcal{F})$ exists. Otherwise we define α_M by a formal application of the Grothendieck–Riemann–Roch formula. Let $c(\alpha_M)$ be the Chern class corresponding to α_M, see Remark 2.5.

For $\sigma \in H^*(S)$ consider, with the same convention if the root does not exist, the class

$$\mu_M(\sigma) = -\pi_{M*}\big(\mathrm{ch}_2\big(\mathcal{F} \otimes \det(\mathcal{F})^{-1/\mathsf{rk}(v)}\big)\pi_S^*(\sigma)\big).$$

We will usually drop the subscript M from the notation.

Theorem 1.2. *Let $n = \frac{1}{2}\dim M$ and let $\mathsf{p} \in H^4(S,\mathbb{Z})$ be the class of a point. For any $\alpha \in K(S)$, class $L \in H^2(S)$ and $u \in \mathbb{C}$ we have*

$$\int_M c(\alpha_M)\mathrm{e}^{\mu(L)+u\mu(\mathsf{p})} = \int_{S^{[n]}} c(\beta_{S^{[n]}})\mathrm{e}^{\mu(L)+u\,\mathsf{rk}(v)\mu(\mathsf{p})},$$

where $\beta \in K(S)$ is any K-theory class such that

$$\mathsf{rk}(\beta) = \frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)}, \quad c_1(\alpha)^2 = c_1(\beta)^2, \quad c_1(\alpha)\cdot L = c_1(\beta)\cdot L, \quad v_2(\beta) = \mathsf{rk}(v)v_2(\alpha). \tag{1.3}$$

As explained in [3, Corollary 5.2] this implies the following closed evaluation of the *Segre numbers* of M:

Corollary 1.3. *Let $\rho = \mathsf{rk}(v)$, $s = \mathsf{rk}(\alpha)$, $n = \frac{1}{2}\dim M$. Then we have*

$$\int_M c(\alpha_M) = \mathrm{Coeff}_{z^n}\big(V_s^{c_2(\alpha)} W_s^{c_1(\alpha)^2} X_s^2\big),$$

where

$$V_s(z) = \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{1-s}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{s}\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\rho-1},$$

$$W_s(z) = \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}s-1}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}(1-s)}\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}-\frac{1}{2}\rho},$$

$$X_s(z) = \left(1+\left(1-\frac{s}{\rho}\right)t\right)^{\frac{1}{2}s^2-s}\left(1+\left(2-\frac{s}{\rho}\right)t\right)^{-\frac{1}{2}s^2+\frac{1}{2}}$$
$$\times\left(1+\left(1-\frac{s}{\rho}\right)\left(2-\frac{s}{\rho}\right)t\right)^{-\frac{1}{2}}\left(1+\left(1-\frac{s}{\rho}\right)t\right)^{-\frac{(\rho-1)^2}{2\rho}s}$$

under the variable change $z = t\big(1+\big(1-\frac{s}{\rho}\big)t\big)^{1-\frac{s}{\rho}}$.

The Segre numbers of the Hilbert scheme of n points on the $K3$ surface S were determined by Marian, Oprea and Pandharipande [11]. In particular, they found the series V_s, W_s, X_s. All that Theorem 1.2 does here is move their result from Hilbert schemes to moduli spaces of sheaves of arbitrary rank. Earlier work on Segre numbers can be found in [1, 8, 9, 10, 14].

1.3 Segre/Verlinde correspondence

Göttsche and Kool conjectured that the Segre numbers of moduli spaces of stable sheaves on surfaces are related by an explicit correspondence to the *Verlinde numbers* of these moduli spaces. For $K3$ surfaces the Verlinde numbers are known explicitly by

$$\chi\big(M, \mu(L) \otimes E^{\otimes r}\big) = \mathrm{Coeff}_{w^n}\big(G_r^{\chi(L)}\, F_r^{\frac{1}{2}\chi(\mathcal{O}_S)}\big),$$

where

$$F_r(w) = (1+v)^{\frac{r^2}{\rho^2}}\left(1 + \frac{r^2}{\rho^2}v\right)^{-1}, \qquad G_r(w) = 1 + v$$

under the variable change $w = v(1+v)^{r^2/\rho^2 - 1}$, and we refer to [3, equation (4)] for the definition of the class $\mu(L) \otimes E^{\otimes r} \in \mathrm{Pic}(M)_{\mathbb{Q}}$. The Verlinde numbers of the Hilbert schemes of points of $K3$ surfaces (and in particular the series F_r, G_r) were first computed in [1]. The computation for moduli spaces of higher rank sheaves reduces to the Hilbert scheme case as shown in [4] using hyperkähler geometry, parallel to Theorem 1.2.

The functions F_r, G_r and V_s, W_s, X_s are related by the following variable change [3]:

$$F_r(w) = V_s(z)^{\frac{s}{\rho}(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2}\, W_s(z)^{-\frac{4s}{\rho}}\, X_s(z)^2,$$
$$G_r(w) = V_s(z)\, W_s(z)^2,$$

where $s = \rho + r$ and $v = t\big(1 - \frac{r}{\rho}t\big)^{-1}$.

Hence with Corollary 1.3 we have proven that the Segre and Verlinde numbers of moduli spaces of stable sheaves on $K3$ surfaces are related by this variable change. This is the $K3$ surface case of the higher-rank Segre–Verlinde correspondence conjectured by Göttsche–Kool [3, Conjecture 1.7].

Corollary 1.4. *The higher-rank Segre–Verlinde correspondence holds for $K3$ surfaces.*

1.4 Plan

In Section 2, we use results from Markman's beautiful article [13] to formulate a universality result for descendent integrals of moduli spaces of stable sheaves on $K3$ surfaces, see Theorem 2.9. This immediately yields Theorem 1.1. In Section 3, we prove Theorem 1.2.

2 Markman's universality

2.1 Basic definitions

Let S be a $K3$ surface and consider the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$(x, y) := -\int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as (r, D, n), we have written $x^\vee = (r, -D, n)$. We will also write

$$\mathsf{rk}(x) = r, \qquad c_1(x) = D, \qquad v_2(x) = n.$$

Given a sheaf or complex E on S the Mukai vector of E is defined by

$$v(E) = \sqrt{\mathrm{td}_S} \cdot \mathrm{ch}(E) \in \Lambda.$$

Let $v \in \Lambda$ be an effective[2] vector, H be an ample divisor on S and let

$$M := M_H(v)$$

be the moduli space of H-stable sheaves with Mukai vector v. The moduli space is smooth and holomorphic-symplectic of dimension $2 + (v, v)$. We further assume that the Mukai vector v is primitive, and the polarization H is v-generic (see [7, Theorem 6.2.5]), so that M is also proper (in particular, semistability is equivalent to stability). We also assume that there exists a universal sheaf $\mathcal{F}$ on $M_H(v) \times S$.

Remark 2.1. The results we state below also hold in the case where there exists only a twisted universal sheaf. More precisely, all statements below can be formulated in terms of the Chern character $\mathrm{ch}(\mathcal{F})$ alone and this class can be defined in the twisted case as well, see [12, Section 3]. The proofs carry over likewise since all ingredients hold in the twisted case as well.

Remark 2.2. More generally, one can also work with σ-stable objects for a Bridgeland stability condition in the distinguished component.

Assume from now on that[3]

$$\dim M = (v, v) + 2 > 2.$$

Consider the morphism $\theta_{\mathcal{F}} \colon \Lambda \to H^2(M_H(v), \mathbb{Z})$ defined by

$$\theta_{\mathcal{F}}(x) = \left[\pi_{M*}\big(\mathrm{ch}(\mathcal{F})\pi_S^*\big(\sqrt{\mathrm{td}_S} \cdot x^\vee\big)\big)\right]_2, \tag{2.1}$$

where $[-]_k$ stands for taking the degree k component of a cohomology class. Then $\theta_{\mathcal{F}}$ restricts to an isomorphism

$$\theta = \theta_{\mathcal{F}}|_{v^\perp} \colon \; v^\perp \xrightarrow{\cong} H^2(M_H(v), \mathbb{Z}) \tag{2.2}$$

which does not depend on the choice of universal family (use that the degree 0 component of the pushforward in (2.1) vanishes) and for which we hence have dropped the subscript $\mathcal{F}$. The isomorphism θ is an isometry with respect to the Mukai pairing on the left, and the pairing given by the Beauville–Bogomolov–Fujiki form on the right. We will identify $v^\perp \subset \Lambda$ with $H^2(M_H(v), \mathbb{Z})$ under this isomorphism.

The universal sheaf $\mathcal{F}$ and hence its Chern character $\mathrm{ch}(\mathcal{F})$ is uniquely determined only up to tensoring by the pullback of a line bundle from M. Following [13], we can pick a canonical normalization as follows:

$$u_v := \exp\left(\frac{\theta_{\mathcal{F}}(v)}{(v, v)}\right) \cdot \mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{td}_S} \;\in H^*(M \times S),$$

where we have suppressed the pullback by the projections to M and S in the first and last term on the right. We will follow similar conventions throughout. It is immediate to check that u_v is independent from the choice of universal family (replace $\mathcal{F}$ by $\mathcal{F} \otimes \pi_M^* \mathcal{L}$ and calculate, see [13, Lemma 3.1]).

Example 2.3. Let $M = S^{[n]}$ be the Hilbert scheme of n points on S. We have $v = 1 - (n-1)\mathsf{p}$, and we always take $\mathcal{F} = I_{\mathcal{Z}}$, the ideal sheaf of the universal subscheme. If $\alpha \in H^2(S)$ is the class of an effective divisor $A \subset S$, then

$$\theta(\alpha) = \pi_{S^{[n]}*}\big(\mathrm{ch}_2(\mathcal{O}_{\mathcal{Z}})\pi_S^*(\alpha)\big)$$

[2] Following [13, Definition 1.1], this means that $v \cdot v \geq -2$ and $\mathsf{rk}(v) \geq 0$, and if $\mathsf{rk}(v) = 0$ then $c_1(v)$ is effective or zero, and if $\mathsf{rk}(v) = c_1(v) = 0$ then $v_2 > 0$.

[3] We return to the case $\dim M = 2$ in Section 2.4.

is the class of the locus of subschemes incident to A. If we denote

$$\delta := -\frac{1}{2}\Delta_{S^{[n]}} = c_1(\pi_{S^{[n]}*}\mathcal{O}_{\mathcal{Z}}) = \pi_{S^{[n]}*}\mathrm{ch}_3(\mathcal{O}_{\mathcal{Z}}),$$

where $\Delta_{S^{[n]}}$ is the class of the locus of non-reduced subschemes, then under the identification (2.2) we have $\delta = -\big(1+(n-1)\mathsf{p}\big)$. Because $\theta_{\mathcal{F}}(v) = -\delta$ the canonical normalization of $\mathrm{ch}(\mathcal{F})$ takes the form

$$u_v = \exp\left(\frac{-\delta}{2n-2}\right)\mathrm{ch}(I_{\mathcal{Z}})\sqrt{\mathrm{td}_S}.$$

2.2 Markman's operator

For $i = 1,2$ let (S_i, H_i, v_i) be the data defining proper fine moduli space of stable sheaves $M_i = M_{H_i}(S_i, v_i)$, and let $\mathcal{F}_i$ be the universal family on $M_i \times S_i$. Consider an isometry of Mukai lattices

$$g\colon\ H^*(S_1,\mathbb{Z}) \to H^*(S_2,\mathbb{Z})$$

such that $g(v_1) = v_2$. Let $K(S)$ be the topological K-group of S endowed with the Euler pairing $(E,F) = -\chi(E^\vee \otimes F)$. We identify g with an isometry

$$g\colon\ K(S_1) \to K(S_2)$$

through the lattice isometry $K(S) \xrightarrow{\cong} H^*(S,\mathbb{Z})$ given by $E \mapsto v(E)$. Hence the following diagram commutes

$$\begin{array}{ccc} K_{\mathrm{top}}(S_1) & \xrightarrow{g} & K_{\mathrm{top}}(S_2) \\ \downarrow v & & \downarrow v \\ H^*(S_1,\mathbb{Z}) & \xrightarrow{g} & H^*(S_2,\mathbb{Z}). \end{array}$$

Similar identification will apply to morphisms g defined over $\mathbb{C}$. The Markman operator associated to g is given by the following result:

Theorem 2.4 (Markman)**.** *For any isometry $g\colon H^*(S_1,\mathbb{C}) \to H^*(S_2,\mathbb{C})$ such that $g(v_1) = v_2$ there exists a unique operator*

$$\gamma(g)\colon\ H^*(M_1,\mathbb{C}) \to H^*(M_2,\mathbb{C})$$

such that

(a) *$\gamma(g)$ is a degree-preserving isometric*[4] *ring-isomorphism,*

(b) $(\gamma(g)\otimes g)(u_{v_1}) = u_{v_2}$.

The operator is called the Markman operator *and given by*

$$\gamma(g) = c_{\dim(M)}\big[-\pi_{13*}\big(\pi_{12}^*((1\otimes g)u_{v_1})^\vee \cdot \pi_{23}^* u_{v_2}\big)\big], \tag{2.3}$$

where π_{ij} is the projection of $M_1 \times S_2 \times M_2$ to the (i,j)-th factor. Moreover, we have

(c) *$\gamma(g_1)\circ\gamma(g_2) = \gamma(g_1g_2)$ and $\gamma(g)^{-1} = \gamma\big(g^{-1}\big)$ if it makes sense.*

(d) $\gamma(g)c_k(T_{M_1}) = c_k(T_{M_2})$.

[4] We endow $H^*(M)$ with the Poincaré pairing: $\langle x, y\rangle = \int_M xy$ for all $x, y \in H^*(M)$.

Remark 2.5. Here the Chern class c_m in (2.3) has the following definition: Let

$$\ell\colon \quad \oplus_i H^{2i}(M,\mathbb{Q}) \to \oplus_i H^{2i}(M,\mathbb{Q})$$

be the universal map that takes the exponential Chern character to Chern classes, so in particular $c(E) = \ell(\mathrm{ch}(E))$ for any vector bundle. Then given $\alpha \in H^*(M)$ we write $c_m(\alpha)$ for $[\ell(\alpha)]_{2m}$.

Remark 2.6. In Theorem 2.4, since the morphism $\gamma(g)$ is a ring isomorphism we have $\gamma(g)1 = 1$. Since $\gamma(g)$ preserves degree and is isometric, it hence sends the class of a point on M_1 to the class of a point on M_2. For any $\sigma \in H^*(M_1)$ we thus observe hat

$$\int_{M_1} \sigma = \int_{M_2} \gamma(g)(\sigma).$$

Proof of Theorem 2.4. If g is an integral isometry, then the statement of the theorem is a combination of Theorems 1.2 and 3.10 of [13]. The proof is involved: Markman establishes that operators $\gamma(g)$ satisfying (a) and (b) exists by considering arbitrary compositions of parallel transport operators and pushforwards by isomorphisms induced by auto-equivalences. Then a small computation starting from an expression for the diagonal class of M_1 in terms of the universal sheaf $\mathcal{F}$ in [12], shows that conditions (a) and (b) for any homomorphism forces the expression (2.3). Hence those homomorphisms are uniquely determined. This last step holds even for homomorphisms defined over $\mathbb{C}$ which satisfy (a) and (b).

In the general case, one defines the operator $\gamma(g)$ by (2.3). Then (a) and (b) holds for a Zariski dense subset of all operators g (i.e., for the integral isometries). Hence it holds for all g. Then by the uniqueness statement one observes (c). Again (d) follows by the Zariski density argument from the integral case (which is [13, Theorem 1.2(6)]). We also refer to [2, Proposition 5.1] for more details on extending the Markman operator from integral isometries to isometries defined over more general coefficient rings. ∎

One can reinterpret the condition $(f \otimes g)(u_{v_1}) = u_{v_2}$ in terms of generators of the cohomology ring. Following [13, equation (3.23)], consider the canonical morphism

$$B\colon \; H^*(S,\mathbb{Q}) \to H^*(M,\mathbb{Q})$$

defined by

$$B(x) = \pi_{M*}(u_v \cdot x^{\vee}).$$

We write $B_k(x)$ for its component in degree $2k$. In particular, $B_0(x) = -(x,v)$ and $B_1(x) = \theta_{\mathcal{F}}(x)$ for all $x \in v^{\perp}$.

Lemma 2.7. *Let $f\colon H^*(M_1,\mathbb{Q}) \to H^*(M_2,\mathbb{Q})$ be a degree-preserving isometric ring isomorphism. Then the following are equivalent:*

(a) $(f \otimes g)(u_{v_1}) = u_{v_2}$,

(b) $f(B(x)) = B(gx)$ *for all* $x \in H^*(S_1,\mathbb{Q})$.

Proof. Since g is an isometry of the Mukai lattice we have for $x \in H^*(S_1)$ the following equality in $H^*(M_2)$:

$$\pi_{M_2*}(u_{v_2} \cdot (gx)^{\vee}) = \pi_{M_2*}\big((1 \otimes g^{-1})u_{v_2} \cdot x^{\vee}\big).$$

Indeed, if we write $u_{v_2} = \sum_i a_i \otimes b_i$ under the Künneth decomposition, then

$$\begin{aligned}\pi_{M_2*}\big((1\otimes g^{-1})(u_{v_2})\cdot x^\vee\big) &= \sum_i a_i \int_{S_1} g^{-1}(b_i)x^\vee = \sum_i -a_i\cdot\big(g^{-1}(b_i)\cdot x\big)\\ &= \sum_i -a_i\cdot(b_i\cdot g(x)) = \sum_i a_i\int_{S_2} b_i g(x)^\vee\\ &= \pi_{M_2*}\big(u_{v_2}\cdot g(x)^\vee\big).\end{aligned}$$

Hence we see that:

$$\begin{aligned}(b) &\iff \forall x\in H^*(S_1,\mathbb{Z})\colon\ f\pi_{M_1*}\big(u_{v_1}\cdot x^\vee\big) = \pi_{M_2*}\big(u_{v_2}\cdot(gx)^\vee\big)\\ &\iff \forall x\in H^*(S_1,\mathbb{Z})\colon\ \pi_{M_2*}\big((f\otimes 1)u_{v_1}\cdot x^\vee\big) = \pi_{M_2*}\big((1\otimes g^{-1})u_{v_2}\cdot x^\vee\big)\\ &\iff (f\otimes 1)(u_{v_1}) = \big(1\otimes g^{-1}\big)(u_{v_2})\\ &\iff (a).\end{aligned}$$

∎

Corollary 2.8. *In the setting of Theorem 2.4, $\gamma(g)B(x) = B(gx)$.*

2.3 Universality

We apply Theorem 2.4 to study descendent integrals over M. Let $k\geq 0$ and let $P(t_{ij}, u_r)$ be a polynomial depending on the variables

$$t_{j,i},\quad j=1,\dots,k,\quad i\geq 1,\qquad\text{and}\qquad u_r,\quad r\geq 1.$$

Let also $A = (a_{ij})_{i,j=0}^k$ be a $(k+1)\times(k+1)$-matrix.

Our main result is the following.

Theorem 2.9 (universality)**.** *There exists $I(P,A)\in\mathbb{Q}$ (depending only on P and A) such that for any $M = M_H(v)$ with $\dim(M) > 2$ and for any $x_1,\dots,x_k\in\Lambda$ with*

$$\begin{pmatrix} v\cdot v & (v\cdot x_i)_{i=1}^k\\ (x_i\cdot v)_{i=1}^k & (x_i\cdot x_j)_{i,j=1}^k\end{pmatrix} = A \tag{2.4}$$

we have

$$\int_M P(B_i(x_j), c_r(T_M)) = I(P,A).$$

In other words, the integral

$$\int_M P(B_i(x_j), c_r(T_M))$$

depends upon the above data only through P, the dimension $\dim M = 2n$, and the pairings $v\cdot x_i$ and $x_i\cdot x_j$ for all i, j.

The proof of Theorem 2.9 will proceed in several steps. We begin with a general vanishing result.

Proposition 2.10. *Let $M = M_H(v)$ be a moduli space of stable sheaves on S of dimension $2n > 2$, and let $x_1,\dots,x_k$, $w\in\Lambda_{\mathbb{C}}$ be given with $w\cdot y = 0$ for all $y\in\{v,x_1,\dots,x_k,w\}$. Then any integral of the form*

$$\int_M \prod_{i=1}^{\ell} B_{s_i}(w)\cdot(\text{monomial in } B_i(x_j) \text{ and } c_r(T_M)) \tag{2.5}$$

for some $s_i\in\mathbb{Z}$ vanishes unless $\ell = 0$.

Proof. We give two proofs of this fact. For the first proof, choose an isometry $g\colon \Lambda_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$ such that

$$g(v) = 1 - (n-1)\mathsf{p}, \qquad w' := g(w) \in H^2(S, \mathbb{C}),$$

where $v \cdot v = 2n-2$. Such an isometry exists since $v \cdot v > 0$ and $\mathrm{SO}(\Lambda_{\mathbb{C}})$ acts transitively on vectors of the same square. By Theorem 2.4(a) for the first and Corollary 2.8 and Theorem 2.4(d) for the second equation, we find that

$$\begin{aligned}
&\int_M \prod_{i=1}^{\ell} B_{s_i}(w) \cdot \big(\text{monomial in } B_i(x_j) \text{ and } c_r(T_M)\big) \\
&\quad = \int_{S^{[n]}} \gamma(g)\left(\prod_{i=1}^{\ell} B_{s_i}(w) \cdot \big(\text{monomial in } B_i(x_j) \text{ and } c_r(T_{S^{[n]}})\big) \right) \\
&\quad = \int_{S^{[n]}} \prod_{i=1}^{\ell} B_{s_i}(w') \cdot \big(\text{monomial in } B_i(gx_j) \text{ and } c_r(T_{S^{[n]}})\big).
\end{aligned}$$

By [1, Theorem 4.1] (or more precisely, the induction method used in the proof), this last integral depends upon w' only through its intersection numbers against products of Chern classes of S and degree-components of gx_j.[5] Since these intersections numbers are all zero, we may replace w' by 0, in which case the claimed vanishing follows immediately. ■

Alternative proof. If $w = 0$ there is nothing to prove, so let $w \neq 0$. Choose $w' \in \Lambda_{\mathbb{C}}$ such that $w \cdot w' = 1$ and $w' \cdot w' = w' \cdot v = 0$. Extend v, w, w' to a basis $\{v, w, w'\} \cup \{e_i\}_{i=4}^{24}$ of $\Lambda_{\mathbb{C}}$. For any j, expand x_j in this basis:

$$x_j = a_1 v + a_2 w + a_3 w' + a_4 e_4 + \cdots + a_{24} e_{24}.$$

Because $x_j \cdot w = 0$, we must have $a_3 = 0$. By an induction on the number of classes x_j, we know the claim of Proposition 2.10 if x_j is a multiple of w.[6] Moreover, if we know the claim for $x_j \in \{u_1, u_2\}$ for some $u_1, u_2 \in \Lambda_{\mathbb{C}}$ then we know it for $x_j = u_1 + u_2$ by expanding the monomial in (2.5). Hence we may replace x_j by $x_j - a_2 w$. In other words, we may assume that $a_2 = 0$. Doing so for all j, we hence see that $w' \in \Lambda_{\mathbb{C}}$ satisfies

$$w' \cdot w = 1, \qquad w' \perp \mathrm{Span}(w', v, x_1, \ldots, x_k).$$

Consider the Lie algebra $\mathfrak{g} = \mathfrak{so}\big(v^{\perp}\big) \cong \wedge^2\big(v^{\perp}\big)$. Theorem 2.4 induces a Lie algebra action $\gamma\colon \mathfrak{g} \to \operatorname{End} H^*(M)$. By Theorem 2.4(a) $\gamma(\mathfrak{g})$ acts by derivations on $H^*(M)$ and acts trivially on $H^{4n}(M)$. (This Lie algebra action is part of the Looijenga–Lunts–Verbistky Lie algebra action, see [13, Lemma 4.13].) Take $w \wedge w' \in \mathfrak{g}$. Since the Lie algebra acts trivial on $H^{4n}(M)$ we have

$$\int_M \gamma(w \wedge w')\left(\prod_{i=1}^{\ell} B_{s_i}(w) \cdot \big(\text{monomial in } B_i(x_j) \text{ and } c_r(T_M)\big) \right) = 0.$$

On the other hand, by Corollary 2.8 we have $\gamma(w \wedge w') B_{s_i}(w) = B_{s_i}(w)$ and $\gamma(w \wedge w') B_i(x_j) = 0$, and by Theorem 2.4$(d)$ we have $\gamma(w \wedge w') c_r(T_M) = 0$. Since $\gamma(w \wedge w')$ acts by derivations, we also get

[5] Since $w' \in H^2(S)$ we always have $(w' \cdot [gx_j]_k) = 0$ for $k = 0, 4$. The vanishing in case $k = 2$ follows from $(w' \cdot gx_j) = 0$.

[6] Because the term $B_i(x_j)$ can be moved to the product $\prod_{i=1}^{\ell} B_{s_i}(w)$ in (2.5).

$$\int_M \gamma(w \wedge w')\Big(\prod_{i=1}^{\ell} B_{s_i}(w)\cdot\big(\text{monomial in } B_i(x_j) \text{ and } c_r(T_M)\big)\Big)$$
$$= \ell \cdot \int_M \prod_{i=1}^{\ell} B_{s_i}(w)\cdot\big(\text{monomial in } B_i(x_j) \text{ and } c_r(T_M)\big). \qquad \blacksquare$$

Lemma 2.11. *In the situation of Theorem 2.9, there exists $y_i \in \Lambda_{\mathbb{C}}$ which have the same intersection matrix as in (2.4), satisfy*

$$\int_M P(B_i(x_j), c_r(T_M)) = \int_M P(B_i(y_j), c_r(T_M))$$

and such that the span $L = \operatorname{Span}(v, y_1, \dots, y_k) \subset \Lambda_{\mathbb{C}}$ is non-degenerate (i.e., the restriction of the inner product of $\Lambda_{\mathbb{C}}$ onto L is non-degenerate).

Proof. Let $L = \operatorname{Span}(v, x_1, \dots, x_k)$. Assume that L is degenerate, i.e., there exists a non-zero $w \in L$ such that $w \cdot x_i = 0$ for all i and $w \cdot v = 0$. Since $v \cdot v \geq 2$, we have that v, w are linearly independent. Hence they can be extended to a basis $u_0, \dots, u_d$ of L with $u_0 = w$ and $u_1 = v$. For every i let $\lambda_i \in \mathbb{C}$ be the unique scalar such that

$$x_i - \lambda_i w \in \operatorname{Span}(u_1, \dots, u_d).$$

We hence obtain

$$\int_M P(B_i(x_j), c_r(T_M)) = \int_M P(B_i(x_j - \lambda_j w) + B_i(w), c_r(T_M))$$
$$\overset{\text{Proposition 2.10}}{=} \int_M P(B_i(x_j - \lambda_j w), c_r(T_M)).$$

Set $y_j = x_j - \lambda_j w$. If $\operatorname{Span}(v, y_1, \dots, y_k)$ is non-degenerate, we are done, otherwise repeat the above process. This process has to stop, since the dimension of the span drops by one in each step. $\blacksquare$

We also require two basic linear algebra lemmata:

Lemma 2.12. *Let V be a finite-dimensional $\mathbb{C}$-vectorspace with a $\mathbb{C}$-linear inner product. Let $v_1, \dots, v_k \in V$ be a list of vectors with Gram matrix*

$$g = \big(g_{ij}\big)_{i,j=1}^{k}, \qquad g_{ij} = \langle v_i, v_j \rangle.$$

Then $\operatorname{rank}(g) \leq \dim(\operatorname{Span}(v_1, \dots, v_k))$. If moreover $\operatorname{Span}(v_1, \dots, v_k)$ is a non-degenerate subvectorspace of V, then $\operatorname{rank}(g) = \dim(\operatorname{Span}(v_1, \dots, v_k))$.

Proof. Let $w_1, \dots, w_\ell \in V$ be a list of vectors such that $h_{ij} = \langle w_i, w_j \rangle$ is invertible. Pairing any linear relation between the w_i's with w_j for $j = 1, \dots, \ell$, and multiplying this system of equations by the inverse of h shows that the $w_1, \dots, w_\ell$ are linearly independent. This proves the first claim. For the second claim, we can choose a subset $\{w_1, \dots, w_d\} \subset \{v_1, \dots, v_k\}$ which forms a basis of $L = \operatorname{Span}(v_1, \dots, v_k)$ and observe that the matrix of the isomorphism $L \to L^\vee$ induced by the inner product with respect to the basis $\{w_i\}$ and the dual basis $\{w_i^*\}$ is the Gram matrix of the w_i. This shows that $\operatorname{rank}(g) \geq \dim L$. $\blacksquare$

Lemma 2.13. *Let V be a finite-dimensional $\mathbb{C}$-vectorspace with a $\mathbb{C}$-linear inner product. Let $v_1, \dots, v_k \in V$ and $w_1, \dots, w_k \in V$ be lists of vectors such that* (*i*) $L = \mathrm{Span}(v_1, \ldots, v_k)$ *is non-degenerate,*

(*ii*) $M = \mathrm{Span}(w_1, \ldots, w_k)$ *is non-degenerate,*

(*iii*) $\langle v_i, v_j \rangle = \langle w_i, w_j \rangle$ *for all* i, j.

Then there exists an isometry $\varphi\colon V \to V$ *such that* $\varphi(v_i) = w_i$ *for all* i.

Proof. By Lemma 2.12 and assumptions (i) and (ii) we know that

$$\dim L = \mathrm{rank}(\langle v_i, v_j\rangle)_{i,j=1}^k = \mathrm{rank}(\langle w_i, w_j\rangle)_{i,j=1}^k = \dim M.$$

Choose a basis of L from the $v_1, \ldots, v_k$, which we can assume is of the form $v_1, \ldots, v_d$, where $d = \dim(L)$. By assumption (i) and Lemma 2.12 the gram matrix $G := (\langle v_i, v_j\rangle)_{i,j=1}^d$ is invertible. But G is also the Gram matrix of $w_1, \ldots, w_d$ by assumption (iii), so the same lemma implies that $w_1, \ldots, w_d$ is linearly independent and hence a basis of M. Define an isometry

$$\varphi\colon\ V \to V$$

by setting $\varphi(v_i) = w_i$ for $i = 1, \ldots, d$, and by letting $\varphi_{L^\perp}\colon L^\perp \to M^\perp$ be an arbitrary isometry. It remains to show that $\varphi(v_i) = w_i$ for $i = d+1, \ldots, k$. For this observe that for any $v \in L$ we have

$$v = \sum_{a=1}^{d} \langle v, v_a\rangle \left(G^{-1}\right)_{ab} v_b$$

and similarly for any $w \in M$. The claim hence follows by writing every v_i in this form, applying φ and using assumption (iii). ∎

We are ready to prove Theorem 2.9.

Proof. Let $(M(v), x_i)$ and $(M(v'), x_i')$ be two pairs with the same intersection matrix A. By Lemma 2.11, we may assume that $v, x_1, \ldots, x_k$ and $v', x_1', \ldots, x_k'$ span a non-degenerate subspace of $\Lambda_{\mathbb{C}}$. Hence, by Lemma 2.13, there exists an isometry

$$g\colon\ H^*(S, \mathbb{C}) \to H^*(S', \mathbb{C})$$

which takes $(v, x_1, \ldots, x_k)$ to $(v', x_1', \ldots, x_k')$. We find that

$$\begin{aligned}\int_{M(v)} P(B_i(x_j), c_r(T_{M(v)})) &\overset{\text{(Theorem 2.4)}}{=} \int_{M(v')} \gamma(g) P(B_i(x_j), c_r(T_{M(v')})) \\ &\overset{\text{(Corollary 2.8)}}{=} \int_{M(v')} P(B_i(g x_j), c_r(T_{M(v')})) \\ &= \int_{M(v')} P(B_i(x_j'), c_r(T_{M(v')})).\end{aligned}$$ ∎

2.4 Case of dimension 2

We discuss how to evaluate integrals

$$\int_M \tau_{k_1}(\gamma_1) \cdots \tau_{k_\ell}(\gamma_\ell) P(c_r(T_M)), \tag{2.6}$$

whenever $M = M_H(v)$ is a 2-dimensional moduli space of stable sheaves, and hence a $K3$ surface. The universal family[7] $\mathcal{F}$ in this case induces a derived auto-equivalence

$$\Phi\colon\ D^b(S) \to D^b(M), \qquad \mathcal{E} \mapsto \pi_{M*}(\pi_S^*(\mathcal{E}) \otimes \mathcal{F}).$$

[7] If only a twisted universal family exists, then we have an equivalence to the derived category of twisted sheaves on M with the corresponding twist, see [6].

The induced action on cohomology

$$\Phi_*\colon\ H^*(S,\mathbb{Z}) \to H^*(M,\mathbb{Z}), \qquad \gamma \mapsto \pi_{M*}(v(\mathcal{F})\cdot \pi_S^*(\gamma))$$

defines an isometry of Mukai lattices (in fact, a Hodge isometry), see [6, Chapter 16] for references for these well-known facts.

We specialize to the case where $\mathsf{rk}(v) > 0$, which is the only one we consider in the applications. Consider the normalized action

$$\widetilde{\Phi}\colon\ H^*(S,\mathbb{Q}) \to H^*(M,\mathbb{Q}), \qquad \gamma \mapsto \pi_{M*}\big(\mathrm{e}^{-c_1(\mathcal{F})/\,\mathsf{rk}(v)}v(\mathcal{F})\cdot \pi_S^*(\gamma)\big).$$

Let us write $\mathrm{ch}(\mathcal{F}) = \mathsf{rk}(v) + \pi_M^*(\ell) + \pi_S^*(c_1(v)) + (\ldots)$, where $\ldots$ stands for terms of degree ≥ 4. Then we have $\widetilde{\Phi} = \mathrm{e}^{-\ell/\,\mathsf{rk}(v)}\Phi_*(\mathrm{e}^{-c_1(v)/\,\mathsf{rk}(v)} \cup (-))$ which shows that $\widetilde{\Phi}$ is still a Hodge isometry. Using the fact that $\widetilde{\Phi}$ is a Hodge isometry implies[8]

$$\widetilde{\Phi}(\mathsf{p}) = \mathsf{rk}(v), \qquad \widetilde{\Phi}(L) = \varphi(L), \qquad \widetilde{\Phi}(1) = \frac{1}{\mathsf{rk}(v)}\mathsf{p}, \tag{2.7}$$

where $\varphi\colon H^2(S,\mathbb{Q}) \to H^2(S,\mathbb{Q})$ is a Hodge isometry.

2.5 Proof of Theorem 1.1

If $\dim M > 2$, the claim follows by Theorem 2.9 since (a) any descendent $\tau_k(\gamma)$ defined as in (1.1) can been written as a polynomial in classes $B_j(x)$, and (b) for any list of vectors $v, x_1, \ldots, x_k \in \Lambda_{\mathbb{C}}$ after an isometry of $\Lambda_{\mathbb{C}}$ we may assume that v is the Mukai vector which defines the Hilbert scheme of n points on a $K3$ surface.[9]

If $\dim M = 2$ and $\mathsf{rk}(v) > 0$, as discussed in Section 2.4 any descendent $\tau_{k_i}(\gamma)$ can be written in terms of polynomials in classes $\widetilde{\Phi}(\alpha)$, where α is effectively determined by γ. Since any integral (2.6) can involve at most two classes of positive degree, this integral can be written as linear combination of the Mukai pairing between classes $\widetilde{\Phi}(\alpha)$ and $\widetilde{\Phi}(\alpha')$ for various α, α'. Since $\widetilde{\Phi}$ is a Hodge isometry, these are just the Mukai pairings between α and α'. This effectively determines the integrals (1.2). We also refer to Section 3.1 for a concrete implementation of this algorithm.

The case where $\dim(M) = 2$ and $\mathsf{rk}(v) = 0$ is similar to the $\dim M = 2$, $\mathsf{rk}(v) > 0$ case, and left to the reader. ∎

3 The Göttsche–Kool conjecture

Let S be a $K3$ surface and let M be a proper fine $2n$-dimensional moduli space of stable sheaves on S of Mukai vector v. Let $\mathcal{F}$ be a universal family. We assume that $\mathsf{rk}(v) > 0$. Our goal is to show that for any $\alpha \in K(S)$, class $L \in H^2(S)$ and $u \in \mathbb{C}$ we have

$$\int_M c(\alpha_M)\mathrm{e}^{\mu(L)+u\mu(\mathsf{p})} = \int_{S^{[n]}} c(\beta_{S^{[n]}})\mathrm{e}^{\mu(L)+u\,\mathsf{rk}(v)\mu(\mathsf{p})},$$

where $\beta \in K(S)$ is as specified in Theorem 1.2.

In Section 3.1, we first tackle the case $\dim M = 2$ separately, and then afterwards prove the $\dim M > 2$ case.

[8] By direct computation, the degree zero component of $\widetilde{\Phi}(\gamma)$ is $\mathsf{rk}(v)\int_S \gamma$. Then observe the degree 1 term of $\widetilde{\Phi}(\mathsf{p})$ vanishes by construction of $\widetilde{\Phi}$. Hence, $\big(\widetilde{\Phi}(\mathsf{p}), \widetilde{\Phi}(\mathsf{p})\big) = 0$ shows the first line. The others follow similarly.

[9] By Eichler's criterion [5, Lemma 7.5], this isometry can be defined over the integers.

3.1 Proof of Theorem 1.2 in case $\dim(M) = 2$

Observe that $S^{[1]} \cong S$, and for $\beta \in K(S)$ and $L \in H^2(S)$ we have

$$\beta_{S^{[1]}} = \mathrm{ch}(\beta) - \chi(\beta), \qquad \mu_{S^{[1]}}(L) = L \in H^2(S), \qquad \mu_{S^{[1]}}(\mathsf{p}) = \mathsf{p}.$$

Hence we need to prove

$$\int_M c(\alpha_M) \mathrm{e}^{\mu(L) + u\mu(\mathsf{p})} = \int_S c(\beta) \mathrm{e}^{L + u\,\mathsf{rk}(v)\mathsf{p}}. \tag{3.1}$$

Recall from Section 2.4 the Hodge isometry $\widetilde{\Phi} \colon H^*(S,\mathbb{Q}) \to H^*(M,\mathbb{Q})$ defined by the universal family $\mathcal{F}$. By comparing the definition of α_M and $\mu(\sigma)$ with the correspondence defining $\widetilde{\Phi}$ we find

$$\begin{aligned} \alpha_M &= -\frac{1}{\sqrt{\mathrm{td}_M}} \widetilde{\Phi}(v(\alpha)), \\ \mu_M(\sigma) &= \left[-\frac{1}{\sqrt{\mathrm{td}_M}} \widetilde{\Phi}\big(\sigma / \sqrt{\mathrm{td}_S}\big) \right]_{\deg(\sigma)}. \end{aligned}$$

In particular, by (2.7) we have $\mu_M(L) = -\widetilde{\Phi}(L)$. Using (2.7) we obtain

$$\begin{aligned} \int_M \mu_M(\mathsf{p}) &= \int_M -(1-\mathsf{p})\, \mathsf{rk}(v) \cdot 1 = \mathsf{rk}(v), \\ \int_M \mu_M(L)^2 &= \int_M \big(-\widetilde{\Phi}(L)\big)^2 = \big(\widetilde{\Phi}(L), \widetilde{\Phi}(L)\big) = (L, L) = \int_S L^2, \\ \int_M c_1(\alpha_M) \mu_M(L) &= \int_M \alpha_M \cup \big(-\widetilde{\Phi}(L)\big) = \int_M \widetilde{\Phi}(v(\alpha)) \cdot \widetilde{\Phi}(L) = \big(\widetilde{\Phi}(v(\alpha)), \widetilde{\Phi}(L)\big) \\ &= (v(\alpha), L) = \int_S c_1(\alpha) \cdot L. \end{aligned}$$

Using (2.7) again we moreover have

$$\begin{aligned} \alpha_M &= -(1-\mathsf{p}) \widetilde{\Phi}(\mathsf{rk}(v) + c_1(\alpha) + v_2(\alpha)) \\ &= -\,\mathsf{rk}(v) \int_S v_2(\alpha) - \varphi(c_1(\alpha)) + \left(-\frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)} + \mathsf{rk}(v) \int_S v_2(\alpha) \right) \mathsf{p}, \end{aligned}$$

and hence (with $\alpha_{M,k}$ be the degree $2k$ component of α_M) we get

$$\int_M c_2(\alpha_M) = \int_M -\alpha_{M,2} + \frac{\alpha_{M,1}^2}{2} = \frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)} - \mathsf{rk}(v) \int_S v_2(\alpha) + \frac{c_1(\alpha)^2}{2}.$$

By inspection one sees now that if β satisfies (1.3), then equation (3.1) holds. This completes the proof. ∎

3.2 Comparing normalizations

From now on assume that

$$\dim M > 2.$$

Let $\alpha \in K(S)$ and consider the definition of α_M using the Grothendieck–Riemann–Roch formula:

$$\alpha_M = -\pi_{M*}\left(v(\alpha) \mathrm{ch}(\mathcal{F}) \sqrt{\mathrm{td}_S} \exp\left(-\frac{c_1(\mathcal{F})}{\mathsf{rk}(v)} \right) \right).$$

The class α_M is easily expressed in terms of Markman's normalization:

Lemma 3.1. *We have*

$$\alpha_M = -B\left(v(\alpha^\vee)\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right)\exp\left(B_1\left(\frac{-\mathsf{p}}{\mathsf{rk}(v)} - \frac{v}{v\cdot v}\right)\right).$$

Proof. Using that $\mathrm{Pic}(M\times S) = \mathrm{Pic}(M)\oplus\mathrm{Pic}(S)$ we can write

$$c_1(\mathcal{F}) = \pi^{M*}(\ell) + \pi_S^*(c_1(v))$$

for some $\ell \in H^2(M)$. By calculating $\theta_{\mathcal{F}}(\mathsf{p})$ one finds $\ell = \theta_{\mathcal{F}}(\mathsf{p})$. Hence

$$\begin{aligned}\alpha_M &= -\pi_{M*}\left(v(\alpha)\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S}\exp\left(-\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right)\exp(\theta_{\mathcal{F}}(\mathsf{p})/(\mathsf{p}\cdot v))\\ &= -B\left(v(\alpha^\vee)\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right)\exp\left(B_1\left(\frac{-\mathsf{p}}{\mathsf{rk}(v)} - \frac{v}{v\cdot v}\right)\right).\end{aligned}$$

■

For $\sigma \in H^*(S)$ recall also the class

$$\mu(\sigma) = -\pi_{M*}\big(\mathrm{ch}_2\big(\mathcal{F}\otimes\det(\mathcal{F})^{-1/\mathsf{rk}(v)}\big)\pi_S^*(\sigma)\big)$$

(defined by the GRR expression if only a semi-universal family exists).

Lemma 3.2. *If $\sigma\in H^*(S)$ is homogeneous, then $\mu(\sigma)$ is the component of degree* $\deg(\sigma)$ *of*

$$-\exp\left(B_1\left(\frac{p}{p\cdot v} - \frac{v}{v\cdot v}\right)\right)B\left(\sigma^\vee\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\sqrt{\mathrm{td}_S}^{-1}\right).$$

Proof. We have that $\mu(\sigma)$ is the degree $\deg(\sigma)$ component of

$$\begin{aligned}-\pi_{M*}\big(\mathrm{ch}\big(\mathcal{F}\otimes\det(\mathcal{F})^{-1/\mathsf{rk}(v)}\big)\pi_S^*(\sigma)\big) &= -\pi_{M*}\big(\mathrm{ch}(\mathcal{F})\exp(-c_1(\mathcal{F})/\mathsf{rk}(v))\pi_S^*(\sigma)\big)\\ &= -\exp\left(\frac{\theta_{\mathcal{F}}(\mathsf{p})}{\mathsf{p}\cdot v} - \frac{\theta_{\mathcal{F}}(v)}{v\cdot v}\right)\exp\left(\frac{\theta_{\mathcal{F}}(v)}{v\cdot v}\right)\pi_{M*}\big(\mathrm{ch}(\mathcal{F})\pi_S^*\big(\sigma^\vee e^{c_1(v)/\mathsf{rk}(v)}\sqrt{\mathrm{td}_S}^{-1}\big)^\vee\sqrt{\mathrm{td}_S}\big)\\ &= -\exp\left(B_1\left(\frac{p}{p\cdot v} - \frac{v}{v\cdot v}\right)\right)B\left(\sigma^\vee\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\sqrt{\mathrm{td}_S}^{-1}\right),\end{aligned}$$

where we used again $c_1(\mathcal{F}) = \pi_M^*\theta_{\mathcal{F}}(\mathsf{p}) + \pi_S^* c_1(V)$. ■

In particular, for $L\in H^2(S)$ we have that

$$\mu(L) = B_1\left(L\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right) - B_1\left(\frac{p}{p\cdot v} - \frac{v}{v\cdot v}\right)$$

and that $\mu(\mathsf{p})$ is a polynomial in $B_1\big(\frac{p}{p\cdot v} - \frac{v}{v\cdot v}\big)$ and $B_i(\mathsf{p})$.

3.3 Dependence

By Theorem 2.9 we conclude that any integral

$$\int_M P(\alpha_{M,k}, \mu(L), \mu(u\mathsf{p})) \tag{3.2}$$

(such as the Segre number) only depends upon P and the intersection pairings in the Mukai lattice of the classes

$$v,\quad \mathsf{p}/\mathsf{rk}(v),\quad v(\alpha)^\vee\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right),\quad L\exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right),\quad u\mathsf{p}. \tag{3.3}$$

Explicitly, the interesting pairings for the first three classes are

(i) $v \cdot v(\alpha)^\vee \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right) = -v_2(\alpha)\cdot \mathsf{rk}(v) + \frac{1}{2}\frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)}(v \cdot v)$,

(ii) $\mathsf{p}/\mathsf{rk}(v) \cdot v(\alpha)^\vee \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right) = -\frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)}$,

(iii) $\left(v(\alpha)^\vee \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right)^2 = v(\alpha)\cdot v(\alpha)$.

The interesting intersections involving L are

(iv) $v \cdot L \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right) = L \cdot c_1(v) - L\cdot c_1(v) = 0$,

$$v(\alpha)^\vee \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right) \cdot L \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right) = v(\alpha)^\vee \cdot L = -c_1(\alpha)\cdot L,$$

$$\left(L \exp\left(\frac{c_1(v)}{\mathsf{rk}(v)}\right)\right)^2 = L^2.$$

The pairings with $u\mathsf{p}$ are $u\,\mathsf{rk}(v)$ times the pairings with $\mathsf{p}/\mathsf{rk}(v)$.

3.4 Moving to the Hilbert scheme

Since (3.2) only depends on the intersection pairings of (3.3) we have that

$$\int_M P(\alpha_{M,k}, \mu(L), \mu(u\mathsf{p})) = \int_{S^{[n]}} P(\beta_{S^{[n]},k}, \mu(L), \mu(u'\mathsf{p}))$$

for any K-theory class $\beta \in K(S)$ and $u' \in \mathbb{C}$ such that the list

$$1-(n-1)\mathsf{p}, \qquad \mathsf{p}, \qquad v(\beta)^\vee, \qquad L, \qquad u'\mathsf{p} \tag{3.4}$$

has the same intersection numbers as the list (3.3). (The list (3.4) is obtained from (3.3) by specializing to $v = 1-(n-1)\mathsf{p}$, the Mukai vector of $S^{[n]}$.)

The interesting parts of the intersections of (3.4) are

(i) $v \cdot v(\beta)^\vee = -v_2(\beta) + \frac{1}{2}\mathsf{rk}(\beta)(2n-2)$,

(ii) $\mathsf{p}\cdot v(\beta)^\vee = -\mathsf{rk}(\beta)$,

(iii) $v(\beta)^\vee \cdot v(\beta)^\vee = v(\beta)\cdot v(\beta)$,

(iv) $v(\beta)^\vee \cdot L = -c_1(\beta)\cdot L$.

Equating (i)–(iv) for M and $S^{[n]}$ we hence get the system

$$-v_2(\alpha)\cdot \mathsf{rk}(v) + \frac{1}{2}\frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)}(v\cdot v) = -v_2(\beta) + \frac{1}{2}\mathsf{rk}(\beta)(2n-2),$$

$$-\frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)} = -\mathsf{rk}(\beta), \qquad v(\alpha)\cdot v(\alpha) = v(\beta)\cdot v(\beta), \qquad -c_1(\alpha)\cdot L = -c_1(\beta)\cdot L.$$

Since $v(\alpha)^2 = c_1(\alpha)^2 - 2\,\mathsf{rk}(\alpha)v_2(\alpha)$, this is equivalent to the system:

$$\mathsf{rk}(\beta) = \frac{\mathsf{rk}(\alpha)}{\mathsf{rk}(v)}, \qquad v_2(\beta) = \mathsf{rk}(v)v_2(\alpha), \qquad c_1(\alpha)^2 = c_1(\beta)^2, \qquad c_1(\alpha)\cdot L = c_1(\beta)\cdot L. \tag{3.5}$$

Moreover, we must have

$$-u' = u'\mathsf{p}\cdot(1-(n-1)\mathsf{p}) = u\mathsf{p}\cdot v = -\mathsf{rk}(v)u.$$

We have proven the following (which immediately implies Theorem 1.2):

Theorem 3.3. *For any polynomial P, we have*

$$\int_M P(\alpha_{M,k}, \mu(L), \mu(u\mathsf{p})) = \int_{S^{[n]}} P(\beta_{S^{[n]},k}, \mu(L), \mu(u\,\mathsf{rk}(v)\mathsf{p}))$$

for any K-theory class $\beta \in K(S)$ such that (3.5) is satisfied.

Acknowledgements

I thank Martijn Kool for bringing Conjecture 5.1 of [3] to my attention and Lothar Göttsche for useful comments. I am also indebted to the referees for careful reading of the paper and pointing out several issues in an earlier version. The author is partially funded by the Deutsche Forschungsgemeinschaft (DFG) – OB 512/1-1.

References

[1] Ellingsrud G., Göttsche L., Lehn M., On the cobordism class of the Hilbert scheme of a surface, *J. Algebraic Geom.* **10** (2001), 81–100, arXiv:math.AG/9904095.

[2] Frei S., Moduli spaces of sheaves on $K3$ surfaces and Galois representations, *Selecta Math. (N.S.)* **26** (2020), paper no. 6, 16 pages, arXiv:1810.06735.

[3] Göttsche L., Kool M., Virtual Segre and Verlinde numbers of projective surfaces, arXiv:2007.11631.

[4] Göttsche L., Nakajima H., Yoshioka K., K-theoretic Donaldson invariants via instanton counting, *Pure Appl. Math. Q.* **5** (2009), 1029–1111, arXiv:math.AG/0611945.

[5] Gritsenko V., Hulek K., Sankaran G.K., Moduli of $K3$ surfaces and irreducible symplectic manifolds, in Handbook of Moduli, Vol. I, *Adv. Lect. Math. (ALM)*, Vol. 24, Int. Press, Somerville, MA, 2013, 459–526, arXiv:1012.4155.

[6] Huybrechts D., Lectures on $K3$ surfaces, *Cambridge Stud. Adv. Math.*, Vol. 158, Cambridge University Press, Cambridge, 2016.

[7] Huybrechts D., Lehn M., The geometry of moduli spaces of sheaves, 2nd ed., *Cambridge Math. Lib.*, Cambridge University Press, Cambridge, 2010.

[8] Lehn M., Chern classes of tautological sheaves on Hilbert schemes of points on surfaces, *Invent. Math.* **136** (1999), 157–207, arXiv:math.AG/9803091.

[9] Marian A., Oprea D., Pandharipande R., Segre classes and Hilbert schemes of points, *Ann. Sci. Éc. Norm. Supér.* **50** (2017), 239–267, arXiv:1507.00688.

[10] Marian A., Oprea D., Pandharipande R., The combinatorics of Lehn's conjecture, *J. Math. Soc. Japan* **71** (2019), 299–308, arXiv:1708.08129.

[11] Marian A., Oprea D., Pandharipande R., Higher rank Segre integrals over the Hilbert scheme of points, *J. Eur. Math. Soc. (JEMS)* **24** (2022), 2979–3015, arXiv:1712.02382.

[12] Markman E., Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, *J. Reine Angew. Math.* **544** (2002), 61–82, arXiv:math.AG/0009109.

[13] Markman E., On the monodromy of moduli spaces of sheaves on $K3$ surfaces, *J. Algebraic Geom.* **17** (2008), 29–99, arXiv:math.AG/0305042.

[14] Voisin C., Segre classes of tautological bundles on Hilbert schemes of surfaces, *Algebr. Geom.* **6** (2019), 186–195, arXiv:1708.06325.