

On q -Isomonodromic Deformations and q -Nekrasov Functions

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Abstract. We construct a fundamental system of a q -difference Lax pair of rank N in terms of 5d Nekrasov functions with $q = t$. Our fundamental system degenerates by the limit $q \rightarrow 1$ to a fundamental system of a differential Lax pair, which yields the Fuji–Suzuki–Tsuda system. We introduce tau functions of our system as Fourier transforms of 5d Nekrasov functions. Using asymptotic expansions of the fundamental system at 0 and ∞ , we obtain several determinantal identities of the tau functions.

Key words: isomonodromic deformations; Nekrasov functions; Painlevé equations; determinantal identities

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1 Introduction

Isomonodromic deformations of linear differential equations [18] have been studied intensively and admit many applications both in mathematics and physics. The recent paper [12] revealed a connection between isomonodromic deformation and the Liouville conformal field theory that the tau function of the sixth Painlevé equation P_{VI} is a Fourier transform of the Virasoro conformal block with central charge $c = 1$. Later, the tau functions of other Painlevé equations P_V , P_{IV} , P_{III} , P_{II} were also recognized as Fourier transforms of irregular conformal blocks [13, 22, 27, 28]. Furthermore, the tau functions of isomonodromic deformation of particular Fuchsian systems of rank N was constructed by semi-degenerate conformal blocks of W_N -algebra with central charge $c = N - 1$ [15].

By AGT correspondence [1], Virasoro conformal blocks correspond to Nekrasov partition functions of 4d supersymmetric gauge theories. Hence, the tau functions of the Painlevé equations have explicit series expansions at regular singular points. Correspondence between partition functions of gauge theories and tau functions of Painlevé equations was investigated in [7]. In [3], using Nakajima–Yoshioka blow-up equations [29] for the pure Nekrasov partition functions, it was proved that the tau function of the Painlevé III_3 (P_{III_3}) in terms of the pure Nekrasov partition function satisfies the bilinear equation for it.

One of the reasons why conformal blocks appear in the construction of isomonodromic tau functions is that when the central charge takes particular values, monodromy invariant finite-systems can be constructed from conformal blocks with the degenerate field [15, 16]. We note that by definition, local monodromies of conformal blocks are independent of their singular points, and connection problem of semi-degenerate W_N -conformal blocks with the degenerate field is reduced to connection problem of the Clausen–Thomae hypergeometric function ${}_N F_{N-1}$. Recall that W_2 is the Virasoro algebra.

A q -analog of isomonodromic deformation of linear differential equations can be considered. It corresponds to study connection preserved deformation of linear q -difference equations, that is, q -difference Lax pairs with rational coefficients of the independent variable. In [21], the q -Painlevé VI (q - P_{VI}) was obtained from the compatibility condition of a Lax pair of rank 2.

In [20], fundamental solutions of the Lax pair were constructed in terms of the q -conformal blocks given as expectation values of the intertwining operators of the Ding–Iohara–Miki algebra [2], and expressed in terms of the q -Nekrasov functions. As a result, a general solution to the q -Painlevé VI is expressed explicitly, and moreover, it was conjectured that tau functions of q -P_{VI} defined as Fourier transforms of the q -conformal blocks satisfy bilinear equations. Later, by taking degeneration of q -P_{VI}, explicit series representations for the tau functions of the q -Painlevé V, III₁, III₂, and III₃ equations were obtained [23]. The series representations of the tau function for q -P_{III₃} expressed by the pure 5d Nekrasov function had been proposed in [5] for surface type $A_7^{(1)'}$ and [4] for surface type $A_7^{(1)}$. The tau functions obtained in [23] are equivalent to those proposed in [4, 5]. A Fredholm determinant representation for the tau functions of q -P_{III₃} was presented in [6].

In this paper, we construct a fundamental system of a Lax pair of rank N in terms of 5d Nekrasov functions with $q = t$. Our fundamental system degenerates by the limit $q \rightarrow 1$ to a fundamental system of a Lax pair $L_{M,N}$ ($M \in \mathbb{Z}_{\geq 1}$) describing isomonodromy deformation of a Fuchsian system with $M + 3$ regular singular points

$$\frac{dY}{dz} = \sum_{i=0}^{M+1} \frac{A_i}{z - x_i} Y,$$

where $Y = Y(z)$, A_i are N by N matrices, and whose *spectral type* is given by the $(M + 3)$ -tuple $(1^N), (1^N), (N - 1, 1), \dots, (N - 1, 1)$.

Here, the spectral type expresses multiplicities of the eigenvalues of the coefficient matrices A_i . The corresponding isomonodromic system was studied from the point of view of UC hierarchy [37]. When $M = 1$, the isomonodromic system is also derived from similarity condition of the Drinfeld–Sokolov hierarchy [11] ($N = 3$), [35] ($N \geq 2$). We call this isomonodromic system the Fuji–Suzuki–Tsuda system. Since a fundamental system of $L_{M,N}$ was constructed by W_N conformal blocks [15], the uniqueness of a fundamental system of a Fuchsian system gives another justification of AGT correspondence. We note that to be rigorous, we have to prove convergence of W_N conformal blocks and 5d Nekrasov partition functions, and moreover, Fourier transforms of them.

A key ingredient to construct a monodromy invariant fundamental system from conformal blocks is the braiding relation interchanging places of the variable x of the degenerate field with other singular points. In differential cases, the braiding relation is derived from W_N conformal field theory. In q -difference cases; however, we have not yet understood braiding relations of q -conformal blocks representation theoretically. Instead, taking advantage of explicit expressions of q -Nekrasov functions, we directly prove contiguity relations of q -Nekrasov functions with special values by generalizing a method used in [20]. Consequently, we solve the connection problem of q -conformal blocks with the degenerate field.

The remainder of this paper is organized as follows. In Section 2, we introduce q -conformal blocks and investigate the connection problem of degenerate q -conformal blocks. In Section 3, we construct a fundamental system of a particular Lax pair of rank N , using degenerate q -conformal blocks. We introduce tau functions of our system as Fourier transforms of q -conformal blocks. Finally, we present determinantal identities of the tau functions.

Notation. Throughout the paper we fix $q \in \mathbb{C}^\times$ such that $|q| < 1$. We set

$$[u] = \frac{1 - q^u}{1 - q}, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

$$(a_1, \dots, a_k; q)_\infty = \prod_{j=1}^k (a_j; q)_\infty, \quad (a; q, q)_\infty = \prod_{j,k=0}^{\infty} (1 - aq^{j+k}).$$

We use the q -gamma function, q -Barnes function and the q -theta function defined by

$$\Gamma_q(u) = \frac{(q; q)_\infty}{(q^u; q)_\infty} (1-q)^{1-u}, \quad G_q(u) = \frac{(q^u; q, q)_\infty}{(q; q, q)_\infty} (q; q)_\infty^{u-1} (1-q)^{-(u-1)(u-2)/2},$$

$$\vartheta(u) = q^{u(u-1)/2} \Theta_q(q^u), \quad \Theta_q(x) = (x, q/x, q; q)_\infty,$$

which satisfy $\Gamma_q(1) = G_q(1) = 1$ and

$$\Gamma_q(u+1) = [u] \Gamma_q(u), \quad G_q(u+1) = \Gamma_q(u) G_q(u), \quad \vartheta(u+1) = -\vartheta(u) = \vartheta(-u).$$

A partition is a finite sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_l)$ such that $\lambda_1 \geq \dots \geq \lambda_l > 0$. Denote the length of the partition by $\ell(\lambda) = l$. The conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_{l'})$ is defined by $\lambda'_j = \#\{i \mid \lambda_i \geq j\}$, $l' = \lambda_1$. We regard a partition as a Young diagram. Namely, we regard a partition λ also as the subset $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i, i \geq 1\}$ of \mathbb{Z}^2 , and denote its cardinality by $|\lambda|$. We denote the set of all partitions by \mathbb{Y} . For $\square = (i, j) \in \mathbb{Z}_{>0}^2$ we set $a_\lambda(\square) = \lambda_i - j$ (the arm length of \square) and $\ell_\lambda(\square) = \lambda'_j - i$ (the leg length of \square). In the last formulas we set $\lambda_i = 0$ if $i > \ell(\lambda)$ (resp. $\lambda'_j = 0$ if $j > \ell(\lambda')$). For a pair of partitions (λ, μ) and $u \in \mathbb{C}$ we set

$$N_{\lambda, \mu}(u) = \prod_{\square \in \lambda} (1 - q^{-\ell_\lambda(\square) - a_\mu(\square) - 1} u) \prod_{\square \in \mu} (1 - q^{\ell_\mu(\square) + a_\lambda(\square) + 1} u),$$

which we call a Nekrasov factor. We set $|\boldsymbol{\lambda}| = \sum_{i=1}^N |\lambda^{(i)}|$ for an element $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ of \mathbb{Y}^N .

2 Contiguity relations

2.1 q -conformal blocks

For $\sigma \in \mathbb{C}^N$, set $\Delta_\sigma = \sum_{i=1}^N (\sigma^{(i)})^2 / 2$. We define $h_i \in \mathbb{C}^N$, $i = 1, \dots, N$, as

$$h_i^{(j)} = \delta_{ij} - \frac{1}{N},$$

which may be regarded as the weights of the vector representation of \mathfrak{sl}_N . We choose the normalization factor as

$$\mathcal{N} \begin{pmatrix} \theta \\ \sigma_2 & \sigma_1 \end{pmatrix} = \frac{\prod_{k, k'=1}^N G_q(1 + \sigma_2^{(k)} - \theta - \sigma_1^{(k')})}{\prod_{1 \leq k < k' \leq N} G_q(1 + \sigma_2^{(k)} - \sigma_2^{(k')}) G_q(1 - \sigma_1^{(k)} + \sigma_1^{(k')})}$$

for $\theta \in \mathbb{C}$, $\sigma_i \in \mathbb{C}^N$, $i = 1, 2$.

Definition 2.1. Let $m \in \mathbb{Z}_{\geq 2}$. For $\theta_k \in \mathbb{C}$ ($1 \leq k \leq m$), and $\sigma_k = (\sigma_k^{(1)}, \dots, \sigma_k^{(N)}) \in \mathbb{C}^N$, $k = 0, \dots, m$, such that $\sum_{j=1}^N \sigma_k^{(j)} = 0$, we define the $m+2$ point q -conformal block by

$$\begin{aligned} \mathcal{F} \begin{pmatrix} \theta_m & \theta_{m-1} & \cdots & \theta_1 \\ \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \cdots & \sigma_1 & \sigma_0 \end{pmatrix}; x_m, \dots, x_1 \\ = \prod_{p=1}^m \mathcal{N} \begin{pmatrix} \theta_p \\ \sigma_p & \sigma_{p-1} \end{pmatrix} q^{N\theta_p \Delta_{\sigma_p}} \prod_{p=1}^m x_p^{\Delta_{\sigma_p} - \Delta_{N\theta_p h_1} - \Delta_{\sigma_{p-1}}} \\ \times \sum_{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{m-1} \in \mathbb{Y}^N} \prod_{p=1}^{m-1} \left(\frac{q^{N\theta_p} x_p}{x_{p+1}} \right)^{|\boldsymbol{\lambda}_p|} \frac{\prod_{p=1}^m \prod_{k, k'=1}^N N_{\lambda_p^{(k)}, \lambda_{p-1}^{(k')}}(q^{\sigma_p^{(k)} - \theta_p - \sigma_{p-1}^{(k')}})}{\prod_{p=1}^{m-1} \prod_{k, k'=1}^N N_{\lambda_p^{(k)}, \lambda_p^{(k')}}(q^{\sigma_p^{(k)} - \sigma_p^{(k')}})}, \end{aligned} \quad (2.1)$$

where $\boldsymbol{\lambda}_p = (\lambda_p^{(1)}, \dots, \lambda_p^{(N)})$, and $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_m = (\emptyset, \dots, \emptyset)$.

Our q -conformal blocks can be obtained as expectation values of intertwining operators of the Ding–Iohara–Miki algebra [2]. We omit a derivation and refer to [20, Section 2].¹ The series part of a q -conformal block is the 5d Nekrasov instanton partition function.

First, we relate a 4-point conformal block to the q -hypergeometric series, which is defined as follows.

Definition 2.2. Let $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_{N-1} \in \mathbb{C}$. A series

$${}^N\phi_{N-1} \left(\begin{matrix} \alpha_1, \dots, \alpha_N \\ \beta_1, \dots, \beta_{N-1} \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(q^{\alpha_1}; q)_k \cdots (q^{\alpha_N}; q)_k}{(q^{\beta_1}; q)_k \cdots (q^{\beta_{N-1}}; q)_k (q; q)_k} z^k$$

is called the q -hypergeometric series.

Let us recall some elementary facts about the Nekrasov factor.

Lemma 2.3. For $\lambda, \mu \in \mathbb{Y}, w \in \mathbb{C}$, we have

- (1) $N_{\lambda, \emptyset}(q^{-1}) \neq 0$ if and only if $\lambda = (1^n)$, $n = 0, 1, 2, \dots$,
- (2) $N_{(1^n), \emptyset}(w) = (q^{-n+1}w; q)_n$,
- (3) $N_{\emptyset, (1^n)}(w) = (w; q)_n$,
- (4) $N_{(1^n), (1^n)}(w) = (q^{-n}w; q)_n (qw; q)_n$.

We need to substitute a special value into the normalization factor as

$$\mathcal{N} \left(\begin{matrix} \frac{1}{N} \\ \sigma_1 - h_i & \sigma_1 \end{matrix} \right).$$

But then it has a zero factor. We redefine the normalization factor in this case as

$$\mathcal{N}' \left(\begin{matrix} \frac{1}{N} \\ \sigma_2 & \sigma_2 + h_i \end{matrix} \right) = \lim_{\epsilon \rightarrow 0} \frac{1}{G_q(\epsilon)} \mathcal{N} \left(\begin{matrix} \frac{1}{N} - \epsilon \\ \sigma_2 & \sigma_2 + h_i \end{matrix} \right).$$

The q -conformal block with the special value above is called *degenerate q -conformal block*. The reduction of the 5d Nekrasov partition function to the q -hypergeometric series was explained in [8, 9], which is stated in our notation as follows.

Proposition 2.4. For $i = 1, \dots, N$, we have

$$\begin{aligned} & \mathcal{F} \left(\begin{matrix} \frac{1}{N} & \theta_1 \\ \sigma_2 & \sigma_2 + h_i & \sigma_0 \end{matrix}; x_1, x_2 \right) \\ &= \mathcal{N} q^{\Delta \sigma_2} \left(\frac{q^{N\theta_1} x_2}{x_1} \right)^{\sigma_2^{(i)} + (1-1/N)/2} \frac{\prod_{k=1}^N \Gamma_q \left(1 - \frac{1}{N} + \sigma_2^{(i)} - \theta_1 - \sigma_0^{(k)} \right)}{\prod_{k=1}^N \Gamma_q \left(1 + \sigma_2^{(i)} - \sigma_2^{(k)} \right)} \\ & \quad \times {}^N\phi_{N-1} \left(\begin{matrix} \left\{ 1 - \frac{1}{N} + \sigma_2^{(i)} - \theta_1 - \sigma_0^{(k)} \right\}_{k=1}^N \\ \left\{ 1 + \sigma_2^{(i)} - \sigma_2^{(k)} \right\}_{k=1, k \neq i}^N \end{matrix}; q, \frac{q^{N\theta_1} x_2}{x_1} \right), \\ & \mathcal{F} \left(\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 - h_i & \sigma_0 \end{matrix}; x_2, x_1 \right) \\ &= \mathcal{N} q^{\Delta \sigma_0} \left(\frac{qx_1}{x_2} \right)^{-\sigma_0^{(i)} + (1-1/N)/2} \frac{\prod_{k=1}^N \Gamma_q \left(1 - \frac{1}{N} + \sigma_2^{(k)} - \theta_1 - \sigma_0^{(i)} \right)}{\prod_{k=1}^N \Gamma_q \left(1 + \sigma_0^{(k)} - \sigma_0^{(i)} \right)} \\ & \quad \times {}^N\phi_{N-1} \left(\begin{matrix} \left\{ 1 - \frac{1}{N} + \sigma_2^{(k)} - \theta_1 - \sigma_0^{(i)} \right\}_{k=1}^N \\ \left\{ 1 + \sigma_0^{(k)} - \sigma_0^{(i)} \right\}_{k=1, k \neq i}^N \end{matrix}; q, \frac{qx_1}{x_2} \right), \end{aligned}$$

¹If we replace $\mathcal{V}_{\theta, w}$ on p. 6 in [20] by the tensor product of the N Fock modules of the Ding–Iohara–Miki algebra, then we will get (2.1), provided that the normalization is suitably chosen.

where

$$\mathcal{N} = q^{N\theta_1\Delta\sigma_2} x_1^{-(1-1/N)/2} x_2^{\Delta\sigma_2 - \Delta_N\theta_1 h_1 - \Delta\sigma_0} \\ \times \frac{\prod_{k,k'=1}^N G_q\left(1 - \frac{1}{N} + \sigma_2^{(k)} - \theta_1 - \sigma_0^{(k')}\right)}{\prod_{1 \leq k < k' \leq N} G_q\left(1 + \sigma_2^{(k)} - \sigma_2^{(k')}\right) G_q\left(1 - \sigma_0^{(k)} + \sigma_0^{(k')}\right)}.$$

Proof. By the definition of q -conformal blocks and Lemma 2.3, we obtain the desired results. \blacksquare

2.2 Connection problem

The connection formula of the q -hypergeometric series is well known. See the equation (4.5.2) on p. 121 in [14] for example. From Proposition 2.4, the connection formula of the degenerate 4-point q -conformal block reads as

$$\mathcal{F}\left(\begin{matrix} \frac{1}{N} & \theta_1 \\ \sigma_2 & \sigma_2 + h_i & \sigma_0 \end{matrix}; x_1, x_2\right) \\ = \sum_{j=1}^N \mathcal{F}\left(\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 - h_j & \sigma_0 \end{matrix}; x_2, x_1\right) \mathcal{B}_{j,i}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| \frac{x_2}{x_1}\right] q^{N\theta_1^2/2 - \theta_1/2} \left(\frac{x_2}{x_1}\right)^{\theta_1}, \quad (2.2)$$

where $\mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right]$ is the N by N matrix given by

$$\mathcal{B}_{j,i}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right] = \frac{\vartheta\left(1 - \frac{1}{N} + \sigma_2^{(i)} + (N-1)\theta_1 - \sigma_0^{(j)} + u\right) \prod_{k=1, k \neq i}^N \vartheta\left(\frac{1}{N} - \sigma_2^{(k)} + \theta_1 + \sigma_0^{(j)}\right)}{\vartheta(N\theta_1 + u) \prod_{k=1, k \neq j}^N \vartheta\left(\sigma_0^{(j)} - \sigma_0^{(k)}\right)}$$

with $x = q^u$. We mention that the connection problem of the q -hypergeometric series was used to derive the three-point function for the q -deformed correlators where conformal blocks are controlled by the q -Virasoro algebra [30].

The connection matrix $\mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right]$ enjoys the following properties.

Lemma 2.5. *We have*

$$\mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| qx\right] = \mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right], \quad (2.3)$$

$$\mathcal{B}\left[\begin{matrix} \theta_1 + 1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right] = (-1)^N \mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right], \quad (2.4)$$

$$\mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 + h_i & \sigma_0 + h_j \end{matrix} \middle| x\right] = \mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right], \quad i, j = 1, \dots, N, \quad (2.5)$$

$$\mathcal{B}\left[\begin{matrix} \theta_1 + 1/N & \frac{1}{N} \\ \sigma_2 & \sigma_0 + h_i \end{matrix} \middle| x\right] = -\mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right], \quad i = 1, \dots, N, \quad (2.6)$$

$$\det \mathcal{B}\left[\begin{matrix} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{matrix} \middle| x\right] = (-1)^{N-1} \frac{\vartheta(u)}{\vartheta(N\theta_1 + u)} \prod_{1 \leq i < j \leq N} \frac{\vartheta(\sigma_2^{(i)} - \sigma_2^{(j)})}{\vartheta(\sigma_0^{(i)} - \sigma_0^{(j)})}. \quad (2.7)$$

Proof. The identities (2.3), (2.4), (2.5), and (2.6) are immediate consequences of the definition of the connection matrix. The expression of the determinant (2.7) of the connection matrix can be verified by straightforward calculations as follows.

By definition, we have

$$\det \mathcal{B} \left[\begin{array}{cc} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{array} \middle| x \right] = \frac{1}{\vartheta(N\theta_1 + u)^N \prod_{i=1}^N \prod_{k \neq i} \vartheta(\sigma_0^{(i)} - \sigma_0^{(k)})} \det(a_{ij}),$$

where

$$a_{ij} = \vartheta\left(1 - \frac{1}{N} + \sigma_2^{(j)} + (N-1)\theta_1 - \sigma_0^{(i)} + u\right) \prod_{k \neq j}^N \vartheta\left(\frac{1}{N} - \sigma_2^{(k)} + \theta_1 + \sigma_0^{(i)}\right).$$

Put $x_{ij} = 1 - \frac{1}{N} + \sigma_2^{(j)} + (N-1)\theta_1 - \sigma_0^{(i)} + u$ and $y_{ij} = \frac{1}{N} - \sigma_2^{(j)} + \theta_1 + \sigma_0^{(i)}$. Using the relation on the theta function:

$$\begin{aligned} & \vartheta(x+y)\vartheta(x-y)\vartheta(u+v)\vartheta(u-v) - \vartheta(x+v)\vartheta(x-v)\vartheta(u+y)\vartheta(u-y) \\ &= \vartheta(x+u)\vartheta(x-u)\vartheta(y+v)\vartheta(y-v), \end{aligned}$$

we obtain for $i, j \geq 2$

$$\begin{aligned} a_{11}a_{ij} - a_{i1}a_{1j} &= (\vartheta(x_{11})\vartheta(y_{1j})\vartheta(x_{ij})\vartheta(y_{i1}) - \vartheta(x_{i1})\vartheta(y_{ij})\vartheta(x_{1j})\vartheta(y_{11})) \prod_{k \neq 1, j} \vartheta(y_{1k})\vartheta(y_{ik}) \\ &= \vartheta(1 + N\theta_1 + u)\vartheta\left(1 - \frac{2}{N} + (N-2)\theta_1 + u + \sigma_2^{(1)} + \sigma_2^{(j)} - \sigma_0^{(1)} - \sigma_0^{(i)}\right) \\ &\quad \times \vartheta(\sigma_2^{(1)} - \sigma_2^{(j)})\vartheta(-\sigma_0^{(1)} + \sigma_0^{(i)}) \prod_{k \neq 1, j} \vartheta(y_{1k})\vartheta(y_{ik}). \end{aligned}$$

Hence, the determinant becomes

$$\det \mathcal{B} \left[\begin{array}{cc} \theta_1 & \frac{1}{N} \\ \sigma_2 & \sigma_0 \end{array} \middle| x \right] = \frac{\prod_{k=2}^N \vartheta(\sigma_2^{(1)} - \sigma_2^{(k)})}{\vartheta(N\theta_1 + u)\vartheta(x_{11})^{N-2} \prod_{i=2}^N \prod_{k \neq i} \vartheta(\sigma_0^{(i)} - \sigma_0^{(k)})} \det(b_{ij}),$$

where

$$b_{ij} = \vartheta\left(1 - \frac{2}{N} + (N-2)\theta_1 + u + \sigma_2^{(1)} + \sigma_2^{(j)} - \sigma_0^{(1)} - \sigma_0^{(i)}\right) \prod_{k \neq 1, j} \vartheta(y_{ik}), \quad 2 \leq i, j \leq N.$$

In the same way, by computing $b_{22}b_{ij} - b_{i2}b_{2j}$, the determinant $\det(b_{ij})$ reduces to a determinant of size $N-2$ whose entries are products of the theta functions. Repeating the computation, we obtain the desired result (2.7). \blacksquare

We note that the connection matrix has periodicity on the variable x and parameters σ_0 , σ_2 , and θ_1 . Furthermore, the determinant has a simple form in terms of the theta function.

We want to prove the following connection formula for 6-point degenerate q -conformal block

$$\begin{aligned} & \mathcal{F} \left(\begin{array}{cccc} \theta_4 & \frac{1}{N} & \theta_2 & \theta_1 \\ \sigma_4 & \sigma_3 & \sigma_3 + h_i & \sigma_1 \end{array} \middle| \sigma_0; x_4, x_2, x_3, x_1 \right) \\ &= \sum_{j=1}^N \mathcal{F} \left(\begin{array}{cccc} \theta_4 & \theta_2 & \frac{1}{N} & \theta_1 \\ \sigma_4 & \sigma_3 & \sigma_1 - h_j & \sigma_1 \end{array} \middle| \sigma_0; x_4, x_3, x_2, x_1 \right) \mathcal{B}_{j,i} \left[\begin{array}{cc} \theta_2 & \frac{1}{N} \\ \sigma_3 & \sigma_1 \end{array} \middle| \begin{array}{c} x_3 \\ x_2 \end{array} \right] \\ &\quad \times q^{N\theta_2^2/2 - \theta_2/2} \left(\frac{x_3}{x_2} \right)^{\theta_2}. \end{aligned} \tag{2.8}$$

The constant with respect to the variables x_1 , x_4 in the 6-point degenerate q -conformal block is q -hypergeometric series. In this case, the connection problem is solved. We solve the connection

problem (2.8) by solving connection problem of each coefficient of x_1, x_4 in the 6-point degenerate q -conformal block. We consider the following series obtained by taking the coefficient of $x_1^{|\lambda|} x_4^{-|\nu|}$ in the 6-point q -conformal block:

$$S_{\lambda, \nu} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right) = \sum_{\mu \in \mathbb{Y}^N} \left(\frac{q^{N\theta_2} x_2}{x_3} \right)^{|\mu|} \prod_{k, k'=1}^N \frac{N_{\lambda_k, \mu_{k'}}(w_{k, k'}) N_{\mu_k, \nu_{k'}}(z_{k, k'})}{N_{\mu_k, \mu_{k'}}(u_{k, k'})}. \quad (2.9)$$

Here we put

$$w_{k, k'} = q^{\sigma_k^{(3)} - \theta_3 - \sigma_{k'}^{(2)}}, \quad z_{k, k'} = q^{\sigma_k^{(2)} - \theta_2 - \sigma_{k'}^{(1)}}, \quad u_{k, k'} = q^{\sigma_k^{(2)} - \sigma_{k'}^{(2)}}.$$

We note that for $\lambda = \nu = (\emptyset, \dots, \emptyset)$,

$$\begin{aligned} \mathcal{F} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right) \\ = \prod_{p=2}^3 \mathcal{N} \left(\begin{array}{cc} \theta_p & \\ \sigma_p & \sigma_{p-1} \end{array} \right) q^{N\theta_p \Delta_{\sigma_p} x_p^{\Delta_{\sigma_p} - \Delta_{N\theta_p h_1} - \Delta_{\sigma_{p-1}}}} S_{\emptyset, \emptyset} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right). \end{aligned}$$

From Proposition 2.4 if we set $\theta_3 = 1/N$, $\sigma_2 = \sigma_3 + h_i$ or $\theta_2 = 1/N$, $\sigma_2 = \sigma_1 - h_i$ in (2.9), then $S_{\emptyset, \emptyset}$ becomes the q -hypergeometric series ${}_N\phi_{N-1}$. In that case, μ in the right hand side of (2.9) restricts to $\mu_i = (1^n)$ ($n = 0, 1, 2, \dots$), $\mu_k = \emptyset$ ($k \neq i$). For the general λ, ν , a tuple μ of partitions also takes the following particular forms.

Let us introduce the following operations on partitions $\lambda \in \mathbb{Y}$:

$$\bar{\lambda} = (\lambda_1 - 1, \dots, \lambda_{\ell(\lambda)} - 1), \quad (2.10)$$

$$r_n(\lambda) = (\lambda_1 + 1, \dots, \lambda_n + 1, \lambda_{n+2}, \dots), \quad n \in \mathbb{Z}_{\geq 0}. \quad (2.11)$$

Here we identify a partition λ with a sequence $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$.

Let us recall the following lemma.

Lemma 2.6. *For $\lambda, \mu \in \mathbb{Y}$, we have*

- (1) $N_{\lambda, \mu}(1) \neq 0$ if and only if $\lambda = \mu$,
- (2) $N_{\lambda, \mu}(q^{-1}) \neq 0$ if and only if $\lambda = r_n(\mu)$, $n = 0, 1, 2, \dots$,
- (3) $N_{\lambda, \mu}(u) = N_{\mu', \lambda'}(u)$.

Lemma 2.6(1) is easily obtained from the definition of the Nekrasov factor and Lemma 2.6(2), (3) were proved in [20] as Lemmas A.2 and A.3, respectively.

Suppose that $\theta_3 = 1/N$, $\sigma_2 = \sigma_3 + h_i$ in (2.9), then the following holds:

$$w_{i, i} = q^{\sigma_3^{(i)} - 1/N - \sigma_2^{(i)}} = q^{-1}, \quad w_{k, k} = q^{\sigma_3^{(k)} - 1/N - \sigma_2^{(k)}} = 1, \quad k \neq i.$$

Hence by Lemma 2.6(1), (2), we obtain

$$\mu_i = r_n(\nu_i), \quad \mu_k = \nu_k, \quad k \neq i.$$

Similarly, suppose that $\theta_2 = 1/N$, $\sigma_2 = \sigma_1 - h_i$ in (2.9), then the following holds:

$$z_{i, i} = q^{\sigma_2^{(i)} - 1/N - \sigma_1^{(i)}} = q^{-1}, \quad z_{k, k} = q^{\sigma_2^{(k)} - 1/N - \sigma_1^{(k)}} = 1, \quad k \neq i.$$

Hence by Lemma 2.6(1), (2), we obtain

$$\mu_i = r_n(\lambda_i), \quad \mu_k = \lambda_k, \quad k \neq i.$$

We also recall the following lemma.

Lemma 2.7 ([20, Lemma A.4]). *Let $\lambda, \mu \in \mathbb{Y}$ and $n \in \mathbb{Z}_{\geq 0}$. Using the notation (2.10) and (2.11), we set*

$$\begin{aligned} \ell &= \ell(\lambda), & k &= \ell(\mu), & \eta &= r_n(\lambda), & \gamma &= (r_n(\mu'))', \\ \tilde{\eta} &= \begin{cases} \bar{\eta}, & n \leq \ell - 1, \\ (\lambda, 1^{n-\ell+1}), & n \geq \ell. \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} \frac{N_{\eta, \lambda}(q^{-1})}{N_{\eta, \eta}(1)} &= \frac{N_{\tilde{\eta}, \bar{\lambda}}(q^{-1})}{N_{\tilde{\eta}, \tilde{\eta}}(1)} (1 - q^{|\tilde{\eta}| - |\lambda|}), \\ N_{\mu, \eta}(u) &= N_{\mu, \tilde{\eta}}(q^{-1}u) \prod_{j=1}^{\mu_\ell} \frac{1 - q^{j-1}u}{1 - q^{-\ell_\mu(\ell, j) + j - 2}u} \prod_{i=1}^{\ell-1} (1 - q^{\ell-i+a_\mu(i,1)}u), \\ N_{\mu, \lambda}(u) &= N_{\mu, \bar{\lambda}}(q^{-1}u) \prod_{j=1}^{\mu_{\ell+1}} \frac{1 - q^{j-1}u}{1 - q^{-\ell_\mu(\ell+1, j) + j - 2}u} \prod_{i=1}^{\ell} (1 - q^{\ell-i+a_\mu(i,1)+1}u), \\ \frac{N_{\mu, \lambda}(u)}{N_{\mu, \eta}(qu)} &= \frac{N_{\mu, \bar{\lambda}}(q^{-1}u)}{N_{\mu, \tilde{\eta}}(u)} (1 - u), \\ \frac{N_{\mu, \lambda}(u)}{N_{\mu, \eta}(qu)} &= \frac{N_{\bar{\mu}, \lambda}(qu)}{N_{\bar{\mu}, \eta}(q^2u)} \frac{1 - q^{|\eta| - |\lambda| + 1 - k}u}{1 - qu}, \\ N_{\gamma, \lambda}(u) &= N_{\gamma, \bar{\lambda}}(q^{-1}u) (1 - q^{\ell+|\gamma| - |\mu| - 1}u) \prod_{j=1}^{\mu_\ell} \frac{1 - q^{j-1}u}{1 - q^{-\ell_\mu(\ell, j) + j - 2}u} \prod_{i=1}^{\ell-1} (1 - q^{\ell-i+a_\mu(i,1)}u). \end{aligned}$$

We denote by $T_{q,z}$, the q -shift operator with respect to the variable z , namely, $T_{q,z}(f(z)) = f(qz)$ for a function $f(z)$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define

$$\lambda + 1^m = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_m + 1, \lambda_{m+1}, \dots).$$

Theorem 2.8. *For i, j ($1 \leq i, j \leq N$), let $m \geq \ell(\lambda_j)$, and*

$$\begin{aligned} \hat{\lambda} &= (\lambda_1, \dots, \lambda_j + 1^m, \dots, \lambda_N), \\ A_{\lambda, \nu}^{(j)}(\theta_2, \sigma_1, \sigma_3) &= \prod_{k=1}^N \frac{N_{\nu_k, \hat{\lambda}_j}(q^{\sigma_3^{(k)} - \theta_2 - \sigma_1^{(j)} + 1 - 1/N})}{N_{\nu_k, \lambda_j}(q^{\sigma_3^{(k)} - \theta_2 - \sigma_1^{(j)} - 1/N})}, \end{aligned}$$

we have

$$\begin{aligned} S_{\hat{\lambda}, \nu} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 - \frac{1}{N} & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 - h_j \end{array}; x_2, x_3 \right) \\ = q^{m - |\nu| + 1 - 1/N + \sigma_3^{(i)} - \theta_2 - \sigma_1^{(j)}} A_{\lambda, \nu}^{(j)}(\theta_2, \sigma_1, \sigma_3) \frac{1}{1 - q^{\sigma_3^{(i)} - \theta_2 - \sigma_1^{(j)} + 1 - 1/N}} \\ \times (q^{|\nu| - m - 1 + 1/N - \sigma_3^{(i)} + \theta_2 + \sigma_1^{(j)}} T_{q, x_2} - 1) S_{\lambda, \nu} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 \end{array}; x_2, x_3 \right), \end{aligned} \quad (2.12)$$

$$\begin{aligned} S_{\hat{\lambda}, \nu} \left(\begin{array}{ccc} \theta_2 - \frac{1}{N} & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_i - h_j & \sigma_1 - h_j \end{array}; x_3, x_2 \right) = \left(\frac{qx_2}{x_3} \right)^m A_{\lambda, \nu}^{(j)}(\theta_2, \sigma_1, \sigma_3) \\ \times \frac{1 - q^{-m - |\lambda| + \sigma_1^{(j)} - \sigma_1^{(i)}} T_{q, x_2}}{1 - q^{\sigma_1^{(j)} - \sigma_1^{(i)}}} S_{\lambda, \nu} \left(\begin{array}{ccc} \theta_2 & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_i & \sigma_1 \end{array}; x_3, x_2 \right), \quad i \neq j, \end{aligned} \quad (2.13)$$

$$\begin{aligned}
& S_{\hat{\lambda}, \nu} \left(\begin{array}{ccc} \theta_2 - \frac{1}{N} & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - 2h_i & \sigma_1 - h_i \end{array}; x_3, x_2 \right) \\
&= \left(\frac{qx_2}{x_3} \right)^{m-1} A_{\lambda, \nu}^{(j)}(\theta_2, \sigma_1, \sigma_3) \prod_{k=1}^N \frac{1 - q^{\sigma_1^{(k)} - \sigma_1^{(i)} + 1}}{1 - q^{\sigma_3^{(k)} - \theta_2 - \sigma_1^{(j)} + 1 - 1/N}} \frac{1 - q^{-m - |\lambda|} T_{q, x_2}}{1 - q} \\
&\quad \times S_{\lambda, \nu} \left(\begin{array}{ccc} \theta_2 & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_i & \sigma_1 \end{array}; x_3, x_2 \right). \tag{2.14}
\end{aligned}$$

Proof. We can verify the equations (2.12), (2.13), and (2.14) by direct computations using Lemma 2.7. We omit the calculation. \blacksquare

We note that if $\lambda = \nu = (\emptyset, \dots, \emptyset)$, then we can take $m = 0$ in Theorem 2.8 and then $\hat{\lambda} = (\emptyset, \dots, \emptyset)$. In that case, the equations (2.12), (2.13), and (2.14) are equal to the contiguity relations for the q -hypergeometric series. Hence, we may regard the contiguity relations (2.12), (2.13), and (2.14) as a generalization of the contiguity relation of the q -hypergeometric series.

Let us denote the coefficient of $x_1^{|\lambda|} x_4^{-|\nu|}$ in the 6-point q -conformal block by

$$\begin{aligned}
& \mathcal{F}_{\lambda, \nu} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right) \\
&= \prod_{p=2}^3 \mathcal{N} \left(\begin{array}{cc} \theta_p & \\ \sigma_p & \sigma_{p-1} \end{array} \right) q^{N\theta_p \Delta_{\sigma_p} \Delta_{\sigma_p} - \Delta_{N\theta_p h_1} - \Delta_{\sigma_{p-1}} |\nu|} x_3^{|\nu|} x_2^{-|\lambda|} S_{\lambda, \nu} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right).
\end{aligned}$$

We note that if $\lambda = \nu = (\emptyset, \dots, \emptyset)$, then it is a 4-point q -conformal block, namely

$$\mathcal{F}_{\emptyset, \emptyset} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right) = \mathcal{F} \left(\begin{array}{ccc} \theta_3 & \theta_2 & \\ \sigma_3 & \sigma_2 & \sigma_1 \end{array}; x_3, x_2 \right).$$

Corollary 2.9. *We have*

$$\begin{aligned}
& \mathcal{F}_{\hat{\lambda}, \nu} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 - \frac{1}{N} & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 - h_j \end{array}; x_2, x_3 \right) \\
&= C (q^{-m + \theta_2 + \sigma_1^{(j)}} T_{q, x_2} - 1) \mathcal{F}_{\lambda, \nu} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 \end{array}; x_2, x_3 \right), \\
& \mathcal{F}_{\hat{\lambda}, \nu} \left(\begin{array}{ccc} \theta_2 - \frac{1}{N} & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_i - h_j & \sigma_1 - h_j \end{array}; x_3, x_2 \right) \\
&= C q^{\theta_2} \left(\frac{x_3}{qx_2} \right)^{1/N} (1 - q^{-m + \sigma_1^{(j)}} T_{q, x_2}) \mathcal{F}_{\lambda, \nu} \left(\begin{array}{ccc} \theta_2 & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_i & \sigma_1 \end{array}; x_3, x_2 \right),
\end{aligned}$$

where

$$C = q^{m - |\nu| - \Delta_{\sigma_3} + (1 - 1/N)/2 - \theta_2 - \sigma_1^{(j)}} x_3^{-m + (N\theta_2 - 1)(1 - 1/N) + \sigma_1^{(j)}} A_{\lambda, \nu}^{(j)}(\theta_2, \sigma_1, \sigma_3).$$

Corollary 2.9 enable us to reduce the connection problem of degenerate $\mathcal{F}_{\hat{\lambda}, \nu}(x_2)$ to the connection problem of degenerate $\mathcal{F}_{\lambda, \nu}(x_2)$. Firstly, we let $C(q^{-m + \theta_2 + \sigma_1^{(1)}} T_{q, x_2} - 1)$ act on the connection formula (2.2) of the degenerate 4-point q -conformal block. Then we have the connection formula

$$\begin{aligned}
& \mathcal{F}_{\hat{\lambda}, \emptyset} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 - \frac{1}{N} & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 - h_1 \end{array}; x_2, x_3 \right) = - \sum_{j=1}^N \mathcal{F}_{\hat{\lambda}, \emptyset} \left(\begin{array}{ccc} \theta_2 - \frac{1}{N} & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_j - h_1 & \sigma_1 - h_1 \end{array}; x_2, x_1 \right) \\
&\quad \times \mathcal{B}_{j, i} \left[\begin{array}{cc} \theta_2 & \frac{1}{N} \left| \frac{x_3}{x_2} \right. \\ \sigma_3 & \sigma_1 \left| x_2 \right. \end{array} \right] q^{N\theta_2^2/2 - 3\theta_2/2 + 1/N} \left(\frac{x_3}{x_2} \right)^{\theta_2 - 1/N},
\end{aligned}$$

where $\hat{\lambda} = (1^m, \emptyset, \dots, \emptyset)$. Substituting $\theta_2 + 1/N$, $\sigma_1 - h_1$ into θ_2 , σ_1 , respectively, we obtain

$$\begin{aligned} \mathcal{F}_{\hat{\lambda}, \emptyset} \left(\begin{array}{ccc} \frac{1}{N} & \theta_2 & \\ \sigma_3 & \sigma_3 + h_i & \sigma_1 \end{array}; x_2, x_3 \right) &= \sum_{j=1}^N \mathcal{F}_{\hat{\lambda}, \emptyset} \left(\begin{array}{ccc} \theta_2 & \frac{1}{N} & \\ \sigma_3 & \sigma_1 - h_j & \sigma_1 \end{array}; x_2, x_1 \right) \\ &\quad \times \mathcal{B}_{j,i} \left[\begin{array}{c} \theta_2 \\ \sigma_3 \end{array} \middle| \frac{x_3}{x_2} \right] q^{N\theta_2/2 - \theta_2/2} \left(\frac{x_3}{x_2} \right)^{\theta_2} \end{aligned}$$

due to the periodicity (2.6) of the connection matrix $\mathcal{B} \left[\begin{array}{c} \theta_2 \\ \sigma_3 \end{array} \middle| \frac{x}{\sigma_1} \right]$. In this way, by using Corollary 2.9 repeatedly and by the symmetry of $\mathcal{F}_{\lambda, \nu}$ with respect to λ, ν , we obtain the connection formulas for any degenerate $\mathcal{F}_{\lambda, \nu}$ whose connection matrix is the same of degenerate $\mathcal{F}_{\emptyset, \emptyset}$. Therefore, we obtain

Theorem 2.10. *We have*

$$\begin{aligned} \mathcal{F} \left(\begin{array}{cccc} \theta_4 & \frac{1}{N} & \theta_2 & \theta_1 \\ \sigma_4 & \sigma_3 & \sigma_3 + h_i & \sigma_1 \end{array}; x_4, x_2, x_3, x_1 \right) \\ = \sum_{j=1}^N \mathcal{F} \left(\begin{array}{cccc} \theta_4 & \theta_2 & \frac{1}{N} & \theta_1 \\ \sigma_4 & \sigma_3 & \sigma_1 - h_j & \sigma_1 \end{array}; x_4, x_3, x_2, x_1 \right) \mathcal{B}_{j,i} \left[\begin{array}{c} \theta_2 \\ \sigma_3 \end{array} \middle| \frac{x_3}{x_2} \right] \\ \times q^{N\theta_2/2 - \theta_2/2} \left(\frac{x_3}{x_2} \right)^{\theta_2}. \end{aligned}$$

3 CFT construction

In this section, we construct a monodromy invariant fundamental solution of a linear q -difference system as a Fourier transform of q -Nekrasov functions multiplied by some factors. Our fundamental system degenerates by the limit $q \rightarrow 1$ to a monodromy invariant fundamental system of a Fuchsian system of rank N with spectral type

$$(1^N), \quad (1^N), \quad (N-1, 1), \quad \dots, \quad (N-1, 1).$$

Another monodromy invariant fundamental solution of this Fuchsian system was constructed in [15] by using the W conformal field theory. The corresponding tau function was expressed as a Fourier transform of the *semi-degenerate* W_N conformal block with the central charge $c = N-1$. Hence, by the uniqueness of a fundamental system of a Fuchsian system, W conformal blocks are equal to 4d Nekrasov functions. We note that in what follows, we assume convergence of 5d Nekrasov partition functions, and moreover, Fourier transforms of them. Analyticity of 5d Nekrasov partition functions in cases different from our case was discussed in [10].

3.1 Lax form

Let R be defined by

$$R = \left\{ n \in \mathbb{Z}^N \middle| \sum_{i=1}^N n^{(i)} = 0 \right\},$$

which may be regarded as the root lattice of \mathfrak{sl}_N . We set

$$\begin{aligned} \tilde{x} &= q^{N \sum_{i=1}^{m+1} \theta_i} x, & \tilde{t}_i &= q^{N \sum_{j=i+1}^{m+1} \theta_j} t_i, & i &= 1, \dots, m+1, \\ s &= \prod_{i=1}^m \prod_{j=1}^N s_{i,j}, & s_i &= \prod_{j=1}^N s_{i,j}, & i &= 1, \dots, m \end{aligned}$$

for variables x , t_i , and $s_{i,j}$, $i = 1, \dots, m$, $j = 1, \dots, N$. We denote $s_i^n = \prod_{j=1}^N s_{i,j}^{n(j)}$ for $n \in R$ and $s^n = \prod_{i=1}^m s_i^{n_i}$ for $n = (n_1, \dots, n_m) \in R^m$.

We define a N by N matrix for $\theta_1, \dots, \theta_{m+1} \in \mathbb{C}$ and $\theta_0, \theta_\infty, \sigma_1, \dots, \sigma_{m+1} \in \mathbb{C}^N$ by

$$Y_{i,j}^{(0,1)}(x, t) = \frac{1}{\tau_i^{(0,1)}(t)} \times \sum_{n=(n_1, \dots, n_m) \in R^m} s^n \mathcal{F} \left(\begin{matrix} & & \frac{1}{N} & & (\theta_\ell)_{\ell=1}^{m+1} \\ \theta_\infty - h_i & \theta_\infty - h_i + h_j & & & (\sigma_\ell + n_\ell)_{\ell=1}^m & \theta_0 \end{matrix} ; \tilde{x}, (\tilde{t}_\ell)_{\ell=1}^{m+1} \right).$$

Here, the function $\tau_i^{(0,1)}(t)$ is the normalization factor so that

$$Y^{(0,1)}(x, t) = (I + Y_1(t)x^{-1} + O(x^{-2})) \text{diag} (x^{-\theta_\infty^{(1)}}, \dots, x^{-\theta_\infty^{(N)}}),$$

where I is the unit matrix of rank N . The matrix $Y^{(0,1)}(x, t)$ is an expansion around $x = \infty$. We define the N by N matrices by

$$Y_{i,j}^{(k,k+1)}(x, t) = \frac{x^{-\sum_{\ell=1}^k \theta_\ell}}{\tau_i^{(k,k+1)}(t)} \sum_{n \in R^m} s^n s_k^{h_j - h_N} \times \mathcal{F} \left(\begin{matrix} & & (\theta_\ell)_{\ell=1}^k & & \frac{1}{N} & & (\theta_\ell)_{\ell=k+1}^{m+1} \\ \theta_\infty - h_i & (\sigma_\ell - h_N + n_\ell)_{\ell=1}^k & \sigma_k + h_j - h_N + n_k & (\sigma_\ell + n_\ell)_{\ell=k+1}^m & & \theta_0 \end{matrix} ; \tilde{t}_\ell)_{\ell=1}^k, \tilde{x}, (\tilde{t}_\ell)_{\ell=k+1}^{m+1} \right),$$

for $k = 1, \dots, m$, and

$$Y_{i,j}^{(m+1,m+2)}(x, t) = \frac{x^{-\sum_{\ell=1}^{m+1} \theta_\ell}}{\tau_i^{(m+1,m+2)}(t)} \sum_{n \in R^m} s^n \mathcal{F} \left(\begin{matrix} & & & & (\theta_\ell)_{\ell=1}^{m+1} & & \frac{1}{N} \\ \theta_\infty - h_i & (\sigma_\ell - h_N + n_\ell)_{\ell=1}^m & \theta_0 - h_j & \theta_0 \end{matrix} ; (\tilde{t}_\ell)_{\ell=1}^{m+1}, \tilde{x} \right),$$

where the functions $\tau_i^{(k,k+1)}(t)$ are given recursively by

$$\tau_i^{(k,k+1)}(t) = \tau_i^{(k-1,k)}(t) q^{N\theta_k^2/2 + \theta_k/2 + N\theta_k \sum_{i=1}^{k-1} \theta_i t_k^{-\theta_k}}. \quad (3.1)$$

By the definition of q -conformal blocks, asymptotic behaviours of the matrices of $Y^{(0,1)}(x, t)$ and $Y^{(m+1,m+2)}(x, t)$ are given by

$$Y^{(0,1)}(x, t) = \widehat{Y}^{(0,1)}(x, t) \text{diag} (x^{-\theta_\infty^{(1)}}, \dots, x^{-\theta_\infty^{(N)}}), \quad (3.2)$$

$$\widehat{Y}^{(0,1)}(x, t) = I + Y_1(t)x^{-1} + O(x^{-2}),$$

$$Y^{(m+1,m+2)}(x, t) = \widehat{Y}^{(m+1,m+2)}(x, t) \text{diag} (x^{-\theta_0^{(1)}}, \dots, x^{-\theta_0^{(N)}}) x^{-\sum_{i=1}^{m+1} \theta_i}, \quad (3.3)$$

$$\widehat{Y}^{(m+1,m+2)}(x, t) = G(t)(I + O(x)),$$

where I is the unit matrix of rank N , $Y_1(t)$ and $G(t)$ are N by N matrices. The off diagonal elements of $Y_1(t)$ and all elements of the coefficient matrix $G(t)$ will be expressed as ratios of tau functions. Furthermore, asymptotic behaviours of the matrices $Y^{(k,k+1)}(x, t)$, $k = 1, \dots, m$, are given by

$$Y^{(k,k+1)}(x, t) = \widehat{Y}^{(k,k+1)}(x, t) \text{diag} (x^{-\sigma_k^{(1)} - 1/N}, \dots, x^{-\sigma_k^{(N-1)} - 1/N}, x^{-\sigma_k^{(N)} + 1 - 1/N}) x^{-\sum_{i=1}^k \theta_i},$$

$$\widehat{Y}^{(k,k+1)}(x, t) = \sum_{n=-\infty}^{\infty} Y_n^{(k,k+1)}(t) x^n. \quad (3.4)$$

We assume that each $\widehat{Y}^{(k,k+1)}(x,t)$ is holomorphic in the domain

$$|t_1| > |t_2| > \cdots > |t_k| > |q^{1+N \sum_{i=1}^k \theta_i x}|, \quad |q^N \sum_{i=1}^k \theta_i x| > |t_{k+1}| > \cdots > |t_{m+1}|, \quad (3.5)$$

which is a natural assumption from the point of view of conformal blocks.

Theorem 3.1. *Let*

$$1 < |q^{-N\theta_i}| < \frac{1}{|q|}, \quad i = 1, \dots, m+1, \quad (3.6)$$

$$|q^{-N \sum_{j=i}^{k+1} \theta_j t_{k+1}}| < |t_i|, \quad i = 1, \dots, m, \quad k = i, \dots, m. \quad (3.7)$$

Then, the matrices $Y^{(k,k+1)}(x,t)$, $k = 0, 1, \dots, m$, form a fundamental solution of the q -linear difference system

$$T_{q,x}(Y(x,t)) = A(x,t)Y(x,t), \quad T_{q,t_i}(Y(x,t)) = B_i(x,t)Y(x,t), \quad i = 1, \dots, m+1, \quad (3.8)$$

with N by N matrices $A(x,t)$, $B_i(x,t)$ satisfying

$$A(x,t) = \frac{A_{m+1}x^{m+1} + \sum_{k=0}^m A_k(t)x^k}{\prod_{k=1}^{m+1} (x - q^{-1-N \sum_{j=1}^{k-1} \theta_j t_k})}, \quad \det A(x,t) = \prod_{k=1}^{m+1} \frac{x - q^{-1-N \sum_{j=1}^k \theta_j t_k}}{x - q^{-1-N \sum_{j=1}^{k-1} \theta_j t_k}}, \quad (3.9)$$

$$A_{m+1} = \text{diag}(q^{-\theta_\infty^{(1)}}, \dots, q^{-\theta_\infty^{(N)}}),$$

$$A_0(t) = (-1)^{m+1} q^{-m-1 - \sum_{i=1}^{m+1} \theta_i} \prod_{k=0}^m (q^{-N \sum_{j=1}^k \theta_j t_{k+1}}) G(t) \text{diag}(q^{-\theta_0^{(1)}}, \dots, q^{-\theta_0^{(N)}}) G(t)^{-1},$$

$$B_i(x,t) = \frac{xI + B_{i,0}(t)}{x - q^{-N \sum_{j=1}^i \theta_j t_i}}, \quad \det B_i(x,t) = \frac{x - q^{-N \sum_{j=1}^{i-1} \theta_j t_i}}{x - q^{-N \sum_{j=1}^i \theta_j t_i}}. \quad (3.10)$$

Proof. Thanks to Theorem 2.10, we have for $k = 0, 1, \dots, m+1$

$$\begin{aligned} Y_{i,j}^{(k,k+1)}(x,t) &= \frac{x^{-\sum_{\ell=1}^k \theta_\ell}}{\tau_i^{(k,k+1)}(t)} \sum_{n \in R^m} s^n s_k^{h_j - h_N} \\ &\times \sum_{p=1}^N \mathcal{F} \left(\begin{array}{ccc} (\theta_\ell)_{\ell=1}^{k+1} & \frac{1}{N} (\theta_\ell)_{\ell=k+2}^{m+1} & \\ \theta_\infty - h_i & (\sigma_\ell - h_N + n_\ell)_{\ell=1}^k & \sigma_{k+1} + n_{k+1} - h_p \quad (\sigma_\ell + n_\ell)_{\ell=k+1}^m \quad \theta_0 \\ (\tilde{t}_\ell)_{\ell=1}^{k+1}, \tilde{x}, (\tilde{t}_\ell)_{\ell=k+2}^{m+1} \end{array} \right) \mathcal{B}_{p,j} \left[\begin{array}{cc} \theta_{k+1} & \frac{1}{N} \\ \sigma_k - h_N + n_k & \sigma_{k+1} + n_{k+1} \end{array} \middle| \frac{t_{k+1}}{q^N \sum_{i=1}^{k+1} \theta_i x} \right] \\ &\times q^{N\theta_{k+1}^2/2 - \theta_{k+1}/2} \left(\frac{t_{k+1}}{q^N \sum_{i=1}^{k+1} \theta_i x} \right)^{\theta_{k+1}}. \end{aligned}$$

By the periodicity (2.4), (2.6) of the connection matrix \mathcal{B} and the definition (3.1) of the function $\tau_i^{(k,k+1)}(t)$, we obtain

$$\begin{aligned} Y_{i,j}^{(k,k+1)}(x,t) &= \frac{x^{-\sum_{\ell=1}^{k+1} \theta_\ell}}{\tau_i^{(k+1,k+2)}(t)} \sum_{p=1}^N \mathcal{B}_{p,j} \left[\begin{array}{cc} \theta_{k+1} & \frac{1}{N} \\ \sigma_k - h_N & \sigma_{k+1} \end{array} \middle| \frac{t_{k+1}}{q^N \sum_{i=1}^{k+1} \theta_i x} \right] \sum_{n \in R^m} s^n s_k^{h_j - h_N} \\ &\times \mathcal{F} \left(\begin{array}{ccc} (\theta_\ell)_{\ell=1}^{k+1} & \frac{1}{N} (\theta_\ell)_{\ell=k+2}^{m+1} & \\ \theta_\infty - h_i & (\sigma_\ell - h_N + n_\ell)_{\ell=1}^k & \sigma_{k+1} + n_{k+1} - h_p \quad (\sigma_\ell + n_\ell)_{\ell=k+1}^m \quad \theta_0 \\ (\tilde{t}_\ell)_{\ell=1}^{k+1}, \tilde{x}, (\tilde{t}_\ell)_{\ell=k+2}^{m+1} \end{array} \right). \end{aligned}$$

Hence, we have

$$Y_{i,j}^{(k,k+1)}(x,t) = s_k^{h_j - h_N} \sum_{p=1}^N Y_{i,p}^{(k+1,k+2)}(x,t) \mathcal{B}_{p,j} \left[\begin{array}{c|c} \theta_{k+1} & \frac{1}{N} \\ \sigma_k - h_N & \sigma_{k+1} \end{array} \middle| \frac{t_{k+1}}{q^N \sum_{i=1}^{k+1} \theta_i x} \right].$$

Therefore we obtain for $k = 0, 1, \dots, m+1$

$$\begin{aligned} Y^{(k,k+1)}(x,t) &= Y^{(k+1,k+2)}(x,t) \mathcal{B} \left[\begin{array}{c|c} \theta_{k+1} & \frac{1}{N} \\ \sigma_k - h_N & \sigma_{k+1} \end{array} \middle| \frac{t_{k+1}}{q^N \sum_{i=1}^{k+1} \theta_i x} \right] \\ &\quad \times \text{diag} (s_k^{h_1 - h_N}, \dots, s_k^{h_{N-1} - h_N}, 1), \end{aligned} \quad (3.11)$$

where $\sigma_0 = \theta_\infty + h_N$, $\sigma_{m+1} = \theta_0$, $t_0 = 1$, and $s_0 = 1$.

Put

$$A(x,t) = T_{q,x}(Y^{(0,1)}(x,t))Y^{(0,1)}(x,t)^{-1}, \quad (3.12)$$

$$B_i(x,t) = T_{q,t_i}(Y^{(0,1)}(x,t))Y^{(0,1)}(x,t)^{-1}, \quad i = 1, \dots, m+1. \quad (3.13)$$

Since all connection matrices are independent of the q -shifts with respect to the variables x and t_i ($i = 1, \dots, m+1$) by (2.3), from (3.1) the matrix functions $A(x,t)$, $B_i(x,t)$ of x are analytically continued on \mathbb{P}^1 as

$$\begin{aligned} A(x,t) &= T_{q,x}(Y^{(0,1)}(x,t))Y^{(0,1)}(x,t)^{-1} = \dots = T_{q,x}(Y^{(m+1,m+2)}(x,t))Y^{(m+1,m+2)}(x,t)^{-1}, \\ B_i(x,t) &= T_{q,t_i}(Y^{(0,1)}(x,t))Y^{(0,1)}(x,t)^{-1} = \dots = T_{q,t_i}(Y^{(m+1,m+2)}(x,t))Y^{(m+1,m+2)}(x,t)^{-1}. \end{aligned}$$

Therefore by the assumption on holomorphicity of $\widehat{Y}^{k,k+1}(x,t)$, the functions $A(x,t)$, $B_i(x,t)$ of x are meromorphic functions of x on \mathbb{P}^1 , namely, rational functions.

As a result, the matrices $Y^{(k,k+1)}(x,t)$ satisfy the q -linear difference system

$$T_{q,x}(Y(x,t)) = A(x,t)Y(x,t), \quad T_{q,t_i}(Y(x,t)) = B_i(x,t)Y(x,t), \quad i = 1, \dots, m+1,$$

with $N \times N$ matrix rational functions $A(x,t)$, $B_i(x,t)$ of x .

The forms of $A(x,t)$ and $B_i(x,t)$ can be deduced from $\det Y^{(0,1)}(x,t)$. The connection formula (3.1) yields

$$\begin{aligned} &\prod_{i=0}^m \frac{(q^{-N \sum_{j=1}^i \theta_j t_{i+1}/x}; q)_\infty}{(q^{-N \sum_{j=1}^{i+1} \theta_j t_{i+1}/x}; q)_\infty} \det Y^{(0,1)}(x,t) \\ &= q^{N \sum_{i=1}^k \theta_i (N \sum_{i=1}^k \theta_i + 1)/2} x^{N \sum_{i=1}^k \theta_i} \prod_{i=0}^{k-1} s_i^{\sum_{j=1}^{N-1} (h_j - h_N)} t_{i+1}^{-N \theta_{i+1}} \prod_{1 \leq i < j \leq N} \frac{\vartheta(\theta_\infty^{(i)} - \theta_\infty^{(j)})}{\vartheta(\theta_k^{(i)} - \theta_k^{(j)})} \\ &\quad \times \prod_{i=k}^m \frac{(q^{-N \sum_{j=1}^i \theta_j t_{i+1}/x}; q)_\infty}{(q^{-N \sum_{j=1}^{i+1} \theta_j t_{i+1}/x}; q)_\infty} \prod_{i=0}^{k-1} \frac{(q^{1+N \sum_{j=1}^{i+1} \theta_j x/t_{i+1}}; q)_\infty}{(q^{1+N \sum_{j=1}^i \theta_j x/t_{i+1}}; q)_\infty} \det Y^{(k,k+1)}(x,t), \end{aligned} \quad (3.14)$$

where $k = 1, \dots, m+1$. The asymptotic behaviours (3.2), (3.3), (3.4) imply that (3.14) is a meromorphic function of x on \mathbb{P}^1 . Since $\widehat{Y}^{(k,k+1)}(x,t)$ is holomorphic in the domain $|q^{-N \sum_{i=1}^k \theta_i t_{k+1}}| < |x| < |q^{-1-N \sum_{i=1}^{k-1} \theta_i t_k}|$ from the assumption (3.5), the meromorphic function (3.14) of x is holomorphic on \mathbb{P}^1 due to the condition (3.7). Hence, it is a constant with respect to x , and therefore, using (3.2) again, we obtain

$$\det Y^{(0,1)}(x,t) = \prod_{k=0}^m \frac{(q^{-N \sum_{i=1}^{k+1} \theta_i t_{k+1}/x}; q)_\infty}{(q^{-N \sum_{i=1}^k \theta_i t_{k+1}/x}; q)_\infty}. \quad (3.15)$$

Then by the definitions (3.12), (3.13) we obtain

$$\det A(x, t) = \prod_{k=1}^{m+1} \frac{x - q^{-1-N} \sum_{i=1}^k \theta_i t_k}{x - q^{-1-N} \sum_{i=1}^{k-1} \theta_i t_k},$$

$$\det B_i(x, t) = \frac{x - q^{-N} \sum_{j=1}^{i-1} \theta_j t_i}{x - q^{-N} \sum_{j=1}^i \theta_j t_i}, \quad i = 1, \dots, m+1.$$

We recall that for each $k = 0, 1, \dots, m+1$, we have

$$A(x, t) = T_{q,x}(Y^{(k,k+1)}(x, t))Y^{(k,k+1)}(x, t)^{-1}.$$

from the definitions (3.12), (3.13) and the connection relations (3.1). Hence, in the domain

$$|q^{-1-N} \sum_{i=1}^k \theta_i t_{k+1}| < |x| < |q^{-1-N} \sum_{i=1}^k \theta_i t_k|,$$

possible poles of $A(x, t)$ are the zeros of $\det(Y^{(k,k+1)}(x, t))$. From (3.14) we can read where the zeros of $\det(Y^{(k,k+1)}(x, t))$ is. The conditions (3.6) and (3.7) imply that $A(x, t)$ only has simple poles at $x = q^{-1-N} \sum_{i=1}^{k-1} \theta_i t_k$, $k = 1, \dots, m+1$.

Therefore, together with the asymptotic behaviour (3.2), we have

$$A = \frac{A_{m+1} + \sum_{k=0}^m A_k(t)x^k}{\prod_{k=1}^{m+1} (x - q^{-1-N} \sum_{j=1}^{k-1} \theta_j t_k)}.$$

The q -difference equation $Y(qx) = A(x)Y(x)$ entails that possible (simple) poles of the analytic continuations of $Y^{(0,1)}(x, t)^{\pm 1}$, $Y^{(m+1,m+2)}(x, t)^{\pm 1}$ are

$$Y^{(0,1)}(x, t): q^{\ell-N} \sum_{j=1}^{k-1} t_k, \quad Y^{(0,1)}(x, t)^{-1}: q^{\ell-N} \sum_{j=1}^{k-1} t_k, q^{\ell-N} \sum_{j=1}^k t_k,$$

$$Y^{(m+1,m+2)}(x, t): q^{-\ell-1-N} \sum_{j=1}^{k-1} t_k, q^{-\ell-1-N} \sum_{j=1}^k t_k,$$

$$Y^{(m+1,m+2)}(x, t)^{-1}: q^{-\ell-1-N} \sum_{j=1}^{k-1} t_k,$$

where $\ell \in \mathbb{Z}_{\geq 0}$, $k = 1, \dots, m+1$. Hence, since

$$B_i(x, t) = T_{q,t_i}(Y^{(0,1)}(x, t))Y^{(0,1)}(x, t)^{-1} = T_{q,t_i}(Y^{(m+1,m+2)}(x, t))Y^{(m+1,m+2)}(x, t)^{-1},$$

$B_i(x, t)$ only has a simple pole at $x = q^{-N} \sum_{j=1}^i \theta_j t_i$. Therefore, we obtain (3.10). \blacksquare

The compatibility conditions

$$T_{q,t_i}(A(x, t))B_i(x, t) = B_i(qx, t)A(x, t), \quad i = 1, \dots, m+1, \quad (3.16)$$

of the Lax form (3.8) with rational functions $A(x, t)$ and $B_i(x, t)$ satisfying (3.9) and (3.10) yield nonlinear q -difference system satisfied by the elements of $A(x, t)$, $B_i(x, t)$ as functions of the variables t 's with the parameters θ 's. When $m = 1$ and $N = 2$, the derived nonlinear q -difference system is the q -difference VI equation [21]. Fortunately, the dependent variables of q -P_{VI} are simply written by the elements of $A(x, t)$ and $B_2(x, t)$ and moreover they are expressed in terms of Fourier transforms of q -conformal blocks, namely, tau functions [20]. However, in the general case, nonlinear q -difference systems derived from compatibility conditions are complicated and we have not found a way to nicely write down equation satisfied by the elements of $A(x, t)$ and $B_i(x, t)$. We refer to [32] where the author investigated the $N = 2$ case and obtained a q -analog of the Garnier system, see also [24, 25, 26, 31]. We also refer to [36] where the authors

verified that the Lax form of a generalization q - $P_{(n,n)}$ of q - P_{VI} obtained from the Drinfeld–Sokolov hierarchy is transformed to our Lax form (3.8) for $m = 1$ by q -Laplace transformations [36, Theorem 3.6].

The rational function $B_i(x, t)$ can be computed from the fundamental solution $Y^{(k,k+1)}(x, t)$. In particular, $B_{i,0}(t)$ has two expressions

$$B_{i,0}(t) = -q^{-N \sum_{j=1}^i \theta_j} t_i I + T_{q,t_i}(Y_1(t)) - Y_1(t) = -q^{-N \sum_{j=1}^i \theta_j} t_i T_{q,t_i}(G(t))G(t)^{-1}, \quad (3.17)$$

derived from the asymptotic behaviours of the fundamental solution around 0 and ∞ . Recall that $Y_1(t)$, $G(t)$ are defined by the asymptotic expansions (3.2) and (3.3). This produces non-trivial relations among elements of $Y_1(t)$ and $G(t)$, which can be viewed as Fourier transforms of $m + 3$ -point q -conformal blocks.

We define the tau function by

$$\begin{aligned} \tau[\theta|s, \sigma, t] &= q^{-N \Delta_{\theta_\infty} \sum_{k=1}^{m+1} \theta_k + N \sum_{k=1}^m \Delta_{N\theta_k h_1} \sum_{j=k+1}^{m+1} \theta_j} \\ &\quad \times \prod_{1 \leq k < k' \leq N} G_q(1 + \theta_\infty^{(k)} - \theta_\infty^{(k')}) G_q(1 - \theta_0^{(k)} + \theta_0^{(k')}) \\ &\quad \times \sum_{n \in R^m} s^n \mathcal{F} \left(\begin{matrix} (\theta_p)_{p=1}^{m+1} \\ \theta_\infty & (\sigma_p + n_p)_{p=1}^m & \theta_0 \end{matrix}; (\tilde{t}_p)_{p=1}^{m+1} \right). \end{aligned}$$

Then

$$\tau[\theta|s, \sigma, t] = \sum_{n \in R^m} s^n \prod_{p=1}^{m+1} t_p^{\Delta_{\sigma_{p-1} + n_{p-1}} - \Delta_{N\theta_p h_1} - \Delta_{\sigma_p + n_p}} C[\theta|\sigma + n] Z[\theta|\sigma + n, t]$$

with the definition

$$\begin{aligned} C[\theta|\sigma] &= \frac{\prod_{p=1}^{m+1} \prod_{k,k'=1}^N G_q(1 + \sigma_{p-1}^{(k)} - \theta_p - \sigma_p^{(k')})}{\prod_{p=1}^m \prod_{k,k'=1}^N G_q(1 + \sigma_p^{(k)} - \sigma_p^{(k')})}, \\ Z[\theta|\sigma, t] &= \sum_{\lambda_1, \dots, \lambda_m \in \mathbb{Y}^N} \prod_{p=1}^m \binom{t_{p+1}}{t_p}^{|\lambda_p|} \frac{\prod_{p=1}^{m+1} \prod_{k,k'=1}^N N_{\lambda_{p-1}^{(k)}, \lambda_p^{(k')}}(q^{\sigma_{p-1}^{(k)} - \theta_p - \sigma_p^{(k')}})}{\prod_{p=1}^m \prod_{k,k'=1}^N N_{\lambda_p^{(k)}, \lambda_p^{(k')}}(q^{\sigma_p^{(k)} - \sigma_p^{(k')}})}. \end{aligned}$$

We note that the normalization factors $\tau_i^{(0,1)}(t)$, $i = 1, \dots, N$, are related to the tau function as

$$\begin{aligned} \tau_i^{(0,1)}(t) &= q^{N \Delta_{\theta_\infty} \sum_{k=1}^{m+1} \theta_k - N \sum_{k=1}^m \Delta_{N\theta_k h_1} \sum_{j=k+1}^{m+1} \theta_j + \Delta_{\theta_\infty - h_i}} \\ &\quad \times \left(\prod_{1 \leq k < k' \leq N} G_q(1 + \theta_\infty^{(k)} - \theta_\infty^{(k')}) G_q(1 - \theta_0^{(k)} + \theta_0^{(k')}) \right)^{-1} \\ &\quad \times \mathcal{N}' \left(\begin{matrix} \frac{1}{N} \\ \theta_\infty - h_i & \theta_\infty \end{matrix} \right) \tau[\theta|s, \sigma, t]. \end{aligned}$$

Let us introduce other tau functions by

$$\begin{aligned} \tau &= \tau[\theta|s, \sigma, t], \quad \tau_{ij} = \tau \left[(\theta_\infty - h_i + h_j, (\theta_p)_{p=1}^{m+2}) | s, \sigma, t \right], \\ \tilde{\tau}_{ij} &= \tau \left[(\theta_\infty - h_i, (\theta_p)_{p=1}^{m+1}, \theta_0 - h_j) | s, \sigma - h_N, t \right] \end{aligned}$$

for $i, j = 1, \dots, N$.

Proposition 3.2. For $p = 1, \dots, m+1$ and $i, j = 1, \dots, N$, $i \neq j$, we have the determinantal identity

$$T_{q,t_p} \left(\frac{\tau_{ij}}{\tau} \right) - \frac{\tau_{ij}}{\tau} = \frac{q^{-N \sum_{k=1}^p \theta_k + \theta_p t_p} D_{ij}}{T_{q,t_p}(\tau) \tau^{N-1}} \begin{vmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \cdots & \tilde{\tau}_{1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{j-11} & \tilde{\tau}_{j-12} & \cdots & \tilde{\tau}_{j-1N} \\ T_{q,t_p}(\tilde{\tau}_{i1}) & T_{q,t_p}(\tilde{\tau}_{i2}) & \cdots & T_{q,t_p}(\tilde{\tau}_{iN}) \\ \tilde{\tau}_{j+11} & \tilde{\tau}_{j+12} & \cdots & \tilde{\tau}_{j+1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix}, \quad (3.18)$$

where

$$D_{ij} = (-1)^{(m+1)(N-1)+1} q^{N(N-1) \sum_{k=1}^{m+1} \theta_k / 2 + \theta_\infty^{(i)} - \theta_\infty^{(j)} + \sum_{k=1}^N k(\theta_\infty^{(k)} - \theta_0^{(k)})} [1 - \theta_\infty^{(i)} + \theta_\infty^{(j)}] \\ \times \prod_{\substack{k=1, \\ k \neq j}}^N [\theta_\infty^{(j)} - \theta_\infty^{(k)}] \prod_{k < k'} \frac{1}{[\theta_\infty^{(k')} - \theta_\infty^{(k)}] [\theta_0^{(k)} - \theta_0^{(k')}]} s \prod_{k=1}^m s_{k,N}^{-N}.$$

Proof. By the relation (3.17), the (i, j) entry of $B_{p,0}(t)$ for $i \neq j$ satisfies

$$T_{q,t_p} (Y_1(t)_{ij}) - Y_1(t)_{ij} = -\frac{q^{-N \sum_{k=1}^p \theta_k t_p}}{\det G(t)} \sum_{k=1}^N T_{q,t_p} (G(t)_{ik}) \widetilde{G(t)}_{jk}, \quad (3.19)$$

where $\widetilde{G(t)}$ is the cofactor matrix of $G(t)$.

By the definition of $Y_1(t)$ and $G(t)$ associated with $Y^{(0,1)}(x, t)$ and $Y^{(m+1, m+2)}(x, t)$, we have for $i \neq j$

$$(Y_1(t))_{ij} = \frac{1}{[1 - \theta_\infty^{(i)} + \theta_\infty^{(j)}]} \prod_{k=1, k \neq i}^N \Gamma_q(\theta_\infty^{(i)} - \theta_\infty^{(k)}) \prod_{k=1}^N \frac{1}{\Gamma_q(1 + \theta_\infty^{(j)} - \theta_\infty^{(k)})} \frac{\tau_{ij}}{\tau}, \quad (3.20)$$

$$G(t)_{ij} = q^{C_{ij}} \prod_{k=1}^{m+1} t_k^{\theta_k} \prod_{k=1, k \neq i}^N \Gamma_q(\theta_\infty^{(i)} - \theta_\infty^{(k)}) \prod_{k=1}^N \frac{1}{\Gamma_q(1 + \theta_0^{(k)} - \theta_0^{(j)})} \frac{\tilde{\tau}_{ij}}{\tau}, \quad (3.21)$$

where

$$C_{ij} = \Delta_{\theta_0} - \Delta_{\theta_\infty} - \theta_0^{(j)} + \theta_\infty^{(i)} - \frac{N}{2} \left(\sum_{k=1}^{m+1} \theta_k \right)^2 + \left(\frac{N}{2} - 1 - N\theta_0^{(j)} \right) \sum_{k=1}^{m+1} \theta_k.$$

Substituting (3.20) and (3.21) into (3.19) together with the expression

$$\det G(t)^{-1} = (-1)^{(m+1)(N-1)} q^{N \sum_{i=1}^{m+1} \theta_i (N \sum_{i=1}^{m+1} \theta_i + 1) / 2} \\ \times \prod_{1 \leq i < j \leq N} \frac{\vartheta(\theta_\infty^{(i)} - \theta_\infty^{(j)})}{\vartheta(\theta_0^{(i)} - \theta_0^{(j)})} \prod_{k=0}^m s_k^{\sum_{i=1}^{N-1} (h_i - h_N)} t_{k+1}^{-N\theta_{k+1}}$$

of $\det(G(t))$, we obtain the formula (3.18). ■

3.2 Schlesinger transformations

Integer shifts on the characteristic exponents of Fuchsian systems preserving their monodromy are called the Schlesinger transformations [33]. In [19], the authors studied the Schlesinger transformations of monodromy preserving deformation including irregular singular case and

showed that the ratio τ'/τ is a determinant of a matrix whose elements are given in terms of the coefficients of fundamental solutions, where τ is the Jimbo–Miwa–Ueno tau function introduced in [18] and τ' is a Schlesinger transform of the original tau function. We also mention a recent work [17] where the authors presented another determinant formula for the ratio of τ -functions by using relations between Hermite’s two approximation problems and particular Schlesinger transformation.

In our case, the connection matrix has periodicities (2.5) and (2.6) with shifts of the parameters. This fact implies that the following shifts of the parameters

$$\begin{aligned} \theta_\infty &\rightarrow \theta_\infty - h_i, & \theta_0 &\rightarrow \theta_0 - h_j, & 1 \leq i, j \leq n, \\ \theta_i &\rightarrow \theta_i + \frac{1}{N}, & & & 1 \leq i \leq m+1, \end{aligned}$$

can be regarded as q -Schlesinger transformations and yield q -difference linear equations with rational coefficients satisfied by $Y^{(k,k+1)}(x, t)$, $k = 0, 1, \dots, m+1$.

Let transformations r_i , $i = 1, \dots, m+1$, and p on the parameters be defined as follows.

$$\begin{aligned} r_i(\theta_\infty) &= \theta_\infty - h_1, & r_i(\sigma_j) &= \sigma_j - h_1, & j = 1, \dots, i-1, & r_i(\sigma_j) &= \sigma_j, & j = i, \dots, m, \\ r_i(\theta_0) &= \theta_0, & r_i(\theta_j) &= \theta_j + \frac{1}{N}\delta_{i,j}, & j = 1, \dots, m+1, \\ r_i(t_j) &= t_j, & j = 1, \dots, i, & r_i(t_j) &= qt_j, & j = i+1, \dots, m+1, \\ p(\theta_\infty) &= \theta_\infty - h_1, & p(\sigma_j) &= \sigma_j - h_1, & j = 1, \dots, m, & p(\theta_0) &= \theta_0 - h_1. \end{aligned}$$

We show below that from the q -Schlesinger transformations r_i and p , we obtain determinantal identities for the tau functions.

Proposition 3.3. *The functions*

$$(-1)^{N \sum_{j=1}^k \theta_j} Y^{(k,k+1)}(x, t) x^{\theta_\infty^{(N)}}, \quad k = 0, 1, \dots, m+1,$$

satisfy the q -difference linear equation

$$r_i(Y(x, t)) = R_i(x, t)Y(x, t)$$

with

$$\begin{aligned} \det R_i(x, t) &= x - q^{-N \sum_{j=1}^i \theta_j} t_i, & R_i(x, t) &= R_0 x + R_{i,0}(t), & R_0 &= \text{diag}(1, 0, \dots, 0), \\ R_{i,0}(x, t) &= r_i(Y_1(t))R_0 + I - R_0 - R_0 Y_1(t) = -r_i(G(t))G(t)^{-1}. \end{aligned} \quad (3.22)$$

Proof. Put

$$R_i(x, t) = r_i(Y^{(0,1)}(x, t))Y^{(0,1)}(x, t)^{-1}x^{1/N}.$$

Then we have

$$R_i(x, t) = \begin{cases} r_i(Y^{(k,k+1)}(x, t))Y^{(k,k+1)}(x, t)^{-1}x^{1/N}, & k = 1, \dots, i-1, \\ -r_i(Y^{(k,k+1)}(x, t))Y^{(k,k+1)}(x, t)^{-1}x^{1/N}, & k = i, \dots, m+1, \end{cases}$$

due to the connection relations (3.1) and the periodicities (2.5), (2.6) of the connection matrix.

By the definition of q -conformal blocks, $R_i(x, t)$ is a meromorphic function on \mathbb{P}^1 , and hence it is a rational function of x . The determinant of $R_i(x, t)$ is computed from $\det Y^{(0,1)}(x, t)$ (3.15). Then, the form of $R_i(x, t)$ is obtained from the asymptotic expansions of $Y^{(0,1)}(x, t)$ and $Y^{(m+1,m+2)}(x, t)$. ■

Since we can express $R_{i,0}(x, t)$ in two different way (3.22), we obtain the following theorem.

Theorem 3.4. *Let $1 \leq i \leq m + 1$.*

1. *For $2 \leq a, b \leq N$, we have*

$$t_i^{1/N} q^{\theta_\infty^{(1)} - \sum_{j=1}^i \theta_j} \frac{D_{ab}}{r_i(\tau) \tau^{N-1}} \begin{vmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \cdots & \tilde{\tau}_{1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{b-11} & \tilde{\tau}_{b-12} & \cdots & \tilde{\tau}_{b-1N} \\ q^{-\theta_0^{(1)}} r_i(\tilde{\tau}_{a1}) & q^{-\theta_0^{(2)}} r_i(\tilde{\tau}_{a2}) & \cdots & q^{-\theta_0^{(N)}} r_i(\tilde{\tau}_{aN}) \\ \tilde{\tau}_{b+11} & \tilde{\tau}_{b+12} & \cdots & \tilde{\tau}_{b+1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\ = -\delta_{ab} \frac{\prod_{j=1}^N \Gamma_q(1 + \theta_\infty^{(b)} - \theta_\infty^{(j)}) [1 - \theta_\infty^{(a)} + \theta_\infty^{(b)}]}{\prod_{j=1, j \neq a}^N \Gamma_q(\theta_\infty^{(a)} - \theta_\infty^{(j)}) [\theta_\infty^{(a)} - \theta_\infty^{(1)}]}.$$

2. *For $2 \leq b \leq N$, we have*

$$t_i^{1/N} q^{\theta_\infty^{(1)} - 1 - \sum_{j=1}^i \theta_j} \frac{D_{1b}}{r_i(\tau) \tau^{N-1}} \begin{vmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \cdots & \tilde{\tau}_{1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{b-11} & \tilde{\tau}_{b-12} & \cdots & \tilde{\tau}_{b-1N} \\ q^{-\theta_0^{(1)}} r_i(\tilde{\tau}_{11}) & q^{-\theta_0^{(2)}} r_i(\tilde{\tau}_{12}) & \cdots & q^{-\theta_0^{(N)}} r_i(\tilde{\tau}_{1N}) \\ \tilde{\tau}_{b+11} & \tilde{\tau}_{b+12} & \cdots & \tilde{\tau}_{b+1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\ = \frac{\tau_{1b}}{\tau} \prod_{j=2}^N [\theta_\infty^{(1)} - \theta_\infty^{(j)} - 1].$$

3. *For $2 \leq a \leq N$, we have*

$$t_i^{1/N} q^{\theta_\infty^{(1)} - \sum_{j=1}^i \theta_j} \frac{D_{a1}}{r_i(\tau) \tau^{N-1}} \begin{vmatrix} q^{-\theta_0^{(1)}} r_i(\tilde{\tau}_{a1}) & q^{-\theta_0^{(2)}} r_i(\tilde{\tau}_{a2}) & \cdots & q^{-\theta_0^{(N)}} r_i(\tilde{\tau}_{aN}) \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} & \cdots & \tilde{\tau}_{2N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\ = -r_i \left(\frac{\tau_{a1}}{\tau} \right) [1 - \theta_\infty^{(a)} + \theta_\infty^{(1)}] \prod_{j=2, j \neq a}^N [\theta_\infty^{(1)} - \theta_\infty^{(j)}].$$

Similarly to the q -Schlesinger transformation r_i , we have the following proposition and theorem on the q -Schlesinger transformation p .

Proposition 3.5. *The functions $Y^{(k, k+1)}(x, t)$, $k = 0, 1, \dots, m+1$, satisfy the q -difference linear equation*

$$p(Y(x, t)) = P(x, t)Y(x, t)x^{-1/N}$$

with

$$\det P(x, t) = x, \quad P(x, t) = P_0 x + P_{i,0}(t), \quad P_0 = \text{diag}(1, 0, \dots, 0), \\ P_{i,0}(x, t) = p(Y_1(t))P_0 + I - P_0 - P_0 Y_1(t) = p(G(t))G(t)^{-1}.$$

Theorem 3.6.

1. For $2 \leq a, b \leq N$, we have

$$\begin{aligned}
 & q^{\theta_\infty^{(1)} - \theta_0^{(1)} - \sum_{j=1}^N \theta_j} \frac{D_{ab}}{p(\tau)\tau^{N-1}} \begin{vmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \cdots & \tilde{\tau}_{1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{b-11} & \tilde{\tau}_{b-12} & \cdots & \tilde{\tau}_{b-1N} \\ 0 & [\theta_0^{(1)} - \theta_0^{(2)}]p(\tilde{\tau}_{a2}) & \cdots & [\theta_0^{(1)} - \theta_0^{(N)}]p(\tilde{\tau}_{aN}) \\ \tilde{\tau}_{b+11} & \tilde{\tau}_{b+12} & \cdots & \tilde{\tau}_{b+1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\
 &= \delta_{ab} \frac{\prod_{j=1}^N \Gamma_q(1 + \theta_\infty^{(b)} - \theta_\infty^{(j)}) [1 - \theta_\infty^{(a)} + \theta_\infty^{(b)}]}{\prod_{j=1, j \neq a}^N \Gamma_q(\theta_\infty^{(a)} - \theta_\infty^{(j)}) [\theta_\infty^{(a)} - \theta_\infty^{(1)}]}.
 \end{aligned}$$

2. For $2 \leq b \leq N$, we have

$$\begin{aligned}
 & q^{\theta_\infty^{(1)} - \theta_0^{(1)} - 1 - \sum_{j=1}^N \theta_j} \frac{D_{1b}}{p(\tau)\tau^{N-1}} \begin{vmatrix} \tilde{\tau}_{11} & \tilde{\tau}_{12} & \cdots & \tilde{\tau}_{1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{b-11} & \tilde{\tau}_{b-12} & \cdots & \tilde{\tau}_{b-1N} \\ 0 & [\theta_0^{(1)} - \theta_0^{(2)}]p(\tilde{\tau}_{12}) & \cdots & [\theta_0^{(1)} - \theta_0^{(N)}]p(\tilde{\tau}_{1N}) \\ \tilde{\tau}_{b+11} & \tilde{\tau}_{b+12} & \cdots & \tilde{\tau}_{b+1N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\
 &= -\frac{\tau_{1b}}{\tau} \prod_{j=2}^N [\theta_\infty^{(1)} - \theta_\infty^{(j)} - 1].
 \end{aligned}$$

3. For $2 \leq a \leq N$, we have

$$\begin{aligned}
 & q^{\theta_\infty^{(1)} - \theta_0^{(1)} - \sum_{j=1}^N \theta_j} \frac{D_{a1}}{p(\tau)\tau^{N-1}} \begin{vmatrix} 0 & [\theta_0^{(1)} - \theta_0^{(2)}]p(\tilde{\tau}_{a2}) & \cdots & [\theta_0^{(1)} - \theta_0^{(N)}]p(\tilde{\tau}_{aN}) \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} & \cdots & \tilde{\tau}_{2N} \\ \vdots & \vdots & & \vdots \\ \tilde{\tau}_{N1} & \tilde{\tau}_{N2} & \cdots & \tilde{\tau}_{NN} \end{vmatrix} \\
 &= p\left(\frac{\tau_{a1}}{\tau}\right) [1 - \theta_\infty^{(a)} + \theta_\infty^{(1)}] \prod_{j=2, j \neq a}^N [\theta_\infty^{(1)} - \theta_\infty^{(j)}].
 \end{aligned}$$

Remark 3.7. In the case of $N = 2$, the determinantal identities above become bilinear equations satisfied by the tau functions. On the other hand, in the previous work [20], we gave conjectural bilinear equations of the tau functions for $N = 2$ and $m = 1$. They are not equal to, and it is not clear whether the bilinear equations obtained in this paper recover the conjectural bilinear equations (equations (3.16)–(3.23) in [20]).

Remark 3.8. When $N = 2$ and $m > 1$, the compatibility condition (3.16) of the Lax pair yields the q -Garnier system. For its $q \rightarrow 1$ limit, namely, the Garnier system, bilinear equations of the tau functions were presented in [34]. It is easily seen from the transformation of the parameters by r_i that the bilinear equations in Theorem 3.4(2), (3) are q -analogs of (part of) the Hirota–Miwa equations (equations (3.47) in [34]).

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