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Computing Regular Meromorphic Differential Forms via Saito's Logarithmic Residues

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Abstract. Logarithmic differential forms and logarithmic vector fields associated to a hypersurface with an isolated singularity are considered in the context of computational complex analysis. As applications, based on the concept of torsion differential forms due to A.G. Aleksandrov, regular meromorphic differential forms introduced by D. Barlet and M. Kersken, and Brieskorn formulae on Gauss–Manin connections are investigated. (i) A method is given to describe singular parts of regular meromorphic differential forms in terms of non-trivial logarithmic vector fields via Saito's logarithmic residues. The resulting algorithm is illustrated by using examples. (ii) A new link between Brieskorn formulae and logarithmic vector fields is discovered and an expression that rewrites Brieskorn formulae in terms of non-trivial logarithmic vector fields is presented. A new effective method is described to compute non trivial logarithmic vector fields which are suitable for the computation of Gauss–Manin connections. Some examples are given for illustration.

Key words: logarithmic vector field; logarithmic residue; torsion module; local cohomology

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Dedicated to Kyoji Saito
on the occasion of his 77^{th} *birthday*

1 Introduction

In 1975, K. Saito introduced, with deep insight, the concept of logarithmic differential forms and that of logarithmic vector fields and studied Gauss–Manin connection associated with the versal deformations of hypersurface singularities of type A_2 and A_3 as applications. These results were published in [33]. He developed the theory of logarithmic differential forms, logarithmic vector fields and the theory of residues and published in 1980 a landmark paper [34]. One of the motivations of his study, as he himself wrote in [34], came from the study of Gauss–Manin connections [5, 32]. Another motivation came from the importance of these concepts he realized. Notably the logarithmic residue, interpreted as a meromorphic differential form on a divisor, is regarded as a natural generalization of the classical Poincaré residue to the singular cases.

In 1990, A.G. Aleksandrov [2] studied Saito theory and gave in particular a characterization of the image of the residue map. He showed that the image sheaf of the logarithmic residues coincides with the sheaf of regular meromorphic differential forms introduced by D. Barlet [5]

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and M. Kersken [15, 16]. We refer the reader to [4, 8, 9, 10, 12, 29, 30] for more recent results on logarithmic residues.

We consider logarithmic differential forms along a hypersurface with an isolated singularity in the context of computational complex analysis. In our previous paper [40], we study torsion modules and give an effective method for computing them. In the present paper, we first consider a method for computing regular meromorphic differential forms. We show that, based on the result of A.G. Aleksandrov mentioned above, representatives of regular meromorphic differential forms can be computed by adapting the method presented in [40] on torsion modules. Main ideas of our approach are the use of the concept of logarithmic residues and that of logarithmic vector fields. Next, we discuss a relation between logarithmic differential forms and Brieskorn formulae [5, 35, 37] and we show that Brieskorn formulae can be rewritten in terms of logarithmic vector fields. Applications to the computation of Gauss–Manin connections are illustrated by using examples.

In Section 2, we briefly recall some basics on logarithmic differential forms, logarithmic residues, Barlet sheaf and torsion differential forms. In Section 3, we first recall the notion of logarithmic vector fields and a result gave in [40] to show that torsion differential forms can be described in terms of non trivial logarithmic vector fields. Next, we recall our previous results to show that non-trivial logarithmic vector fields can be computed by using a polar method and local cohomology. Lastly in Section 3, we present Theorem 3.11 which say that regular meromorphic differential forms can be explicitly computed by modifying our previous algorithm on torsion differential forms. In Section 4, we give some examples to illustrate the proposed method of computing non-trivial logarithmic vector fields and regular meromorphic differential forms. In Section 5, we consider Brieskorn formulae on Gauss–Manin connections. We show that Brieskorn formulae described in terms of logarithmic differential forms can be rewritten in terms of non-trivial logarithmic vector fields. We give a new method for computing non-trivial logarithmic vector fields which is suitable in use to compute a connection matrix of Gauss–Manin connections. Finally, we show that the use of integral dependence relations provides a new effective tool for computing saturations of Gauss–Manin connection.

2 Logarithmic differential forms and residues

In this section, we briefly recall the concept of logarithmic differential forms and that of logarithmic residues and fix notation. We refer the reader to [34] for details. Next we recall the result of A.G. Aleksandrov on regular meromorphic differential forms. Then, we recall a result of G.-M. Greuel on torsion modules.

Let X be an open neighborhood of the origin O in $\mathbb{C}^n$. Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions and $\mathcal{O}_{X,O}$ the stalk at O of the sheaf $\mathcal{O}_X$.

2.1 Logarithmic residues

Let f be a holomorphic function defined on X. Let $S = \{x \in X \mid f(x) = 0\}$ denote the hypersurface defined by f.

Definition 2.1. Let ω be a meromorphic differential q-form on X, which may have poles only along S. The form ω is a logarithmic differential form along S if it satisfies the following equivalent four conditions:

(i) $f\omega$ and $f\mathrm{d}\omega$ are holomorphic on X.

(ii) $f\omega$ and $\mathrm{d}f \wedge \omega$ are holomorphic on X.

(iii) There exist a holomorphic function $g(x)$ and a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on X, such that:

(a) $\dim_{\mathbb{C}}(S \cap \{x \in X \mid g(x) = 0\}) \leq n-2$,

(b) $g\omega = \dfrac{\mathrm{d}f}{f} \wedge \xi + \eta$.

(iv) There exists an $(n-2)$-dimensional analytic set $A \subset S$ such that the germ of ω at any point $p \in S - A$ belongs to $\frac{\mathrm{d}f}{f} \wedge \Omega_{X,p}^{q-1} + \Omega_{X,p}^{q}$, where $\Omega_{X,p}^{q}$ denotes the module of germs of holomorphic q-forms on X at p.

For the equivalence of the condition above, see [34]. Let $\Omega_X^q(\log S)$ denote the sheaf of logarithmic q-forms along S. Let $\mathcal{M}_S$ be the sheaf on S of meromorphic functions, let Ω_S^q be the sheaf on S of holomorphic q-forms defined to be

$$\Omega_S^q = \Omega_X^q / \big(f\Omega_X^q + \mathrm{d}f \wedge \Omega_X^{q-1}\big).$$

Definition 2.2. The residue map $\mathrm{res}\colon \Omega_X^q(\log S) \longrightarrow \mathcal{M}_S \otimes_{\mathcal{O}_X} \Omega_S^{q-1}$ is defined as follows: For $\omega \in \Omega_S^q(\log S)$, by definition, there exist g, ξ and η such that

(a) $\dim_{\mathbb{C}}(S \cap \{x \in X \mid g(x) = 0\}) \leq n-2$, and

(b) $g\omega = \dfrac{\mathrm{d}f}{f} \wedge \xi + \eta$.

Then the residue of ω is defined to be $\mathrm{res}(\omega) = \frac{\xi}{g}\Big|_S$ in $\mathcal{M}_S \otimes_{\mathcal{O}_X} \Omega_S^{q-1}$.

Note that it is easy to see that the image sheaf of the residue map res of the subsheaf $\frac{\mathrm{d}f}{f} \wedge \Omega_X^{q-1} + \Omega_X^q$ of $\Omega_X^q(\log S)$ is equal to $\Omega_X^{q-1}\Big|_S$:

$$\mathrm{res}\left(\frac{\mathrm{d}f}{f} \wedge \Omega_X^{q-1} + \Omega_X^q\right) = \Omega_X^{q-1}\Big|_S .$$

See also [34] for details on logarithmic residues. The concept of residues for logarithmic differential forms can be actually regarded as a natural generalization of the classical Poincaré residue.

2.2 Barlet sheaf and torsion differential forms

In 1978, by using results of F. El Zein on fundamental classes, D. Barlet introduced in [5] the notion of the sheaf ω_S^q of regular meromorphic differential forms in a quite general setting. He showed that for the case $q = n-1$, the sheaf ω_S^{n-1} coincides with the Grothendieck dualizing sheaf and ω_S^q can also be defined in the following manner.

Definition 2.3. Let S be a hypersurface in $X \subset \mathbb{C}^n$. Let ω_S^{n-1} be the Grothendieck dualizing sheaf $\mathrm{Ext}^1_{\mathcal{O}_X}\big(\mathcal{O}_S, \Omega_X^n\big)$. Then, the sheaf of regular meromorphic differential forms ω_S^q, $q = 0, 1, \ldots, n-2$ on S is defined to be

$$\omega_S^q = \mathrm{Hom}_{\mathcal{O}_S}\big(\Omega_S^{n-1-q}, \omega_S^{n-1}\big).$$

In 1990, A.G. Aleksandrov [2] obtained the following result.

Theorem 2.4. *For any $q \geq 0$, there is an isomorphism of $\mathcal{O}_S$ modules*

$$\mathrm{res}\big(\Omega_X^q(\log S)\big) \cong \omega_S^{q-1}.$$

See [2] or [3] for the proof.

Let $\mathrm{Tor}(\Omega_S^q)$ denote the sheaf of torsion differential q-forms of Ω_S^q.

Example 2.5. Let X be an open neighborhood of the origin O in $\mathbb{C}^2$. Let $f(x,y)=x^2-y^3$ and $S=\{(x,y)\in X \mid f(x,y)=0\}$. Then, for stalk at the origin of the sheaves of logarithmic differential forms, we have

$$\Omega^1_{X,O}(\log S)\cong \mathcal{O}_{X,O}\left(\frac{\mathrm{d}f}{f},\frac{\beta}{f}\right), \qquad \Omega^2_{X,O}(\log S)\cong \mathcal{O}_{X,O}\left(\frac{\mathrm{d}x\wedge \mathrm{d}y}{f}\right),$$

where $\mathcal{O}_{X,O}$ is the stalk at the origin of the sheaf $\mathcal{O}_X$ of holomorphic functions and $\beta=2y\mathrm{d}x-3x\mathrm{d}y$. The differential form β, as an element of $\Omega^1_S=\Omega^1_X/\big(\mathcal{O}_X\mathrm{d}f+f\Omega^1_X\big)$, is a torsion. The differential form $y\beta$ is also a torsion. Since the defining function f is quasi-homogeneous, the dimension of the vector space $\mathrm{Tor}\big(\Omega^1_S\big)$ is equal to the Milnor number $\mu=2$ of S [18, 47]. Therefore we have $\mathrm{Tor}\big(\Omega^1_S\big)\cong\mathcal{O}_{X,O}(\beta)\cong\mathbb{C}(\beta,y\beta)$.

In 1988 [1], A.G. Aleksandrov studied logarithmic differential forms and residues and proved in particular the following.

Theorem 2.6. *Let $S=\{x\in X \mid f(x)=0\}$ be a hypersurface in $X\subset\mathbb{C}^n$. For $q=0,1,\dots,n$, there exists an exact sequence of sheaves of $\mathcal{O}_X$ modules,*

$$0\longrightarrow \frac{\mathrm{d}f}{f}\wedge\Omega^{q-1}_X+\Omega^q_X\longrightarrow\Omega^q_X(\log S)\xrightarrow{\cdot f}\mathrm{Tor}\big(\Omega^q_S\big)\longrightarrow 0.$$

The result above yields the following observation: $\mathrm{Tor}\big(\Omega^q_S\big)$ plays a key role to study the structure of $\mathrm{res}\big(\Omega^q_X(\log S)\big)$.

2.3 Vanishing theorem

In 1975, in his study [13] on Gauss–Manin connections G.-M. Greuel proved the following results on torsion differential forms.

Theorem 2.7. *Let $S=\{x\in X \mid f(x)=0\}$ be a hypersurface in X with an isolated singularity at $O\in\mathbb{C}^n$. Then,*

- *(i)* $\mathrm{Tor}\big(\Omega^q_S\big)=0$, $q=0,1,\dots,n-2$.
- *(ii)* $\mathrm{Tor}\big(\Omega^{n-1}_S\big)$ *is a skyscraper sheaf supported at the origin O.*
- *(iii) The dimension, as a vector space over $\mathbb{C}$, of the torsion module* $\mathrm{Tor}\big(\Omega^{n-1}_S\big)$ *is equal to $\tau(f)$, the Tjurina number of the hypersurface S at the origin defined to be*

$$\tau(f)=\dim_{\mathbb{C}}\left(\mathcal{O}_{X,O}\Big/\left(f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\dots,\frac{\partial f}{\partial x_n}\right)\right),$$

where $\big(f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\dots,\frac{\partial f}{\partial x_n}\big)$ is the ideal in $\mathcal{O}_{X,O}$ generated by $f,\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\dots,\frac{\partial f}{\partial x_n}$.

Note that the first result was obtained by U. Vetter in [46] and the last result above is a generalization of a result of O. Zariski [47]. G.-M. Greuel obtained much more general results on torsion modules. See [13, Proposition 1.11, p. 242].

Assume that the hypersurface S has an isolated singularity at the origin. We thus have, by combining the results of G.-M. Greuel above and of A.G. Aleksandrov presented in the previous section, the following:

- *(i)* $\Omega^q_{X,O}(\log S)=\frac{\mathrm{d}f}{f}\wedge\Omega^{q-1}_{X,O}+\Omega^q_{X,O}$, $q=1,2,\dots,n-2$,
- *(ii)* $0\longrightarrow\frac{\mathrm{d}f}{f}\wedge\Omega^{n-2}_{X,O}+\Omega^{n-1}_{X,O}\longrightarrow\Omega^{n-1}_{X,O}(\log S)\xrightarrow{\cdot f}\mathrm{Tor}\big(\Omega^{n-1}_S\big)\longrightarrow 0.$

Accordingly we have the following.

Proposition 2.8. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface in X with an isolated singularity at $O \in \mathbb{C}^n$. Then, $\omega_S^q = \Omega_X^q$, $q = 0, 1, \ldots, n-3$ holds.*

Proof. Since $\mathrm{res}\big(\Omega_X^q(\log S)\big) = \Omega_X^{q-1}\Big|_S$, $q = 1, 2, \ldots, n-2$, the result of A.G. Aleksandrov presented in the last section yields the result. ■

3 Description via logarithmic residues

In this section, we recall results given in [40] to show that torsion differential forms can be described in terms of non-trivial logarithmic vector fields. We also recall basic ideas and the framework for computing non-trivial logarithmic vector fields. As an application, we give a method for computing logarithmic residues.

3.1 Logarithmic vector fields

A vector field v on X with holomorphic coefficients is called logarithmic along the hypersurface S, if the holomorphic function $v(f)$ is in the ideal (f) generated by f in $\mathcal{O}_X$. Let $\mathcal{D}\mathrm{er}_X(-\log S)$ denote the sheaf of modules on X of logarithmic vector fields along S [34].

Let $\omega_X = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$. For a holomorphic vector field v, let $i_v(\omega_X)$ denote the inner product of ω_X by v.

Proposition 3.1. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin. Then, $\Omega_{X,O}^{n-1}(\log S)$ is isomorphic to $\mathcal{D}\mathrm{er}_{X,O}(-\log S)$, more precisely*

$$\Omega_{X,O}^{n-1}(\log S) = \left\{ \frac{i_v(\omega_X)}{f} \,\middle|\, v \in \mathcal{D}\mathrm{er}_{X,O}(-\log S) \right\}$$

holds.

Proof. Let $\beta = i_v(\omega_X)$, and set $\omega = \frac{\beta}{f}$. Then, $f\omega = \beta$ is a holomorphic differential form. Therefore, the meromorphic differential $n-1$ form ω is logarithmic if and only if $\mathrm{d}f \wedge \frac{\beta}{f}$ is a holomorphic differential n-form. Since $\mathrm{d}f \wedge \beta = \mathrm{d}f \wedge i_v(\omega_X) = v(f)\omega_X$, we have $\mathrm{d}f \wedge \frac{\beta}{f} = \frac{v(f)}{f}\omega_X$. Hence, the condition above means $v(f)$ is in the ideal $(f) \subset \mathcal{O}_{X,O}$ generated by f. This completes the proof. ■

A germ of logarithmic vector field v generated over $\mathcal{O}_{X,O}$ by

$$f\frac{\partial}{\partial x_i}, \qquad i = 1, 2, \ldots, n, \qquad \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}, \qquad 1 \leq i < j \leq n,$$

is called trivial.

Lemma 3.2. *Let v be a germ of a logarithmic vector field. Then, the following conditions are equivalent:*

(i) $\omega = \dfrac{i_v(\omega_X)}{f}$ *belongs to* $\dfrac{\mathrm{d}f}{f} \wedge \Omega_{X,O}^{n-2} + \Omega_{X,O}^{n-1}$,

(ii) *v is a trivial vector field.*

Proof. The logarithmic differential form $\omega = \frac{i_v(\omega_X)}{f}$ is in $\Omega_{X,O}^{n-1} + \frac{\mathrm{d}f}{f} \wedge \Omega_{X,O}^{n-2}$ if and only if the numerator $i_v(\omega_X)$ is in $f\Omega_{X,O}^{n-1} + \mathrm{d}f \wedge \Omega_{X,O}^{n-2}$. The last condition is equivalent to the triviality of the vector field v, which completes the proof. ■

For $\beta \in \Omega^{n-1}_{X,O}$, let $[\beta]$ denote the Kähler differential form in $\Omega^{n-1}_{S,O}$ defined by β, that is, $[\beta]$ is the equivalence class in $\Omega^{n-1}_{X,O}/\big(f\Omega^{n-1}_{X,O} + \mathrm{d}f \wedge \Omega^{n-2}_{X,O}\big)$ of β.

The lemma above amount to say that, for logarithmic vector fields v, $[i_v(\omega_X)]$ is a non-zero torsion differential form in $\mathrm{Tor}\big(\Omega^{n-1}_{S,O}\big)$ if and only if v is a non-trivial logarithmic vector field.

We say that germs of two logarithmic vector fields $v, v' \in \mathcal{D}\mathrm{er}_{X,O}(-\log S)$ are equivalent, denoted by $v \sim v'$, if $v - v'$ is trivial. Let $\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\!\sim$ denote the quotient by the equivalence relation $\sim$. (See [39].)

Now consider the following map

$$\Theta\colon\ \mathcal{D}\mathrm{er}_{X,O}(-\log S)/\!\sim\ \longrightarrow \Omega^{n-1}_{X,O}/\big(f\Omega^{n-1}_{X,O} + \mathrm{d}f \wedge \Omega^{n-2}_{X,O}\big)$$

defined to be $\Theta([v]) = [i_v(\omega_X)]$, where $[v]$ is the equivalence class in $\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\!\sim$ of v. It is easy to see that the map Θ is well-defined. We arrive at the following description of the torsion module.

Theorem 3.3 ([40])**.** *The map*

$$\Theta\colon\ \mathcal{D}\mathrm{er}_{X,O}(-\log S)/\!\sim\ \longrightarrow \mathrm{Tor}\big(\Omega^{n-1}_{S}\big)$$

is an isomorphism.

3.2 Polar method

In [39], based on the concept of polar variety, logarithmic vector fields are studied and an effective and constructive method is considered. Here in this section, following [27, 39] we recall some basics and give a description of non-trivial logarithmic vector fields.

Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity. In what follows, we assume that $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ is a regular sequence and the common locus $V\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big) \cap X$ is the origin O. See [19] for an algorithm of testing zero-dimensionality of varieties at a point.

Let $\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big) : \big(\frac{\partial f}{\partial x_1}\big)$ denote the ideal quotient, in the local ring $\mathcal{O}_{X,O}$, of $\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big)$ by $\big(\frac{\partial f}{\partial x_1}\big)$. We have the following.

Lemma 3.4. *Let $a(x)$ be a germ of holomorphic function in $\mathcal{O}_{X,O}$. Then, the following are equivalent:*

(*i*) $a(x) \in \left(f, \dfrac{\partial f}{\partial x_2}, \dfrac{\partial f}{\partial x_3}, \ldots, \dfrac{\partial f}{\partial x_n}\right) : \left(\dfrac{\partial f}{\partial x_1}\right).$

(*ii*) *There exists a germ of logarithmic vector field v in $\mathcal{D}\mathrm{er}_{X,O}(-\log S)$ such that*

$$v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_{n-1}(x)\frac{\partial}{\partial x_{n-1}} + a_n(x)\frac{\partial}{\partial x_n},$$

where $a_2(x), \ldots, a_n(x) \in \mathcal{O}_{X,O}$.

Note that in [24, 27], by utilizing local cohomology and Grothendieck local duality, an effective method of computing a set of generators over the local ring $\mathcal{O}_{X,O}$ of the module of logarithmic vector fields is given. See the next section.

Lemma 3.5. *Assume that $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ is a regular sequence. Let v' be a logarithmic vector fields in $\mathcal{D}\mathrm{er}_{X,O}(-\log S)$ of the form*

$$v' = a_2(x)\frac{\partial}{\partial x_2} + a_3(x)\frac{\partial}{\partial x_3} + \cdots + a_n(x)\frac{\partial}{\partial x_n}.$$

Then, v' is trivial.

Lemmas 3.4 and 3.5 immediately yield the following.

Proposition 3.6. *Let $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ be a regular sequence. Let v be a germ of logarithmic vector field along S of the form*

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_{n-1}(x)\frac{\partial}{\partial x_{n-1}} + a_n(x)\frac{\partial}{\partial x_n}.$$

Then, the following conditions are equivalent:

(i) v *is trivial,*

(ii) $a_1(x) \in \left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

Therefore, we have the following.

Theorem 3.7 ([39]). *$\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\sim$ is isomorphic to*

$$\left(\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right) : \left(\frac{\partial f}{\partial x_1}\right)\right)\Big/\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

To be more precise, let A be a basis as a vector space of the quotient

$$\left(\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right) : \left(\frac{\partial f}{\partial x_1}\right)\right)\Big/\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Then the corresponding logarithmic vector fields,

$$v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_{n-1}(x)\frac{\partial}{\partial x_{n-1}} + a_n(x)\frac{\partial}{\partial x_n}, \qquad a(x) \in A$$

give rise to a basis of $\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\sim$.

3.3 Local cohomology and duality

In this section, we briefly recall some basics on local cohomology and Grothendieck local duality. We give an outline for computing non-trivial logarithmic vector fields. We refer to [40] for details.

Let $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ denote the local cohomology supported at the origin O of the sheaf Ω^n_X of holomorphic n-forms. Then, the stalk $\mathcal{O}_{X,O}$ and the local cohomology $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ are mutually dual as locally convex topological vector spaces.

The duality is given by the point residue pairing:

$$\mathrm{Res}_{\{O\}}(*,*) \colon\ \mathcal{O}_{X,O} \times \mathcal{H}^n_{\{O\}}(\Omega^n_X) \longrightarrow \mathbb{C}.$$

Let $W_{\Gamma(f)}$ denote the set of local cohomology classes in $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ that are annihilated by f, $\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$:

$$W_{\Gamma(f)} = \left\{\varphi \in \mathcal{H}^n_{\{O\}}(\Omega^n_X) \;\middle|\; f\varphi = \frac{\partial f}{\partial x_2}\varphi = \cdots = \frac{\partial f}{\partial x_n}\varphi = 0\right\}.$$

Then, a complex analytic version of Grothendieck local duality on residue implies that the pairing

$$\mathcal{O}_{X,O}\Big/\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right) \times W_{\Gamma(f)} \longrightarrow \mathbb{C}$$

is non-degenerate.

Let $\mu(f)$ and $\mu(f|_{H_{x_1}})$ denote the Milnor number of f and that of a hyperplane section $f|_{H_{x_1}}$ of f, where $f|_{H_{x_1}}$ is the restriction of f to the hyperplane $H_{x_1} = \{x \in X \mid x_1 = 0\}$. Then, the classical Lê–Teissier formula [17, 43] and the Grothendieck local duality imply the following:

$$\dim_{\mathbb{C}} W_{\Gamma(f)} = \mu(f) + \mu(f|_{H_{x_1}}).$$

Let $\gamma\colon W_{\Gamma(f)} \longrightarrow W_{\Gamma(f)}$ be a map defined by $\gamma(\varphi) = \frac{\partial f}{\partial x_1} * \varphi$ and let $W_{\Gamma(f)}$ be the image of the map γ:

$$W_{\Delta(f)} = \left\{ \frac{\partial f}{\partial x_1} * \varphi \,\middle|\, \varphi \in W_{\Gamma(f)} \right\}.$$

Let $\mathrm{Ann}_{\mathcal{O}_{X,O}}(W_{\Delta(f)})$ be the annihilator in $\mathcal{O}_{X,O}$ of the set $W_{\Delta(f)}$ of local cohomology classes. We have the following.

Lemma 3.8 ([39]). $\mathrm{Ann}_{\mathcal{O}_{X,O}}(W_{\Delta(f)}) = \left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}\right) : \left(\frac{\partial f}{\partial x_1}\right)$.

Proof. See [20, 39, 41]. ∎

Recall that the ideal quotient $\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}\right) : \left(\frac{\partial f}{\partial x_1}\right)$ is coefficient ideal w.r.t. $\frac{\partial}{\partial x_1}$ of logarithmic vector fields along S. The lemma above says that the coefficient ideal can be described in terms of local cohomology $W_{\Delta(f)}$.

Let $W_{T(f)}$ be the kernel of the map γ. By definition we have

$$W_{T(f)} = \left\{ \varphi \in \mathcal{H}^n_{\{O\}}(\Omega^n_X) \,\middle|\, f\varphi = \frac{\partial f}{\partial x_1}\varphi = \frac{\partial f}{\partial x_2}\varphi = \cdots = \frac{\partial f}{\partial x_n}\varphi = 0 \right\}.$$

Since the pairing

$$\mathcal{O}_{X,O} \Big/ \left(f, \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right) \times W_{T(f)} \longrightarrow \mathbb{C}$$

is non-degenerate by Grothendieck local duality, $\dim_{\mathbb{C}}(W_{T(f)})$ is equal to

$$\tau = \dim_{\mathbb{C}} \left(\mathcal{O}_{X,O} \Big/ \left(f, \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right) \right),$$

the Tjurina number.

From the exactness of the sequence

$$0 \longrightarrow W_{T(f)} \longrightarrow W_{\Gamma(f)} \longrightarrow W_{\Delta(f)} \longrightarrow 0,$$

we have

$$\dim_{\mathbb{C}} W_{\Delta(f)} = \mu(f) - \tau(f) + \mu(f|_{H_{x_1}}).$$

The argument above also implies the following.

Corollary 3.9 ([39]).

$$\dim_{\mathbb{C}} \left(\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\sim \right) = \tau.$$

Notice that the dimension of $W_{\Delta(f)}$ that measures the way of vanishing of coefficients of logarithmic vector fields depends on the choice of a system of coordinates, or a hyperplane. In order to analyze complex analytic properties of logarithmic vector fields, as we observed in [39], it is important to select an appropriate system of coordinates or a generic hyperplane. We return to this issue afterwards at the end of this section.

Now let $H^n_{[O]}(\mathcal{O}_X) = \lim_{k\to\infty} \mathrm{Ext}^n_{\mathcal{O}_X}\big(\mathcal{O}_{X,O}/(x_1, x_2, \ldots, x_n)^k, \mathcal{O}_X\big)$ be the sheaf of algebraic local cohomology and let

$$H_{\Gamma(f)} = \left\{ \phi \in H^n_{[O]}(\mathcal{O}_X) \,\middle|\, f\phi = \frac{\partial f}{\partial x_2}\phi = \cdots = \frac{\partial f}{\partial x_n}\phi = 0 \right\},$$
$$H_{\Delta(f)} = \left\{ \frac{\partial f}{\partial x_1}\phi \,\middle|\, \phi \in H_{\Gamma(f)} \right\}.$$

Then, the following holds

$$W_{\Gamma(f)} = \{\phi \cdot \omega_X \,|\, \phi \in H_{\Gamma(f)}\}, \qquad W_{\Delta(f)} = \{\phi \cdot \omega_X \,|\, \phi \in H_{\Delta(f)}\}.$$

In [41], algorithms for computing algebraic local cohomology classes and some relevant algorithms are given. Accordingly, $H_{\Gamma(f)}, H_{\Delta(f)}$ are computable. Note also that a standard basis of the ideal quotient $\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big) : \big(\frac{\partial f}{\partial x_1}\big)$ can be computed by using $H_{\Delta(f)}$ in an efficient manner [41].

Now we present an outline of a method for constructing a basis, as a vector space, of the quotient space $\big(\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big) : \big(\frac{\partial f}{\partial x_1}\big)\big) / \big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big)$.

We fix a term ordering $\succ$ on $H^n_{[O]}(\mathcal{O}_X)$ and its inverse term ordering $\succ^{-1}$ on the local ring $\mathcal{O}_{X,O}$.

Step 1: Compute a basis $\Phi_{\Gamma(f)}$ of $H_{\Gamma(f)}$.

Step 2: Compute a monomial basis $M_{\Gamma(f)}$ of the quotient space $\mathcal{O}_{X,O}/\big(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\big)$, with respect to $\succ^{-1}$, by using $\Phi_{\Gamma(f)}$.

Step 3: Compute $\frac{\partial f}{\partial x_n}\phi$ of each $\phi \in \Phi_{\Gamma(f)}$ and compute a basis $\Phi_{\Delta(f)}$ of $H_{\Delta(f)}$.

Step 4: Compute a standard basis SB of the ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Delta(f)})$ by using $\Phi_{\Delta(f)}$.

Step 5: Compute the normal form $\mathrm{NF}_{\succ^{-1}}\big(x^\lambda s(x)\big)$ of $x^\lambda s(x)$ for $x^\lambda \in M_{\Gamma(f)}, s(x) \in \mathrm{SB}$.

Step 6: Compute a basis A, as a vector space, of $\mathrm{Span}_{\mathbb{C}}\big\{\mathrm{NF}_{\succ^{-1}}(x^\lambda s(x)) \,|\, x^\lambda \in M_{\Gamma(f)}, s(x) \in \mathrm{SB}\big\}$.

Then, we have the following:

$$\mathrm{Span}_{\mathbb{C}}(\mathrm{A}) \cong \left(\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right) : \left(\frac{\partial f}{\partial x_1}\right)\right) \Big/ \left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Note that, by utilizing algorithms given in [22], the method proposed above can be extended to treat parametric cases, the case where the input data contain parameters.

In order to obtain non-trivial logarithmic vector fields, it is enough to do the following.

For each $a(x) \in A$, compute $a_2(x), a_3(x), \ldots, a_n(x), b(x) \in \mathcal{O}_{X,O}$, such that

$$a(x)\frac{\partial f}{\partial x_1} + a_2(x)\frac{\partial f}{\partial x_2} + \cdots + a_{n-1}(x)\frac{\partial f}{\partial x_{n-1}} + a_n(x)\frac{\partial f}{\partial x_n} - b(x)f(x) = 0.$$

Then,

$$a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_{n-1}(x)\frac{\partial}{\partial x_{n-1}} + a_n(x)\frac{\partial}{\partial x_n}, \qquad a(x) \in A$$

gives rise to the desired set of non-trivial logarithmic vector fields.

The step above can be executed efficiently by using an algorithm described in [21]. See also [40] for details.

Before ending this section, we turn to the issue on the genericity. For this purpose, let us recall a result of B. Teissier on this subject.

Let $p' = (p'_1, p'_2, \dots, p'_n)$ be a non-zero vector and let $[p']$ denote the corresponding point in the projective space $\mathbb{P}^{n-1}$. We identify the hyperplane

$$H_{p'} = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n \mid p'_1 x_1 + p'_2 x_2 + \dots + p'_n x_n = 0\}$$

with the point $[p']$ in $\mathbb{P}^{n-1}$. In [43, 44], B. Teissier introduced an invariant $\mu^{(n-1)}(f)$ as

$$\mu^{(n-1)}(f) = \min_{[p'] \in \mathbb{P}^{n-1}} \mu(f|_{H_{p'}}),$$

where $f|_{H_{p'}}$ is the restriction of f to $H_{p'}$ and $\mu(f|_{H_{p'}})$ is the Milnor number at the origin O of the hyperplane section $f|_{H_{p'}}$ of f. He also proved that the set

$$U = \{[p'] \in \mathbb{P}^{n-1} \mid \mu(f|_{H_{p'}}) = \mu^{(n-1)}(f)\}$$

is a Zariski open dense subset of $\mathbb{P}^{n-1}$.

Accordingly, in order to obtain good representations of logarithmic vector fields, it is desirable to use a generic system of coordinate or a generic hyperplane $H_{p'}$ that satisfies the condition $\mu(f|_{H_{p'}}) = \mu^{(n-1)}(f)$.

In a previous paper [25], methods for computing limiting tangent spaces were studied and an algorithm of computing $\mu(f|_{H_{p'}})$, $p' \in \mathbb{P}^{n-1}$ was given. In [23, 26], more effective algorithms for computing $\mu^{(n-1)}$ were given. Utilizing the results in [23, 26], an effective method for computing logarithmic vector fields that takes care of the genericity condition is designed in [27, 40]. See also [42] for related results.

3.4 Regular meromorphic differential forms

Now we are ready to consider a method for computing regular meromorphic differential forms. For simplicity, we first consider a 3-dimensional case. Assume that a non-trivial logarithmic vector field v is given:

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + a_3(x)\frac{\partial}{\partial x_3}.$$

Let $v(f) = b(x)f(x)$ and $\beta = i_v(\omega_X)$, where $\omega_X = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. We have $\beta = a_1(x)\mathrm{d}x_2 \wedge \mathrm{d}x_3 - a_2(x)\mathrm{d}x_1 \wedge \mathrm{d}x_3 + a_3(x)\mathrm{d}x_1 \wedge \mathrm{d}x_2$. We introduce differential forms ξ and η as

$$\xi = -a_2(x)\mathrm{d}x_3 + a_3(x)\mathrm{d}x_2, \qquad \eta = b(x)\mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

Let $g(x) = \frac{\partial f}{\partial x_1}$. Then, the following holds

$$g(x)\beta = \mathrm{d}f \wedge \xi + f(x)\eta.$$

Accordingly, the logarithmic differential form $\omega = \frac{\beta}{f}$ satisfies

$$g(x)\omega = \frac{\mathrm{d}f}{f} \wedge \xi + \eta.$$

We may assume that the coordinate system (x_1, x_2, x_3) is generic [27] and $g(x)$ satisfies the condition (a), (b) of (iii) in Definition 2.1.

Since $g(x) = \frac{\partial f}{\partial x_1}$, we have, by definition, the following:

$$\mathrm{res}\left(\frac{\beta}{f}\right) = \left.\frac{\xi}{\frac{\partial f}{\partial x_1}}\right|_S .$$

Notice that the differential form ξ above is directly defined from the coefficients of the logarithmic vector field v.

Proposition 3.10. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin $O \in X \subset \mathbb{C}^n$. Assume that the coordinate system $(x_1, x_2, \ldots, x_n)$ is generic so that $\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\right)$ is a regular sequence and $g(x) = \frac{\partial f}{\partial x_1}$ satisfies the condition (a), (b) of (iii) in Definition 2.1. Let*

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

be a germ of non-trivial logarithmic vector field along S. Let $v(f) = b(x)f(x)$, $\beta = i_v(\omega_X)$. Let ξ, η denote the differential form defined to be

$$\begin{aligned}\xi = &-a_2(x)\mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n + a_3(x)\mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n - \cdots \\ &+ (-1)^{(n+1)}a_n(x)\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \cdots \wedge \mathrm{d}x_{n-1}, \\ \eta = \ &b(x)\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \cdots \wedge \mathrm{d}x_n.\end{aligned}$$

Then,

$$g(x)\frac{\beta}{f} = \frac{\mathrm{d}f}{f} \wedge \xi + \eta \qquad \text{and} \qquad \mathrm{res}\left(\frac{\beta}{f}\right) = \left.\frac{\xi}{\frac{\partial f}{\partial x_1}}\right|_S$$

hold.

Note that, in 1984, M. Kersken [16] obtained related results on regular meromorphic differential forms. The statement in Proposition 3.10 above is a refinement a result of M. Kersken.

Theorem 3.11. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin $O \in X \subset \mathbb{C}^n$. Let $V = \{v_1, v_2, \ldots, v_\tau\}$ be a set of non-trivial logarithmic vector fields such that the class $[v_1], [v_2], \ldots, [v_\tau]$ constitute a basis of the vector space $\mathcal{D}\mathrm{er}_{X,O}(-\log S)/\sim$, where τ stands for the Tjurina number of f. Let $\xi_1, \xi_2, \ldots, \xi_\tau$ be the differential forms correspond to $v_1, v_2, \ldots, v_\tau$ defined in Proposition 3.10.*

Then, any logarithmic residue in $\mathrm{res}\big(\Omega^{n-1}(\log S)\big)$, or a regular meromorphic differential form γ in ω_S^{n-2} can be represented as

$$\gamma = \left.\left(\frac{1}{\frac{\partial f}{\partial x_1}}(c_1\xi_1 + c_2\xi_2 + \cdots + c_\tau\xi_\tau)\right)\right|_S + \alpha,$$

where $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, \tau$, and $\alpha \in \Omega_X^{n-2}\big|_S$.

4 Examples

In this section, we give examples of computation for illustration. Data is an extraction from [40]. Let $f_0(z, x, y) = x^3 + y^3 + z^4$ and let $f_t(z, x, y) = f_0(z, x, y) + txyz^2$, where t is a deformation parameter. We regard z as the first variable. Then, f_0 is a weighted homogeneous polynomial with respect to a weight vector $(3, 4, 4)$ and f_t is a μ-constant deformation of f_0, called U_{12}

singularity. The Milnor number $\mu(f_t)$ of U_{12} singularity is equal to 12. In contrast, the Tjurina number $\tau(f_t)$ depends on the parameter t. In fact, if $t=0$, then $\tau(f_0)=12$ and if $t\neq 0$, then $\tau(f_t)=11$. In the computation, we fix a term order $\succ^{-1}$ on $\mathcal{O}_{X,O}$ which is compatible with the weight vector $(3,4,4)$.

We consider these two cases separately.

Example 4.1 (weighted homogeneous U_{12} singularity)**.** Let $f_0(z,x,y)=x^3+y^3+z^4$. Then, $\mu(f_0)=\tau(f_0)=12$. The monomial basis M with respect to the term ordering $\succ^{-1}$ of the quotient space $\mathcal{O}_{X,O}/(f_0,\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y})$ is

$$\mathrm{M}=\left\{x^iy^jz^k\,|\,i=0,1,\ j=0,1,\ k=0,1,2,3\right\}.$$

The standard basis Sb of the ideal quotient $\left(f_0,\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y}\right):\left(\frac{\partial f_0}{\partial z}\right)$ is

$$\mathrm{Sb}=\left\{x^2,y^2,z\right\}.$$

The normal form in $\mathcal{O}_{X,O}/\left(f_0,\frac{\partial f_0}{\partial x},\frac{\partial f_0}{\partial y}\right)$ of x^2, y^2 and z are

$$\mathrm{NF}_{\succ^{-1}}\left(x^2\right)=\mathrm{NF}_{\succ^{-1}}\left(y^2\right)=0,\qquad \mathrm{NF}_{\succ^{-1}}(z)=z.$$

Therefore, $\mathrm{A}=\{x^iy^jz^k\,|\,i=0,1,\ j=0,1,\ k=1,2,3\}$. Notice that A consists of 12 elements. It is easy to see that the Euler vector field

$$v=4x\frac{\partial}{\partial x}+4y\frac{\partial}{\partial y}+3z\frac{\partial}{\partial z}$$

that corresponds to the element $z\in\mathrm{A}$ is a non-trivial logarithmic vector field. Therefore, the torsion module of the hypersurface $S_0=\left\{(x,y,z)\,|\,x^3+y^3+z^4=0\right\}$ is given by

$$\mathrm{Tor}\left(\Omega^2_{S_0}\right)=\left\{x^iy^jz^ki_v(\omega_X)\,|\,i=0,1,\ j=0,1,\ k=1,2,3\right\},$$

where $\omega_X=\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y$.

Let $\xi=-4x\mathrm{d}y+4y\mathrm{d}x$. Then $\mathrm{res}\left(\frac{i_v(\omega_X)}{f}\right)=\frac{\xi}{4z^3}\Big|_S$. Computation of other logarithmic residues are same.

The following is also an extraction from [40].

Example 4.2 (semi quasi-homogeneous U_{12} singularity)**.** Let $f(x,y,z)=x^3+y^3+z^4+txyz^2$, $t\neq 0$. Then, $\mu(f)=12$, $\tau(f)=11$ and $\mu(f|H_z)=4$. We have $\dim_{\mathbb{C}}H_{\Gamma(f)}=16$, $\dim_{\mathbb{C}}H_{\Delta(f)}=5$. Let $\succ$ be a term ordering on $H^3_{[O]}(\mathcal{O}_X)$ which is compatible with the weight vector $(4,4,3)$.

A basis $\Phi_{\Gamma(f)}$ of $H_{\Gamma(f)}$ is given by

$$\left\{\begin{bmatrix}1\\xyz\end{bmatrix},\begin{bmatrix}1\\xyz^2\end{bmatrix},\begin{bmatrix}1\\x^2yz\end{bmatrix},\begin{bmatrix}1\\xy^2z\end{bmatrix},\begin{bmatrix}1\\xyz^3\end{bmatrix},\begin{bmatrix}1\\x^2yz^2\end{bmatrix},\begin{bmatrix}1\\xy^2z^2\end{bmatrix},\begin{bmatrix}1\\x^2y^2z\end{bmatrix},\begin{bmatrix}1\\xyz^4\end{bmatrix},\right.$$
$$\begin{bmatrix}1\\x^2yz^3\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xy^3z\end{bmatrix},\begin{bmatrix}1\\xy^2z^3\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\x\ 3yz\end{bmatrix},\begin{bmatrix}1\\x^2y^2z^2\end{bmatrix},\begin{bmatrix}1\\x^2yz^4\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xy^3z^2\end{bmatrix},$$
$$\begin{bmatrix}1\\xy^2z^4\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\x^3yz^2\end{bmatrix},\begin{bmatrix}1\\x^2y^2z^3\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\x^4yz\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xy^4z\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xyz^5\end{bmatrix},$$
$$\left.\begin{bmatrix}1\\x^2y^2z^4\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\x^4yz^2\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xy^4z^2\end{bmatrix}-\frac{t}{3}\begin{bmatrix}1\\xyz^6\end{bmatrix}\right\}.$$

The monomial basis M with respect to the term ordering $\succ^{-1}$ of the quotient $\mathcal{O}_{X,O}/\big(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\big)$ is

$$\mathrm{M} = \big\{x^i y^j z^k \,|\, i = 0,1,\ j = 0,1,\ k = 0,1,2,3\big\}.$$

A basis $\Phi_{\Delta(f)}$ of $H_{\Delta(f)}$ is given by

$$\left\{ \begin{bmatrix} 1 \\ xyz \end{bmatrix}, \begin{bmatrix} 1 \\ xyz^2 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2yz \end{bmatrix}, \begin{bmatrix} 1 \\ xy^2z \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^2z \end{bmatrix} + \frac{t}{6}\begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} \right\}.$$

We see from this data that the standard basis of the ideal quotient $\big(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\big) : \big(\frac{\partial f}{\partial z}\big)$ in the local ring $\mathcal{O}_{X,O}$ is

$$\mathrm{Sb} = \left\{ z^2 - \frac{t}{6}xy, xz, yz, x^2, y^2 \right\}.$$

From Sb and M, we have

$$\mathrm{A} = \left\{ z^2 - \frac{t}{6}xy, xz, yz, z^3, xz^2, yz^2, xyz, xz^3, yz^3, xyz^2, xyz^3 \right\}.$$

These 11 elements in A are used to construct non-trivial logarithmic vector fields and regular meromorphic differential forms. We give the results of computation.

(i) Let $a = 6z^2 - txy$. Then,

$$v = \frac{d_1}{27 + t^3z^2}\frac{\partial}{\partial x} + \frac{d_2}{27 + t^3z^2}\frac{\partial}{\partial y} + \big(6z^2 - txy\big)\frac{\partial}{\partial z}$$

is a non-trivial logarithmic vector field, where

$$d_1 = 216xz - 6t^2y^2z - 2t^4x^2yz, \qquad d_2 = 216yz + 24t^2x^2z + 10t^3yz^3 - 2t^4xy^2z.$$

(ii) Let $a = xz$. Then,

$$v = \frac{d_1}{27 + t^3z^2}\frac{\partial}{\partial x} + \frac{d_2}{27 + t^3z^2}\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$$

is a non-trivial logarithmic vector field, where

$$d_1 = 36x^2 - 6yz^2 - 6t^2xy^2, \qquad d_2 = 36xy + 2t^2x^3 - 4t^2y^3 - 2t^2z^4.$$

We omit the other nine cases. As described in Theorem 3.11, regular meromorphic differential forms can be constructed directly from these data.

5 Brieskorn formula

In 1970, B. Brieskorn studied the monodromy of Milnor fibration and developed the theory of Gauss–Manin connection [7]. He proved the regularity of the connection and proposed an algebraic framework for computing the monodromy via Gauss–Manin connection. He gave in particular a basic formula, now called Brieskorn formula, for computing Gauss–Manin connection.

We show in this section a link between Brieskorn formula, torsion differential forms and logarithmic vector fields. We present an alternative method for computing non-trivial logarithmic vector fields. The resulting algorithm can be used as a basic tool for studying Gauss–Manin connections. We also present some examples for illustration.

5.1 Brieskorn lattice and Gauss–Manin connection

We briefly recall some basics on Brieskorn lattice and Brieskorn formula. We refer to [6, 7, 37]. Let $f(x)$ be a holomorphic function on X with an isolated singularity at the origin $O \in X$, where X is an open neighborhood of O in $\mathbb{C}^n$. Let

$$H_0' = \Omega^{n-1}_{X,O}/\big(\mathrm{d}f \wedge \Omega^{n-2}_{X,O} + \mathrm{d}\Omega^{n-2}_{X,O}\big), \qquad H_0'' = \Omega^{n}_{X,O}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-2}_{X,O}.$$

Then, $\mathrm{d}f \wedge H_0' \subset H_0''$. A map $D\colon \mathrm{d}f \wedge H_0' \longrightarrow H_0''$ is defined as follows:

$$D(\mathrm{d}f \wedge \varphi) = [\mathrm{d}\varphi], \qquad \varphi \in \Omega^{n-1}_{X,O}.$$

Let $\varphi = \sum_{i=1}^n (-1)^{i+1} h_i(x) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n$. Then

$$\mathrm{d}f \wedge \varphi = \left(\sum_{i=1}^n h_i(x) \frac{\partial f}{\partial x_i} \right) \omega_X,$$

where $\omega_X = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$. Therefore in terms of the coordinate we have the following, known as Brieskorn formula

$$D(\mathrm{d}f \wedge \varphi) = \left(\sum_i^n \frac{\partial h_i}{\partial x_i} \right) \omega_X.$$

Example 5.1. Let $f(x,y) = x^2 - y^3$ and $S = \{(x,y) \in X \,|\, f(x,y) = 0\}$ where $X \subset \mathbb{C}^2$ is an open neighborhood of the origin O. The Jacobi ideal J of f is $\big(x, y^2\big) \subset \mathcal{O}_{X,O}$ and $M = \{1, y\}$ is a monomial basis of the quotient $\mathcal{O}_{X,O}/J$. Let τ denote the Tjurina number. Then, since f is a weighted homogeneous polynomial, we have $\tau = \mu = 2$ (see Example 2.5).

Let $v = \frac{1}{6}\big(3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\big)$ be the Euler vector field. Then, v is logarithmic along S.

Let $\beta = i_v(\omega_X)$. Then, $\beta = \frac{1}{6}(3x\mathrm{d}y - 2y\mathrm{d}x)$. Since $v(f) = f$, we have $\mathrm{d}f \wedge \beta = f\omega_X$, where $\omega_X = \mathrm{d}x \wedge \mathrm{d}y$. By Brieskorn formula, we have

$$D(f\omega_X) = D(\mathrm{d}f \wedge \beta) = \frac{5}{6}\omega_X.$$

Note that the formula above is equivalent $\mathrm{d}\big(\frac{\beta}{f^\lambda}\big) = 0$, with $\lambda = \frac{5}{6}$.

Likewise, for $y\beta$, we have $\mathrm{d}f \wedge (y\beta) = f(x,y)y\omega_X$ and

$$D(f(x,y)y\omega_X) = D(\mathrm{d}f \wedge (y\beta)) = \frac{7}{6}y\omega_X,$$

which is equivalent to $\mathrm{d}\big(\frac{y\beta}{f^\lambda}\big) = 0$, with $\lambda = \frac{7}{6}$.

Since $Df = fD + 1$ as operators, we have

$$fD(\omega_X) = -\frac{1}{6}\omega_X, \qquad fD(y\omega_X) = \frac{1}{6}y\omega_X.$$

Notice that β, $y\beta$ are non-zero torsion differential forms in Ω^1_S and v, yv are non-trivial logarithmic vector fields along S. Note also that $yv(f) = yf$. Notably, Brieskorn formula described in terms of differential forms can be rewritten in terms of non-trivial logarithmic vector fields v and yv which satisfy $v(f) = f$ and $yv(f) = yf$ respectively.

Let $S = \{x \in X \,|\, f(x) = 0\}$ be the hypersurface with an isolated singularity at the origin $O \in X$ defined by f. Consider, for instance, a trivial vector field $v' = \frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_2}$. Since $v'(f) = 0$ and $\frac{\partial}{\partial x_1}\big(\frac{\partial f}{\partial x_2}\big) + \frac{\partial}{\partial x_2}\big(-\frac{\partial f}{\partial x_1}\big) = 0$ hold, we have a trivial relation $D((0 \cdot \omega_X) = 0 \cdot \omega_X$. It is easy to see in general that, from a trivial vector field Brieskorn formula only gives the trivial relation.

The observation above leads the following.

Proposition 5.2. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin $O \in X$, where $X \subset \mathbb{C}^n$. Let*

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

be a germ of non-trivial logarithmic vector field along S. Let $v(f) = b(x)f(x)$ Then,

$$D(f(x)b(x)\omega_X) = \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}\right)\omega_X$$

holds, where $\omega_X = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$.

Proof. Let $\beta = i_v(\omega_X)$. Since $\mathrm{d}f \wedge \beta = v(f)\omega_X$, we have $\mathrm{d}f \wedge \beta = \left(\sum_{i=1}^{n} a_i(x)\frac{\partial f}{\partial x_i}\right)\omega_X$. Since $v(f) = b(x)f(x)$, Brieskorn formula implies the result. ■

Notice that the action of Df on $b(x)\omega_X$ in the formula above is completely written in terms of non-trivial logarithmic vector field v such that $v(f) = b(x)f$. To the best of our knowledge, this simple observation has not been explicitly stated in literature on Gauss–Manin connections.

Now we present an alternative method for computing the module of germs of non-trivial logarithmic vector fields.

Step 1: Compute a monomial basis M of the quotient space

$$\mathcal{O}_{X,O}\Big/\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Step 2: Compute a standard basis Sb of the ideal quotient

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) : (f).$$

Step 3: Compute a basis B of the vector space by using Sb and M

$$\left(\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) : (f)\right)\Big/\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Step 4: For each $b(x) \in$ B, compute a logarithmic vector field along S such that

$$v(f) = b(x)f(x).$$

The method above computes a basis of non-trivial logarithmic vector fields. Each step can be effectively executable, as in [40], by utilizing algorithms described in [20, 21, 22, 41].

Note that, the number of non-trivial logarithmic vector fields in the output is equals to the Tjurina number $\tau(f)$. See also [18].

Let

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

be a germ of non-trivial logarithmic vector field along S, such that $v(f) = b(x)f(x)$. Then from Proposition 5.2, we have

$$D(f(x)b(x)\omega_X) = \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}\right)\omega_X.$$

Therefore, the proposed method can be used as a basic procedure for computing a connection matrix of Gauss–Manin connection.

One of the advantages of the proposed method lies in the fact that the resulting algorithm also can handle parametric cases.

5.2 Examples

Let us recall that $x^3 + y^7 + txy^5$ is the standard normal form of semi quasi-homogeneous E_{12} singularity. The weight vector is $(7, 3)$ and the weighted degree of the quasi-homogeneous part is equal to 21 and the weighted degree of the upper monomial txy^5 is equal to 22. We examine here, by contrast, the case where the weighted degree of an upper monomial is bigger than 22.

Example 5.3. Let $f(x, y) = x^3 + y^7 + txy^6$, where t is a parameter. Notice that the polynomial f is not weighted homogeneous. The weighted degree of the upper monomial txy^6 is equal to 25, which is bigger than that of txy^5. Accordingly f is a quasi homogeneous function. The Milnor number μ is equal to 12.

Let H_J denote the set of local cohomology classes in $H^2_{[0,0]}(\mathcal{O}_X)$ that are killed by the Jacobi ideal $J = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$:

$$H_J = \left\{ \psi \in H^2_{[0,0]}(\mathcal{O}_X) \,\middle|\, \frac{\partial f}{\partial x}\psi = \frac{\partial f}{\partial y}\psi = 0 \right\}.$$

Then, by using an algorithm given in [22, 41], a basis as a vector space of H_J is computed as

$$\begin{bmatrix} 1 \\ xy \end{bmatrix}, \begin{bmatrix} 1 \\ xy^2 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^3 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y \end{bmatrix}, \begin{bmatrix} 1 \\ xy^4 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^2 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^5 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^3 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^6 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^4 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^5 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ x^2y^6 \end{bmatrix} - \frac{6}{7}t \begin{bmatrix} 1 \\ xy^7 \end{bmatrix} + \frac{2}{7}t^2 \begin{bmatrix} 1 \\ x^3y \end{bmatrix},$$

where [ ] stands for Grothendieck symbol.

It is easy to see that every local cohomology classes in H_J is killed by f, that is $f \cdot \varphi = 0$, $\forall\, \varphi \in H_J$. Therefore, f is in the ideal $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \subset \mathcal{O}_{X,O}$.

Therefore, by a classical result of K. Saito [31], f is in fact quasi-homogeneous. The Tjurina number τ is equal to the Milnor number $\mu = 12$. A monomial basis M of $\mathcal{O}_{X,O}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is

$$\mathrm{M} = \{1, y, y^2, x, y^3, xy, y^4, xy^2, y^5, xy^3, xy^4, xy^5\}.$$

Since a standard basis Sb of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) : (f)$ is $\{1\}$, a basis B of the vector space $\left(\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) : (f)\right)/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is equal to M that consists of $\tau = 12$ elements.

By using an algorithm given in [21], we compute a logarithmic vector field which plays the role of Euler vector field. The result of computation is the following:

$$v = \frac{d_1}{3(49 + 12t^3y^4)}\frac{\partial}{\partial x} + \frac{d_2}{3(49 + 12t^3y^4)}\frac{\partial}{\partial y},$$

where

$$d_1 = 49x + 8t^2y^5 + 12t^3xy^4, \qquad d_2 = 21y - 4tx + 4t^3y^5.$$

The vector field v enjoys $v(f) = f$. Note also that for the case $t = 0$, we have

$$v = \frac{1}{21}\left(7x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}\right).$$

We emphasize here the fact that, the algorithm in [27] for computing logarithmic vector fields can handle parametric cases. Since $v(f) = f$ holds, the other non-trivial logarithmic vector fields can be obtained from v. In fact, for $x^iy^j \in M$, we have $x^iy^jv(f) = x^iy^jf$.

Therefore, thanks to Brieskorn formula, Gauss–Manin connection can be determined explicitly by using these non-trivial logarithmic vector fields,

Remark 5.4. Recall that, according to Grothendieck local duality theorem, the vector space H_J can be regarded as a dual space to $\mathcal{O}_{X,O}/J$. Since these local cohomology classes given above constitute a dual basis of the monomial basis M of the quotient space $\mathcal{O}_{X,O}/J$, the normal form of a holomorphic function w.r.t. $\mathcal{O}_{X,O}/J$ can be computed by using the basis of H_J in an efficient manner, without using division algorithms [41].

Therefore the use of local cohomology classes in reduction steps allows us to design an effective procedure for computing the connection matrix of Gauss–Manin connection.

J. Scherk studied in [35] the following case.

Example 5.5. Let $f(x,y) = x^5 + x^2y^2 + y^5$. Then, the Milnor number $\mu(f)$ is equal to 11 and the Tjurina number $\tau(f)$ is equal to 10. A monomial basis M of $\mathcal{O}_{X,O}/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is $\mathrm{M} = \left\{1, x, x^2, x^3, x^4, x^5, xy, y, y^2, y^3, y^4\right\}$. A standard basis Sb of the ideal quotient $\left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}\right) : (f)$ is $\{x, y\}$. A basis B of the vector space $\left(\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) : (f)\right)/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is

$$\mathrm{B} = \left\{x, x^2, x^3, x^4, x^5, xy, y, y^2, y^3, y^4\right\}.$$

Since $\mathrm{Sb} \cap \mathrm{B} = \{x, y\}$, we first compute non-trivial logarithmic vector fields associated to x and y.

(*i*) For $b(x,y) = x$, we have

$$v = \frac{d_1}{5(4-25xy)}\frac{\partial}{\partial x} + \frac{d_2}{5(4-25xy)}\frac{\partial}{\partial y},$$

where $d_1 = 4x^2 - 25x^3y - 5y^3$, $d_2 = 6xy - 25x^2y^2$.

Since $v(f) = xf$, by a direct computation, we have for instance

$$D(f(x,y)x\omega_X) = \left(\frac{7}{10}x - \frac{3\times 25}{16}y^4\right)\omega_X \mod \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Since $x^i v(f) = x^{i+1}f$, $i = 1,2,3,4$ and $yv(f) = xyf$ hold, we can compute the action of Df on $x^{i+1}\omega_X$ and $xy\omega_X$ by using the vector field v above.

(*ii*) For $b(x,y) = y$, we have

$$v = \frac{d_1}{5(4-25xy)}\frac{\partial}{\partial x} + \frac{d_2}{5(4-25xy)}\frac{\partial}{\partial y},$$

where $d_1 = 6xy - 25x^2y^2$, $d_2 = 4y^2 - 25xy^3 - 5x^3$ and

$$D(f(x,y)y\omega_X) = \left(\frac{7}{10}y - \frac{3\times 25}{16}x^4\right)\omega_X \mod \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Since the vector field v above satisfies $v(f) = yf$, we also have $y^j v(f) = y^{j+1}f$, $j = 1,2,3$.

We can use these relations to compute the action of Df on $y^{j+1}\omega_X$, $j = 1,2,3$. In this way, we obtain $\tau = 10$ fundamental relations.

Since the Milnor number μ is equal to 11, these 10 relations are not enough to compute a connection matrix of the Gauss–Manin connection. We have to compute the saturation.

Now recall the classical result on integral closure due to J. Brianọn and H. Skoda [38]. From the Brianọn–Skoda theorem, we see that the function f^2 is in the ideal $J = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. In [35],

J. Scherk computed the following relation explicitly and exploited it as the starting point for computing $D(f^2\omega_X)$ and $D(fD(f\omega_X))$:

$$25(4-25xy)f^2 = \{(20x-125x^2y)f + 4x^3y^2 - 5xy^5 - 25x^4y^3\}\frac{\partial f}{\partial x} + \{(20y-125xy^2)f + 6x^2y^3 - 25x^3y^4\}\frac{\partial f}{\partial y}.$$

Here we propose a slightly different approach. By using an algorithm given in [28], we can compute the following integral dependence relation

$$25(4-25xy)f^2 = 10x\left(\frac{\partial f}{\partial x}\right)f + 10y\left(\frac{\partial f}{\partial y}\right)f + d_{2,0}\left(\frac{\partial f}{\partial x}\right)^2 + d_{1,1}\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) + d_{0,2}\left(\frac{\partial f}{\partial y}\right)^2,$$

where

$$d_{2,0} = 2x^2 - 25x^3y - 10y^3, \qquad d_{1,1} = 11xy - 50x^2y^2, \qquad d_{0,2} = 2y^2 - 25xy^3 - 10x^3.$$

Compare to the relation used by Scherk, the integral dependence relation given above represent much more precise relations between f^2, $f\big(\frac{\partial f}{\partial x}\big)$, $f\big(\frac{\partial f}{\partial y}\big)$, $\big(\frac{\partial f}{\partial x}\big)^2$, $\big(\frac{\partial f}{\partial x}\big)\big(\frac{\partial f}{\partial y}\big)$, $\big(\frac{\partial f}{\partial y}\big)^2$. Thanks to this property, the use of the integral dependence relation, or the integral equation leads an effective method for computing $D\big(f^2\omega_X\big)$ and $D(fD(f\omega_X))$.

Note that in [28], we consider integral dependence relations in the context of symbolic computation and introduced a concept of generalized integral dependence relations. From this point of view relations obtained from non-trivial logarithmic vector fields can be interpreted as generalized integral dependence relations. These relations can also be computed by using the algorithms described in [28].

Let $f(x)$ be a holomorphic function defined on $X \subset \mathbb{C}^n$. Assume that the degree of integral equation, or the integral number of f over the Jacobi ideal in the local ring $\mathcal{O}_{X,O}$ is equal to two. Let

$$f(x)^2 + \sum_{i=1}^{n} a_i(x)f(x)\frac{\partial f}{\partial x_i}(x) + \sum_{j\geq i} a_{i,j}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x) = 0$$

be the integral equation of f. Then, from the Brieskorn formula, we have

$$\begin{aligned} D(f(x)^2\omega_X) &= -D\left\{\sum_{i=1}^{n}\left(a_i(x)f(x) + \sum_{j\geq i} a_{i,j}(x)\frac{\partial f}{\partial x_j}(x)\right)\frac{\partial f}{\partial x_i}(x)\omega_X\right\} \\ &= -\left\{\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i(x)f(x) + \sum_{j\geq i} a_{i,j}(x)\frac{\partial f}{\partial x_j}(x)\right)\right\}\omega_X \end{aligned}$$

which is equal to

$$-\left\{\sum_{i=1}^{n} a_i(x)\frac{\partial f}{\partial x_i}(x) + \sum_{j\geq i}\frac{\partial a_{i,j}}{\partial x_i}(x)\frac{\partial f}{\partial x_j} + R(x)\right\}\omega_X,$$

where

$$R(x) = \left(\sum_{i=1}^{n}\frac{\partial a_i}{\partial x_i}(x)\right)f(x) + \sum_{j\geq i} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}(x).$$

If, there exist holomorphic functions $c_i(x)$, $i = 1, 2, \dots, n$ such that

$$R(x) = \sum_{i=1}^{n} c_i(x) \frac{\partial f}{\partial x_i}(x),$$

then, we have for instance the following relation that can be used as a starting point of the computation of a saturation

$$D^2(f(x)^2\omega_X) = -\left\{ \sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial x_i}(x) + \frac{\partial c_i}{\partial x_i}(x) \right) + \sum_{j \geq i} \frac{\partial^2 a_{i,j}}{\partial x_j \partial x_i}(x) \right\} \omega_X.$$

Computing Gauss–Manin connections is a quite difficult problem [11, 14, 35, 36, 45]. We expect that the approach presented in this paper provides a method to reduce difficulty to some extent.

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References

[1] Aleksandrov A.G., A de Rahm complex of nonisolated singularities, *Funct. Anal. Appl.* **22** (1988), 131–133.

[2] Aleksandrov A.G., Nonisolated hypersurface singularities, in Theory of Singularities and its Applications, *Adv. Soviet Math.*, Vol. 1, Amer. Math. Soc., Providence, RI, 1990, 211–246.

[3] Aleksandrov A.G., Logarithmic differential forms, torsion differentials and residue, *Complex Var. Theory Appl.* **50** (2005), 777–802.

[4] Aleksandrov A.G., Tsikh A.K., Théorie des résidus de Leray et formes de Barlet sur une intersection complète singulière, *C. R. Acad. Sci. Paris Sér. I Math.* **333** (2001), 973–978.

[5] Barlet D., Le faisceau ω'_X sur un espace analytique X de dimension pure, in Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977), *Lecture Notes in Math.*, Vol. 670, Springer, Berlin, 1978, 187–204.

[6] Brasselet J.P., Sebastiani M., Brieskorn and the monodromy, *J. Singul.* **18** (2018), 84–104.

[7] Brieskorn E., Die Monodromie der isolierten Singularitäten von Hyperflächen, *Manuscripta Math.* **2** (1970), 103–161.

[8] Brunella M., Some remarks on indices of holomorphic vector fields, *Publ. Mat.* **41** (1997), 527–544.

[9] Corrêa M., da Silva Machado D., Residue formulas for logarithmic foliations and applications, *Trans. Amer. Math. Soc.* **371** (2019), 6403–6420, arXiv:1611.01203.

[10] Corrêa M., da Silva Machado D., GSV-index for holomorphic Pfaff systems, *Doc. Math.* **25** (2020), 1011–1027, arXiv:1611.09376.

[11] Douai A., Très bonnes bases du réseau de Brieskorn d'un polynôme modéré, *Bull. Soc. Math. France* **127** (1999), 255–287.

[12] Granger M., Schulze M., Normal crossing properties of complex hypersurfaces via logarithmic residues, *Compos. Math.* **150** (2014), 1607–1622, arXiv:1109.2612.

[13] Greuel G.M., Der Gauss–Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, *Math. Ann.* **214** (1975), 235–266.

[14] Guimaraes A.G., Polinômio de Bernstein–Sato de uma hipersuperficie com singularidade isolada, Ph.D. Thesis, ICMC-USP, São Carlos, 2002.

[15] Kersken M., Der Residuenkomplex in der lokalen algebraischen und analytischen Geometrie, *Math. Ann.* **265** (1983), 423–455.

[16] Kersken M., Reguläre Differentialformen, *Manuscripta Math.* **46** (1984), 1–25.

[17] Lê D.T., Calcul du nombre de cycles évanouissants d'une hypersurface complexe, *Ann. Inst. Fourier (Grenoble)* **23** (1973), 261–270.

[18] Michler R., Torsion of differentials of hypersurfaces with isolated singularities, *J. Pure Appl. Algebra* **104** (1995), 81–88.

[19] Nabeshima K., Tajima S., Testing zero-dimensionality of varieties at a point, *Math.Comput.Sci.*, to appear, arXiv:1903.12365.

[20] Nabeshima K., Tajima S., Computing Tjurina stratifications of μ-constant deformations via parametric local cohomology systems, *Appl. Algebra Engrg. Comm. Comput.* **27** (2016), 451–467.

[21] Nabeshima K., Tajima S., Solving extended ideal membership problems in rings of convergent power series via Gröbner bases, in Mathematical Aspects of Computer and Information Sciences, *Lecture Notes in Comput. Sci.*, Vol. 9582, Springer, Cham, 2016, 252–267.

[22] Nabeshima K., Tajima S., Algebraic local cohomology with parameters and parametric standard bases for zero-dimensional ideals, *J. Symbolic Comput.* **82** (2017), 91–122, arXiv:1508.06724.

[23] Nabeshima K., Tajima S., Computing μ^*-sequences of hypersurface isolated singularities via parametric local cohomology systems, *Acta Math. Vietnam.* **42** (2017), 279–288.

[24] Nabeshima K., Tajima S., Computation methods of logarithmic vector fields associated to semi-weighted homogeneous isolated hypersurface singularities, *Tsukuba J. Math.* **42** (2018), 191–231.

[25] Nabeshima K., Tajima S., A new method for computing the limiting tangent space of an isolated hypersurface singularity via algebraic local cohomology, in Singularities in Generic Geometry, *Adv. Stud. Pure Math.*, Vol. 78, Math. Soc. Japan, Tokyo, 2018, 331–344.

[26] Nabeshima K., Tajima S., Alternative algorithms for computing generic μ^*-sequences and local Euler obstructions of isolated hypersurface singularities, *J. Algebra Appl.* **18** (2019), 1950156, 13 pages.

[27] Nabeshima K., Tajima S., Computing logarithmic vector fields and Bruce–Roberts Milnor numbers *via* local cohomology classes, *Rev. Roumaine Math. Pures Appl.* **64** (2019), 523–540.

[28] Nabeshima K., Tajima S., Generalized integral dependence relations, in Mathematical Aspects of Computer and Information Sciences, *Lecture Notes in Computer Science*, Vol. 11989, Springer, Cham, 2020, 48–63.

[29] Pol D., On the values of logarithmic residues along curves, *Ann. Inst. Fourier (Grenoble)* **68** (2018), 725–766, arXiv:1410.2126.

[30] Pol D., Characterizations of freeness for equidimensional subspaces, *J. Singul.* **20** (2020), 1–30, arXiv:1512.06778.

[31] Saito K., Quasihomogene isolierte Singularitäten von Hyperflächen, *Invent. Math.* **14** (1971), 123–142.

[32] Saito K., Calcul algébrique de la monodromie, *Astérisque* **7** (1973), 195–211.

[33] Saito K., On the uniformization of complements of discriminant loci, in Hyperfunctions and Linear Partial Differential Equations, *RIMS Kôkyûroku*, Vol. 287, Res. Inst. Math. Sci. (RIMS), Kyoto, 1977, 117–137.

[34] Saito K., Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.

[35] Scherk J., On the Gauss–Manin connection of an isolated hypersurface singularity, *Math. Ann.* **238** (1978), 23–32.

[36] Scherk J., On the pole order and Hodge filtrations of isolated hypersurface singularities, *Canad. Math. Bull.* **36** (1993), 368–372.

[37] Schulze M., Algorithms for the Gauss–Manin connection, *J. Symbolic Comput.* **32** (2001), 549–564.

[38] Skoda H., Briançon J., Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbf{C}^n$, *C. R. Acad. Sci. Paris Sér. A* **278** (1974), 949–951.

[39] Tajima S., On polar varieties, logarithmic vector fields and holonomic D-modules, in Recent Development of Micro-Local Analysis for the Theory of Asymptotic Analysis, *RIMS Kôkyûroku Bessatsu*, Vol. 40, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013, 41–51.

[40] Tajima S., Nabeshima K., An algorithm for computing torsion differential forms associated to an isolated hypersurface singularity, *Math.Comput.Sci.*, to appear.

[41] Tajima S., Nakamura Y., Nabeshima K., Standard bases and algebraic local cohomology for zero dimensional ideals, in Singularities – Niigata–Toyama 2007, *Adv. Stud. Pure Math.*, Vol. 56, Math. Soc. Japan, Tokyo, 2009, 341–361.

[42] Tajima S., Shibuta T., Nabeshima K., Computing logarithmic vector fields along an ICIS germ via Matlis duality, in Computer Algebra in Scientific Computing, *Lecture Notes in Computer Science*, Vol. 12291, Springer, Cham, 2020, 543–562.

[43] Teissier B., Cycles évanescents, sections planes et conditions de Whitney, *Astérisque* **7** (1973), 285–362.

[44] Teissier B., Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces, *Invent. Math.* **40** (1977), 267–292.

[45] van Straten D., The spectrum of hypersurface singularities, arXiv:2002.00519.

[46] Vetter U., Äußere Potenzen von Differentialmoduln reduzierter vollständiger Durchschnitte, *Manuscripta Math.* **2** (1970), 67–75.

[47] Zariski O., Characterization of plane algebroid curves whose module of differentials has maximum torsion, *Proc. Nat. Acad. Sci. USA* **56** (1966), 781–786.