The Lattice Sine-Gordon Equation as a Superposition Formula for an NLS-Type System

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Abstract. We treat the lattice sine-Gordon equation and two of its generalised symmetries as a compatible system. Elimination of shifts from the two symmetries of the lattice sine-Gordon equation yields an integrable NLS-type system. An auto-Bäcklund transformation and a superposition formula for the NLS-type system is obtained by elimination of shifts from the lattice sine-Gordon equation and its down-shifted version. We use the obtained formulae to calculate a superposition of two and three elementary solutions.

Key words: quad-equation; NLS-type system; auto-Bäcklund transformation

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1 Introduction

Integrable differential-difference equations (DΔEs) with one continuous and one discrete variable are known to be closely related with integrable partial differential equations (PDEs). In particular, many integrable DΔEs may be interpreted as Bäcklund transformations of some PDEs [11]. For example, the integrable Volterra-type equations

$$\frac{\partial}{\partial x}u_l = f(u_{l-1}, u_l, u_{l+1}) \tag{1.1}$$

are known to be related to the NLS-type PDEs [14] by means of elimination of shifts. More precisely, if one considers (1.1) and its simplest generalised symmetry

$$\frac{\partial}{\partial t}u_l = g(u_{l-2}, u_{l-1}, u_l, u_{l+1}, u_{l+2}), \tag{1.2}$$

then elimination of u_{l-2}, u_{l+1}, u_{l+2} from the system of (1.1) and (1.2) yields a two component system of the NLS-type on the quantities u_{l-1} and u_l. A by-product of this calculation is an invertible auto-transformation of the resulting system of PDEs generated by (1.1).

As far as construction of exact solutions is concerned, a more important class of transformations is auto-Bäcklund transformations with a spectral parameter which are usually non-invertible. A direct calculation of such transformation is a tedious task. The knowledge of other structures associated with integrability, e.g., a Lax pair or Painlevé structure, may significantly speed up the search for such a transformation [16].

In [5] we showed that a system obtained by elimination of shifts from (1.1) and (1.2), where at the same time equations (1.1) and (1.2) represent symmetries of a quad-equation, automatically possesses an auto-Bäcklund transformation with spectral parameter. The method was illustrated with using the lattice H_1 equation [2] (lattice potential KdV equation) and a pair of its simplest symmetries.

In this paper we use the method of [5] to find an auto-Bäcklund transformation and the superposition formula for solutions of the system

$$\begin{aligned} p_t &= -p_{xx} + q_x p_x^2 - q_x, \\ q_t &= q_{xx} + p_x q_x^2 - p_x \end{aligned} \tag{1.3}$$

using its connection with the lattice sine-Gordon equation and the Volterra-type equations.

System (1.3) is reminiscent of the well known derivatives NLS system, yet it belongs to a different cluster of NLS-type systems labelled as (d) in [12]. All systems in the cluster are related by transformations that preserve local conservation laws. Therefore, having a procedure for generating solutions for (1.3) allows one to generate solutions for other systems in the cluster as well.

1.1 Symmetries of the lattice sine-Gordon equation

We consider the lattice sine-Gordon (lsG) equation

$$\alpha(u_{l,m}u_{l+1,m+1} - u_{l+1,m}u_{l,m+1}) + \beta u_{l,m}u_{l+1,m}u_{l,m+1}u_{l+1,m+1} = 1, \tag{1.4}$$

where α and β are arbitrary parameters. It was first derived by Hirota in a slightly different form by the method of dependent variable transformation [9]. The unknown function u is assumed to depend on the three continuous variables t, x, and y as well as two discrete variables l and m. The dependence on the continuous variables is determined through the *generalised symmetries* of equation (1.4) while discrete variable l labels the components of the symmetries, and m counts iterations of the auto-Bäcklund transformation.

Equation (1.4) degenerates into the lattice Liouville equation [10, 15]

$$\alpha(u_{l,m}u_{l+1,m+1} - u_{l+1,m}u_{l,m+1}) = 1 \tag{1.5}$$

when $\beta = 0$. This connection turns out to be useful for the construction of symmetries of (1.4). First, we point out that equation (1.5) is *Darboux integrable*. That is, it possesses integrals with respect to both discrete variables. The integrals

$$w_{l,m} = \frac{u_{l,m}}{u_{l+1,m} + u_{l-1,m}}, \qquad \bar{w}_{l,m} = \frac{u_{l,m}}{u_{l,m+1} + u_{l,m-1}}, \tag{1.6}$$

satisfy the relations

$$(S_m - 1)w_{l,m} = 0, \qquad (S_l - 1)\bar{w}_{l,m} = 0,$$

on solutions of (1.5). Here S_l and S_m stand for the respective shift operators, e.g., $S_l(w_{l,m}) = w_{l+1,m}$. These integrals can be easily derived from the determinantal structure of (1.5) (see, e.g., [6]).

Then, the construction of the higher symmetries of (1.5) is rather algorithmic due to a method presented in [3] which allows one to construct operators that map the integrals of a Darboux integrable equation to its symmetries. Since the lattice Liouville equation is Darboux integrable, the structure of its symmetries is determined by such operators, i.e., the symmetries can be cast into either of the forms

$$\partial_x u_{l,m} = MF(w), \qquad \partial_x u_{l,m} = \bar{M}F(\bar{w}), \tag{1.7}$$

where w and $\bar{w}$ are the integrals of (1.5) or constants. Skipping technical details, we present the operators explicitly:

$$M = w_{l,m}(u_{l-1,m}S_l - u_{l+1,m}), \qquad \bar{M} = \bar{w}_{l,m}(u_{l,m-1}S_m - u_{l,m+1}).$$

Since equation (1.5) is invariant with respect to the involution $u_{l\pm i,m} \to u_{l\mp i,m}$, there exists another set of operators

$$\mathcal{M} = w_{l,m}\big(u_{l+1,m}S_l^{-1} - u_{l-1,m}\big),$$
$$\bar{\mathcal{M}} = \bar{w}_{l,m}\big(u_{l,m+1}S_m^{-1} - u_{l,m-1}\big)$$

with the same property, that is $\mathcal{M}(F(w))$ and $\bar{\mathcal{M}}(F(\bar{w}))$ are the symmetries of (1.5). Obviously all the statements regarding the integrals and symmetries of (1.4) and (1.5) involve two identical parts: one involving variables $u_{l+i,m}$ and the other variables $u_{l,m+i}$. Therefore, for convenience, the second part will be omitted whenever it is not essential.

Equations (1.7) are too general to be all integrable. Nevertheless, for some choices of function F this is the case, in particular, when F is chosen such that (1.7) is also a symmetry of (1.4). The simplest choice $F = 1$ delivers the equation

$$\partial_x u_{l,m} = \frac{u_{l,m}(u_{l-1,m} - u_{l+1,m})}{u_{l-1,m} + u_{l+1,m}}, \tag{1.8}$$

which is the simplest generalised symmetry of (1.4). That is, on solutions of (1.4) and (1.8) the relation

$$\partial_x\left(\alpha(u_{l,m}u_{l+1,m+1} - u_{l+1,m}u_{l,m+1}) + \beta u_{l,m}u_{l+1,m}u_{l,m+1}u_{l+1,m+1}\right) = 0$$

is satisfied identically.

Further, the fact that (1.8) is a symmetry of (1.5) implies that on solutions of these equations the derivative ∂_x commutes with the shift S_l, which in turn, implies that $\partial_x w_{l,m}$ can be expressed in terms of $w_{l,m}$ itself and its shifts. Therefore the integral $w_{l,m}$ provides us with a Miura-type transformation from equation (1.8) into the modified Volterra equation

$$\partial_x w_{l,m} = 2w_{l,m}^2(w_{l-1,m} - w_{l+1,m}). \tag{1.9}$$

The complete classification of the Volterra-type equations can be found in [17]. As it is pointed out in [4], the Volterra-type equations possess *local* master symmetries. For example, a master symmetry of (1.9) is the equation

$$\partial_y w_{l,m} = l\partial_x w_{l,m} + w_{l,m}^2(w_{l-1,m} + w_{l+1,m}). \tag{1.10}$$

The commutator $(\partial_y\partial_x - \partial_x\partial_y)w_{l,m}$, calculated on solutions of (1.9) and (1.10) gives the simplest generalised symmetry of (1.9). The locality of master symmetries significantly simplifies their construction. Moreover, it has been observed that for the Volterra-type equations the master symmetries bear resemblance of the equation itself. The present case is no exception: a master symmetry of (1.8) is given by $\partial_y u_{l,m} = l\partial_x u_{l,m}$. Hence the simplest generalised symmetry of (1.8) has the form

$$\partial_t u_{l,m} = (\partial_y\partial_x - \partial_x\partial_y)u_{l,m},$$

which is explicitly given by the equation

$$\partial_t u_{l,m} = \frac{4u_{l-1,m}u_{l,m}^2u_{l+1,m}(u_{l+2,m} - u_{l-2,m})}{(u_{l-2,m} + u_{l,m})(u_{l-1,m} + u_{l+1,m})^2(u_{l+2,m} + u_{l,m})}. \tag{1.11}$$

Both equations (1.8) and (1.11) are the symmetries of (1.4). So, the remainder of the paper focusses on a system of equations comprising of (1.8), (1.11), and (1.4). We now assume that $\beta \neq 0$ hence by rescaling of the dependent variable we can set $\beta = 1$:

$$\alpha(u_{l,m}u_{l+1,m+1} - u_{l+1,m}u_{l,m+1}) + u_{l,m}u_{l+1,m}u_{l,m+1}u_{l+1,m+1} = 1. \tag{1.12}$$

Remark 1.1. In the preceding calculations, we derived the symmetries (1.8) and (1.11) in a way that highlights the connection with the lattice Liouville equation and its integrals. One could employ different approaches, e.g., calculate them directly or extract from previously published articles [7, 8]. Also, it is worth mentioning that the integrals (1.6) are related to the conserved densities of equations (1.8) and (1.11), i.e., $\partial_x \ln w_{l,m}, \partial_t \ln w_{l,m} \in \operatorname{Im}(S_l - 1)$, which is in parallel with the continuous case. The connection between conserved densities and discrete Miura-type transformations has been noted before, see, e.g., [1].

1.2 Connection with NLS-type systems

As we already mentioned, the compatibility of a Volterra-type equation with its generalised symmetry yields an NLS-type system through elimination of shifts [14]. In the present case, if we express the variables $u_{l+1,m}$, $u_{l+2,m}$, and $u_{l-2,m}$ from equation (1.8) to obtain

$$\begin{aligned} u_{l+1,m} &= \frac{u_{l-1,m}(u_{l,m} - \partial_x u_{l,m})}{u_{l,m} + \partial_x u_{l,m}}, \qquad u_{l+2,m} = \frac{u_{l,m}(u_{l+1,m} - \partial_x u_{l+1,m})}{u_{l+1,m} + \partial_x u_{l+1,m}}, \\ u_{l-2,m} &= \frac{u_{lm}(u_{l-1,m} + \partial_x u_{l-1,m})}{u_{l-1,m} - \partial_x u_{l-1,m}} \end{aligned} \tag{1.13}$$

and substitute them into (1.11), we obtain a system of two equations on $u_{l-1,m}$ and $u_{l,m}$. Note that substitution of the expression for $u_{l+1,m}$ into the one for $u_{l+2,m}$ produces quite a cumbersome formula. Nevertheless, the resulting system turns out to be quite compact:

$$\begin{aligned} \partial_t u_{l-1,m} &= -\partial_x^2 u_{l-1,m} + \frac{(\partial_x u_{l-1,m})^2}{u_{l-1,m}} \left(\frac{\partial_x u_{l,m}}{u_{l,m}} + 1 \right) - \frac{u_{l-1,m} \partial_x u_{l,m}}{u_{l,m}}, \\ \partial_t u_{l,m} &= \partial_x^2 u_{l,m} + \frac{(\partial_x u_{l,m})^2}{u_{l,m}} \left(\frac{\partial_x u_{l-1,m}}{u_{l-1,m}} - 1 \right) - \frac{u_{l,m} \partial_x u_{l-1,m}}{u_{l-1,m}}. \end{aligned} \tag{1.14}$$

Since both pairs $(u_{l-1,m}, u_{l,m})$ and $(u_{l,m}, u_{l+1,m})$ satisfy the same system, then (1.14) admits an auto-transformation given explicitly by

$$(u_{l-1,m}, u_{l,m}) \to \left(u_{l,m}, \frac{u_{l-1,m}(u_{l,m} - \partial_x u_{l,m})}{u_{l,m} + \partial_x u_{l,m}} \right).$$

System (1.14) takes a particularly simple form (1.3) in the variables

$$u_{l-1,m} = \mathrm{e}^p, \qquad u_{l,m} = \mathrm{e}^q.$$

As is the case with equations (1.8) and (1.11), the integral $w_{l,m}$ yields a Miura-type transformation between the respective NLS-type systems. More precisely, if we eliminate variables $u_{l-2,m}$ and $u_{l+1,m}$ from the expressions for $w_{l-1,m}$ and $w_{l,m}$ by means of (1.13), we obtain

$$w_{l-1,m} = \frac{1}{2} \frac{u_{l-1,m} - \partial_x u_{l-1,m}}{u_{l,m}}, \qquad w_{l,m} = \frac{1}{2} \frac{u_{l,m} + \partial_x u_{l,m}}{u_{l-1,m}}. \tag{1.15}$$

Substitution (1.15) transforms (1.14) into the combination of the derivative NLS system and its translational symmetry:

$$\begin{aligned} \partial_t w_{l-1,m} &= -\partial_x^2 w_{l-1,m} - 4\partial_x \left(w_{l-1,m}^2 w_{l,m} \right) + 2\partial_x w_{l-1,m}, \\ \partial_t w_{l,m} &= \partial_x^2 w_{l,m} - 4\partial_x \left(w_{l,m}^2 w_{l-1,m} \right) + 2\partial_x w_{l,m}. \end{aligned}$$

Therefore, the auto-Bäcklund transformation and superposition formulae obtained in the following section can also be used to build solutions of the derivative NLS, but not the other way around, as transformation (1.15) is not invertible. We will refer to system (1.14) as a modified derivative NLS (mdNLS) system.

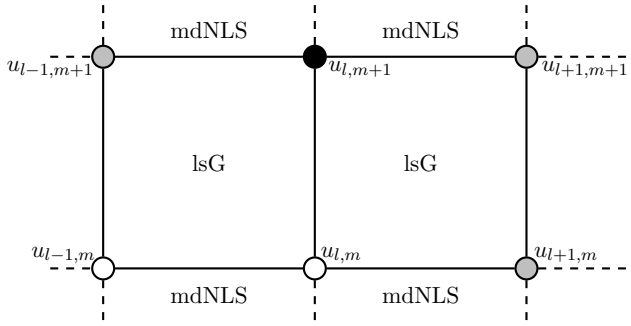
2 Auto-Bäcklund transformation and superposition formula

Here we apply the method of [5] to derive exact solutions of (1.14). The idea of using (1.12) as a superposition formula for solutions of (1.14) is based on the fact that the symmetries of (1.12) only depend on shifts with respect to one of the variables. So, equation (1.12) possesses another set of symmetries of the form (1.8) and (1.11) with the substitution $u_{l+i,m} \to u_{l+i,m+1}$ applied. Elimination of shifts in this system, again, produces a system of the form (1.14). The process of constructing a solution consists of the following steps.

Starting with a seed solution $(u_{l-1,m}, u_{l,m})$, which often is a trivial solution, we would like to calculate another solution, labelled as $(u_{l-1,m+1}, u_{l,m+1})$. Let us consider the down-shifted, with respect to l, version of equation (1.12):

$$\alpha(u_{l-1,m}u_{l,m+1} - u_{l,m}u_{l-1,m+1}) + u_{l-1,m}u_{l,m}u_{l-1,m+1}u_{l,m+1} = 1. \tag{2.1}$$

Equations (1.12), (1.14), and (2.1) can be schematically depicted as



In this diagram the white vertices represent a seed solution, the grey vertices represent the quantities that can be eliminated from the equations, and the black vertex represents the quantity that needs to be calculated. In more detail, the rightmost vertices $u_{l+1,m}$ and $u_{l+1,m+1}$ can be eliminated from (1.12) by means of $(1.13)_1$. This gives an expression containing the x-derivatives of $u_{l,m}$ and $u_{l,m+1}$. Further, expressing $u_{l-1,m+1}$ from (2.1) and substituting in (1.12) we obtain an equation on a single quantity $u_{l,m+1}$ which determines the x-dynamics of the solution. The resulting equation reads

$$\big(\alpha^2 - 1\big)\partial_x u_{l,m+1} = -A_{l,m}\partial_x u_{l,m} + \alpha B_{l,m}, \tag{2.2}$$

where

$$A_{l,m} = \frac{(u_{l-1,m}u_{l,m+1} - \alpha)(\alpha u_{l,m+1}u_{l-1,m} - 1)}{u_{l,m}u_{l-1,m}}, \qquad B_{l,m} = \frac{u_{l-1,m}^2 u_{l,m+1}^2 - 1}{u_{l-1,m}}.$$

The t-dependence of $u_{l,m+1}$ is determined by eliminating the x-derivatives in the respective mdNLS system by means of (2.2). This yields the following equation

$$\begin{aligned}\big(\alpha^2 - 1\big)\partial_t u_{l,m+1} = {}& -A_{l,m}\partial_x^2 u_{l,m} + \frac{A_{l,m}(u_{l-1,m} - \partial_x u_{l-1,m})(\partial_x u_{l,m})^2}{u_{l,m}u_{l-1,m}} \\ & + \frac{\alpha B_{l,m}\partial_x u_{l,m}\partial_x u_{l-1,m}}{u_{l,m}u_{l-1,m}} - \frac{\big(\alpha^2+1\big)u_{l,m+1}\partial_x u_{l-1,m}}{u_{l-1,m}} \\ & + \frac{\alpha\big(\alpha^2+1\big)C_{l,m}\partial_x u_{l,m}}{\big(\alpha^2-1\big)u_{l,m}} - \frac{4\alpha^2 u_{l,m+1}\partial_x u_{l,m}}{\big(\alpha^2-1\big)u_{l,m}} - \frac{\alpha\big(\alpha^2+1\big)}{\alpha^2-1}B_{l,m},\end{aligned} \tag{2.3}$$

where

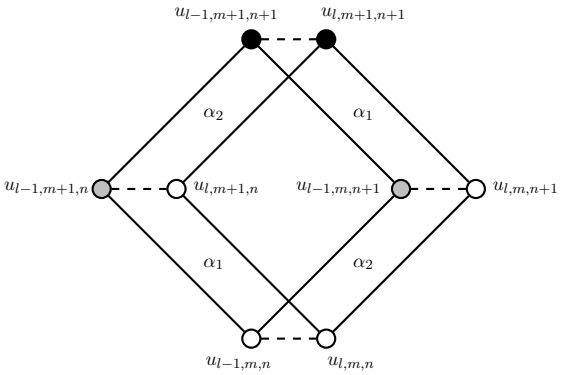
$$C_{l,m} = \frac{1 + u_{l-1,m}^2 u_{l,m+1}^2}{u_{l-1,m}}.$$

Remark 2.1. The requirement of compatibility of (2.2) and (2.3) yields the relation

$$\frac{\partial_x u_{l,m}}{u_{l,m}}\left(u_{l-1,m}A_{l,m}P_{l,m}-\alpha B_{l,m}Q_{l,m}\right)+\alpha C_{l,m}Q_{l,m}=0,$$

where $P_{l,m}$ and $Q_{l,m}$ are the differences between the left- and right-hand sides of the respective equations in system (1.14). Thus, as expected, relations (2.2) and (2.3) are compatible modulo system (1.14).

Once the solution $(u_{l-1,m+1},u_{l,m+1})$ is found, we can proceed to construct more involved solutions in a purely algebraic manner. This step is the standard step in generating the multi-soliton solutions from an auto-Bäcklund transformation and commutative Bianchi diagram. Assuming that transformations with parameters α_1 and α_2 commute, i.e., if we repeat the procedure of the previous step twice: first with parameter α_1 then α_2, and second, with α_2 then α_1 – we get the same result. This simple assumption yields a formula of superposition of solutions for system (1.14). The Bianchi diagram adapted for this calculation is given below:



Here we have slightly modified notation by introducing an additional discrete variable n which counts auto-Bäcklund transformations with respect to parameter α_2. The respective formula of superposition is obtained by solving a system of four copies of (1.12):

$$\begin{aligned}
&\alpha_1(u_{l-1,m,n}u_{l,m+1,n}-u_{l-1,m+1,n}u_{l,m,n})+u_{l-1,m+1,n}u_{l-1,m,n}u_{l,m,n}u_{l,m+1,n}=1,\\
&\alpha_2(u_{l-1,m+1,n}u_{l,m+1,n+1}-u_{l-1,m+1,n+1}u_{l,m+1,n})+u_{l-1,m+1,n+1}u_{l-1,m+1,n}u_{l,m+1,n}u_{l,m+1,n+1}=1,\\
&\alpha_2(u_{l-1,m,n}u_{l,m,n+1}-u_{l-1,m,n+1}u_{l,m,n})+u_{l-1,m,n+1}u_{l-1,m,n}u_{l,m,n}u_{l,m,n+1}=1,\\
&\alpha_1(u_{l-1,m,n+1}u_{l,m+1,n+1}-u_{l-1,m+1,n+1}u_{l,m,n+1})+u_{l-1,m+1,n+1}u_{l-1,m,n+1}u_{l,m,n+1}u_{l,m+1,n+1}=1
\end{aligned}$$

for $(u_{l-1,m+1,n+1},u_{l,m+1,n+1})$. Namely, it is given by

$$\begin{aligned}
u_{l-1,m+1,n+1}&=\frac{\alpha_2^2-\alpha_1^2-u_{l-1,m,n}\left(\alpha_2\left(1-\alpha_1^2\right)u_{l,m,n+1}-\alpha_1\left(1-\alpha_2^2\right)u_{l,m+1,n}\right)}{\left(\alpha_2^2-\alpha_1^2\right)u_{l,m,n+1}u_{l-1,m,n}u_{l,m+1,n}+\alpha_1\left(1-\alpha_2^2\right)u_{l,m,n+1}-\alpha_2\left(1-\alpha_1^2\right)u_{l,m+1,n}},\\
u_{l,m+1,n+1}&=\frac{u_{l,m,n}\left(\alpha_1u_{l,m,n+1}-\alpha_2u_{l,m+1,n}\right)}{\alpha_1u_{l,m+1,n}-\alpha_2u_{l,m,n+1}}.
\end{aligned}\tag{2.4}$$

The process of constructing solutions by means of the obtained auto-Bäcklund transformation and superposition formula is summarised as follows.

Summary 2.2. Given a seed solution $(u_{l-1,m},u_{l,m})$ of (1.14) we first solve a system of (2.2) and (2.3) to determine the component $u_{l,m+1}$. The other component $u_{l-1,m+1}$, is calculated algebraically from (2.1). Further, we set

$$\begin{aligned}
(u_{l-1,m+1,n},u_{l,m+1,n})&=(u_{l-1,m+1},u_{l,m+1})|_{\alpha=\alpha_1},\\
(u_{l-1,m,n+1},u_{l,m,n+1})&=(u_{l-1,m+1},u_{l,m+1})|_{\alpha=\alpha_2}
\end{aligned}\tag{2.5}$$

and substitute these expressions in the superposition formulae (2.4) to calculate the solution $(u_{l-1,m+1,n+1},u_{l,m+1,n+1})$.

3 Examples

If we start with a constant solution

$$u_{l-1,m} = u_{l,m} = 1,$$

then equations (2.2) and (2.3) turn into the system

$$\partial_x u_{l,m+1} = \frac{\alpha}{\alpha^2-1}\big(u_{l,m+1}^2 - 1\big),$$
$$\partial_t u_{l,m+1} = -\frac{\alpha\big(\alpha^2+1\big)}{\big(\alpha^2-1\big)^2}\big(u_{l,m+1}^2 - 1\big),$$

whose solution $u_{l,m+1}$ can be expressed in terms of the functions

$$u_{l,m+1} = f(\alpha,\beta) = \frac{f_+(\alpha,\beta)}{f_-(\alpha,\beta)}, \tag{3.1}$$

where

$$f_\pm(\alpha,\beta) = 1 \pm \beta \exp\left(\frac{2\alpha x}{\alpha^2-1} - \frac{2\alpha\big(\alpha^2+1\big)t}{\big(\alpha^2-1\big)^2}\right).$$

The other component of the solution is found from (2.1) and given by

$$u_{l-1,m+1} = \frac{1-\alpha f(\alpha,\beta)}{f(\alpha,\beta)-\alpha}. \tag{3.2}$$

Applying substitutions (2.5) to functions (3.1), (3.2), and then substituting the resulting expressions in (2.1) we obtain the solution

$$u_{l-1,m+1,n+1} = \frac{\alpha_1^2-\alpha_2^2-\alpha_1\big(1-\alpha_2^2\big)f(\alpha_1,\beta_1)+\alpha_2\big(1-\alpha_1^2\big)f(\alpha_2,\beta_2)}{\big(\alpha_1^2-\alpha_2^2\big)f(\alpha_1,\beta_1)f(\alpha_2,\beta_2)+\alpha_2\big(1-\alpha_1^2\big)f(\alpha_1,\beta_1)-\alpha_1\big(1-\alpha_2^2\big)f(\alpha_2,\beta_2)},$$
$$u_{l,m+1,n+1} = \frac{\alpha_1 f(\alpha_2,\beta_2)-\alpha_2 f(\alpha_1,\beta_1)}{\alpha_1 f(\alpha_1,\beta_1)-\alpha_2 f(\alpha_2,\beta_2)}. \tag{3.3}$$

Note that the components $u_{l-1,m}$ and $u_{l,m}$ are not conserved densities of equation (1.14). Therefore, a quantity of interest is the simplest conserved density given by the formula

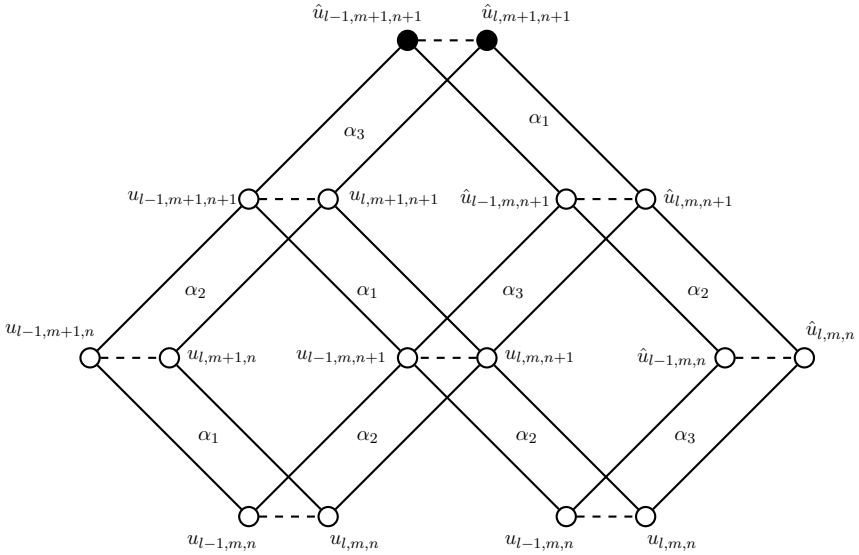
$$\rho = (\partial_x \ln u_{l-1,m+1,n+1})(\partial_x \ln u_{l,m+1,n+1}).$$

For real α_i and imaginary β_i, this quantity represents an interaction of two solitons: the profile consists of two distinct solitons with one increasing in amplitude while overtaking the other and then restoring its shape after the interaction. Solution (3.3) can be re-written in a more compact form in terms of the skew-symmetric Levi-Chivita symbol ε^{ij}:

$$u_{l-1,m+1,n+1} = \frac{-2\varepsilon^{ij}\alpha_i\big(\alpha_i+\big(\alpha_j^2-1\big)f_i\big)}{\varepsilon^{ij}\big(\alpha_j^2-\alpha_i^2\big)f_i f_j + 2\big(\alpha_i^2-1\big)\alpha_j f_i},$$
$$u_{l,m+1,n+1} = -\frac{\varepsilon^{ij}\alpha_j f_i}{\varepsilon^{ij}\alpha_i f_i},$$

where we assume summation over repeated indices and denote $f_m = f(\alpha_m,\beta_m)$.

As a concluding example we construct a three-soliton solution of equation (1.14). The respective stack of Bianchi diagrams (Bianchi lattice) (see, e.g., [13]) is given by



Here the result of the application of a Bäcklund transformation with parameter α_3 to the solution $(u_{l-1,m,n}, u_{l,m,n})$ is denoted as $(\hat{u}_{l-1,m,n}, \hat{u}_{l,m,n})$. The diagram gives a three-soliton solution in the form

$$\hat{u}_{l-1,m+1,n+1} = \frac{\varepsilon^{ijk}\alpha_i\big(\alpha_j\big(\alpha_i^2-\alpha_j^2\big)\big(\alpha_k^2-1\big)f_j-\big(\alpha_i^2-1\big)\big(\alpha_k^2-\alpha_j^2\big)\big)f_i}{\varepsilon^{ijk}\alpha_k\big(\big(\alpha_i^2-\alpha_j^2\big)\big(\alpha_k^2-1\big)f_j-\alpha_j\big(\alpha_i^2-1\big)\big(\alpha_k^2-\alpha_j^2\big)\big)f_i},$$

$$\hat{u}_{l,m+1,n+1} = \frac{\varepsilon^{ijk}\alpha_k\big(\alpha_i^2-\alpha_j^2\big)f_if_j}{\varepsilon^{ijk}\alpha_i\big(\alpha_j^2-\alpha_k^2\big)f_i},$$

where $i,j,k=1,\dots,3$.

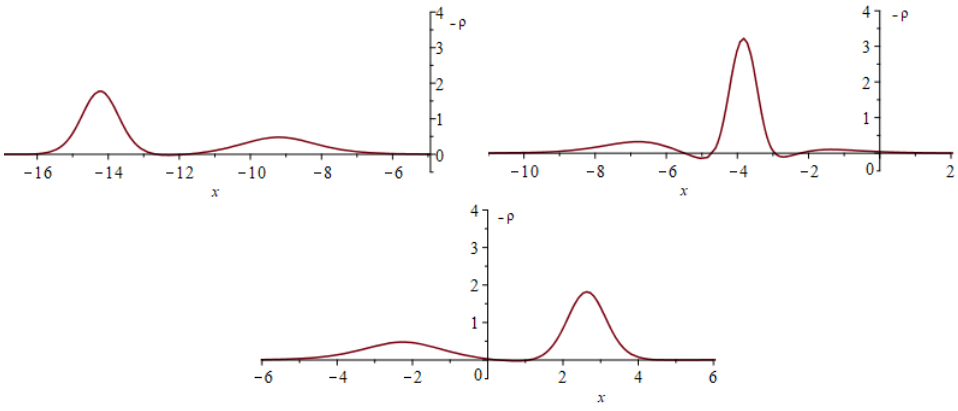


Figure 1. Interaction of two solitons $\big(\alpha_1=2,\ \alpha_2=3,\ \beta_1=\beta_2=5\sqrt{-1}\big)$.

As we previously pointed out, the term *soliton* solution applies to the conserved density

$$\rho = (\partial_x \ln \hat{u}_{l-1,m+1,n+1})(\partial_x \ln \hat{u}_{l,m+1,n+1})$$

rather than individual components $\hat{u}_{l-1,m+1,n+1}$ and $\ln \hat{u}_{l,m+1,n+1}$. The graphs of $-\rho$ for the values of time $t=-6.5$, $t=-0.5$, and $t=3$ are as follows

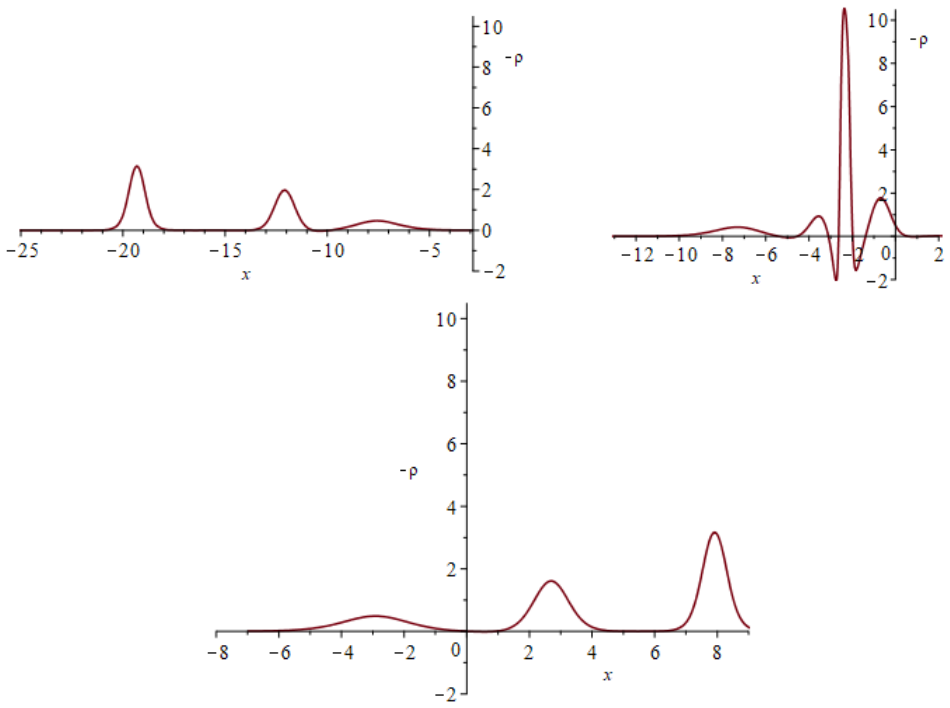


Figure 2. Interaction of three solitons ($\alpha_1 = 3/2$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\beta_1 = \beta_2 = \beta_3 = 7\mathrm{i}$).

4 Conclusion

In this article we have looked at implications of treating the lattice sine-Gordon equation and two of its generalised symmetries as a compatible system. This viewpoint yields, almost automatically, the auto-Bäcklund transformation and algebraic superposition formula for a modified derivative NLS system. The efficacy of the formulae has been verified by constructing two and three soliton solutions.

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