

The Generic Superintegrable System on the 3-Sphere and the $9j$ Symbols of $\mathfrak{su}(1, 1)$

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Abstract. The $9j$ symbols of $\mathfrak{su}(1, 1)$ are studied within the framework of the generic superintegrable system on the 3-sphere. The canonical bases corresponding to the binary coupling schemes of four $\mathfrak{su}(1, 1)$ representations are constructed explicitly in terms of Jacobi polynomials and are seen to correspond to the separation of variables in different cylindrical coordinate systems. A triple integral expression for the $9j$ coefficients exhibiting their symmetries is derived. A double integral formula is obtained by extending the model to the complex three-sphere and taking the complex radius to zero. The explicit expression for the vacuum coefficients is given. Raising and lowering operators are constructed and are used to recover the relations between contiguous coefficients. It is seen that the $9j$ symbols can be expressed as the product of the vacuum coefficients and a rational function. The recurrence relations and the difference equations satisfied by the $9j$ coefficients are derived.

Key words: $\mathfrak{su}(1, 1)$ algebra; $9j$ symbols; superintegrable systems

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1 Introduction

The objective of this paper is to show how the framework provided by the generic superintegrable system on the 3-sphere can be used to study the $9j$ coefficients of $\mathfrak{su}(1, 1)$. In addition to providing a new interpretation for these coefficients, this approach, which can be viewed as a treatment of the problem in the position representation, allows for an explicit construction of the canonical bases involved in the $9j$ problem and a direct derivation of the properties of the $9j$ symbols without reference to Clebsch–Gordan or Racah coefficients.

The $9j$ symbols arise as recoupling coefficients in the combination of four irreducible $\mathfrak{su}(1, 1)$ representations of the positive-discrete series. These coefficients and their equivalent $\mathfrak{su}(2)$ analogues have traditionally found applications in molecular [26] and nuclear [28] physics but have also appeared in the study of spin networks related to quantum gravity [22]. Over the past years, they have been the object of a number of publications, many of which study the $9j$ coefficients from the point of view of special functions. For example, the connection between $9j$ coefficients and orthogonal polynomials in two variables has been studied by Van der Jeugt [31], Suslov [21] and more recently by Hoare and Rahman [12] who used the $9j$ coefficients as a starting point to their study of bivariate Krawtchouk polynomials [6, 8]. A number of explicit multi-sums expressions have also been investigated by Ališauskas and Jucys [1, 2], Rosengren [23, 24, 25] and Srinivasa Rao and Rajeswari [27]. Also worth mentioning is the original approach of Granovskii and Zhedanov [10] which opened a path to a new method for deriving generating functions and convolution identities for orthogonal polynomials [17, 30].

In the present paper, we shall indicate how the $9j$ problem can be studied in the position representation using the connection between the coupling of four $\mathfrak{su}(1, 1)$ representations and the

generic superintegrable system on the three-sphere. This system has been discussed by Kalnins, Kress and Miller in [13]. It is governed by the Hamiltonian

$$H = \sum_{1 \leq i < k \leq 4} J_{ik}^2 + r^2 \sum_{1 \leq \ell \leq 4} \frac{a_\ell}{s_\ell^2}, \quad a_\ell = \alpha_\ell^2 - 1/4, \quad (1.1)$$

where $\alpha_\ell > -1$ and is defined on the 3-sphere of square radius $r^2 = s_1^2 + s_2^2 + s_3^2 + s_4^2$. Here the operators J_{ik} stand for the familiar angular momentum generators

$$J_{ik} = i(s_i \partial_{s_k} - s_k \partial_{s_i}), \quad 1 \leq i < k \leq 4. \quad (1.2)$$

The system described by (1.1) is both superintegrable and exactly solvable. It has five algebraically independent second order constants of motion that generate a quadratic algebra [14].

It will first be shown that the Hamiltonian (1.1) coincides with the total Casimir operator for the combination of four $\mathfrak{su}(1, 1)$ representations and that its constants of motion correspond to the intermediate Casimir operators associated to each possible pairing of the four representations; these results extend the author's previous work [9]. Using this framework, the canonical orthonormal bases of the $9j$ problem, which correspond to the joint diagonalization of different pairs of commuting intermediate Casimir operators, will be constructed as solutions of the Schrödinger equation associated to (1.1) separated in different cylindrical coordinate systems; these solutions will be given in terms of Jacobi polynomials. The coordinate realization of the canonical bases and the underlying quantum mechanical framework will yield an expression for the $9j$ coefficients in terms of an integral on the 3-sphere exhibiting their symmetries. By extending the model to the complex 3-sphere and taking the complex radius to zero, the expression for the $9j$ symbols in terms of a double integral found by Granovskii and Zhedanov [10] shall be recovered. This formula will be used to obtain an explicit hypergeometric formula for the special case corresponding to the “vacuum” $9j$ coefficients. The coordinate realization will also allow for the construction of raising and lowering operators based on the structure relations of the Jacobi polynomials. These operators will then be used to derive directly the relations between contiguous $9j$ symbols, which are usually obtained by manipulations of Clebsch–Gordan or Racah coefficients (given in terms of the Hahn or the Racah polynomials [31]). From these relations, it will be possible to conclude that the $9j$ coefficients can be expressed as a product of the vacuum coefficients and of functions that are rational (and not polynomial as stated in [12]). The fact that the raising and lowering operators factorize the corresponding intermediate Casimir operators shall be used to obtain the action of the intermediate Casimirs on the basis states. This will also lead to both the difference equations and the recurrence relations satisfied by the $9j$ coefficients.

The organization of the paper is as follows.

- Section 2: generic system on the 3-sphere from four $\mathfrak{su}(1, 1)$ representations, exact solutions, canonical basis vectors of the $9j$ problem, triple integral representation, symmetries of the $9j$ coefficients.
- Section 3: double integral formula, explicit vacuum $9j$ coefficients.
- Section 4: raising/lowering operators, relations between contiguous $9j$ symbols.
- Section 5: difference equations and recurrence relations for $9j$ symbols.

2 The $9j$ problem for $\mathfrak{su}(1, 1)$ in the position representation

In this section the $9j$ problem for the positive-discrete series of $\mathfrak{su}(1, 1)$ representations is examined in the position representation. The total Casimir operator for the addition of four representations is identified with the Hamiltonian of the generic superintegrable system on S^3 and the

intermediate Casimir operators are identified with its symmetries. The canonical basis vectors of the $9j$ problem are constructed as wavefunctions separated in different coordinate systems and the $9j$ coefficients are expressed as the overlap coefficients between these bases.

2.1 The addition of four representations and the generic system on S^3

Consider the operators

$$K_0^{(i)} = \frac{1}{4} \left(-\partial_{s_i}^2 + s_i^2 + \frac{a_i}{s_i^2} \right), \quad K_{\pm}^{(i)} = \frac{1}{4} \left((s_i \mp \partial_{s_i})^2 - \frac{a_i}{s_i^2} \right), \quad i = 1, \dots, 4, \quad (2.1)$$

which form four mutually commuting sets of generators satisfying the $\mathfrak{su}(1, 1)$ commutation relations

$$[K_0^{(i)}, K_{\pm}^{(i)}] = \pm K_{\pm}^{(i)}, \quad [K_-^{(i)}, K_+^{(i)}] = 2K_0^{(i)}.$$

The operators (2.1) provide a realization of the positive-discrete series of $\mathfrak{su}(1, 1)$ representations on the space of square-integrable functions on the positive real line. A set of basis vectors $e_{n_i}^{(\nu_i)}$, $n_i = 0, 1, \dots$, for these representations specified by a positive real number ν_i taking the value

$$\nu_i = \frac{\alpha_i + 1}{2}, \quad (2.2)$$

is given in terms of Laguerre polynomials [16] according to

$$e_{n_i}^{(\nu_i)}(s_i) = (-1)^{n_i} \sqrt{\frac{2\Gamma(n_i + 1)}{\Gamma(n_i + \alpha_i + 1)}} e^{-s_i^2/2} s_i^{\alpha_i + 1/2} L_{n_i}^{(\alpha_i)}(s_i^2), \quad n_i = 0, 1, \dots, \quad (2.3)$$

where $\Gamma(z)$ is the gamma function [4]. These basis vectors are orthonormal with respect to the scalar product [16]

$$\int_0^\infty e_{n_i}^{(\nu_i)}(s_i) e_{n'_i}^{(\nu_i)}(s_i) ds_i = \delta_{n_i n'_i},$$

and the action of the generators on the basis vectors is given by

$$\begin{aligned} K_+^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= \sqrt{(n_i + 1)(n_i + 2\nu_i)} e_{n_i + 1}^{(\nu_i)}(s_i), \\ K_-^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= \sqrt{n_i(n_i + 2\nu_i - 1)} e_{n_i - 1}^{(\nu_i)}(s_i), \\ K_0^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= (n_i + \nu_i) e_{n_i}^{(\nu_i)}(s_i), \end{aligned}$$

which corresponds to the usual action defining the irreducible representations of the positive-discrete series [33]. In the realization (2.1), it is easily verified that the Casimir operator of $\mathfrak{su}(1, 1)$ which has the expression

$$Q^{(i)} = [K_0^{(i)}]^2 - K_+^{(i)} K_-^{(i)} - K_0^{(i)},$$

acts as a multiple of the identity, i.e.

$$Q^{(i)} = \nu_i(\nu_i - 1),$$

for $i = 1, \dots, 4$. The four sets (2.1) can be used to define a fifth set of $\mathfrak{su}(1, 1)$ generators through

$$K_0 = \sum_{1 \leq i \leq 4} K_0^{(i)}, \quad K_{\pm} = \sum_{1 \leq i \leq 4} K_{\pm}^{(i)}.$$

The above operators realize a representation of $\mathfrak{su}(1,1)$ on the space $\bigotimes_{i=1}^4 V^{(\nu_i)}$, where $V^{(\nu_i)}$ is the space spanned by the basis vectors (2.3). It is directly checked that the total Casimir operator associated to this realization is

$$Q = H/4, \quad (2.4)$$

where H is the Hamiltonian of the generic superintegrable system on the 3-sphere given by (1.1).

When considering the tensor product of several representations, it is natural to consider the intermediate Casimir operators associated to each possible pairing of representations. These intermediate Casimir operators are defined by

$$Q^{(ij)} = [K_0^{(i)} + K_0^{(j)}]^2 - [K_+^{(i)} + K_+^{(j)}][K_-^{(i)} + K_-^{(j)}] - [K_0^{(i)} + K_0^{(j)}], \quad 1 \leq i < j \leq 4,$$

and have the expression

$$Q^{(ij)} = \frac{1}{4} \left(J_{ij}^2 + \frac{a_i s_j^2}{s_i^2} + \frac{a_j s_i^2}{s_j^2} + a_i + a_j - 1 \right), \quad 1 \leq i < j \leq 4, \quad (2.5)$$

where J_{ij} are the angular momentum operators (1.2). By construction, the intermediate Casimir operators $Q^{(ij)}$ commute with the total Casimir operator Q and hence the intermediate Casimir operators (2.5) are the symmetries of the Hamiltonian (1.1). It is directly checked that the intermediate Casimir operators $Q^{(ij)}$, $Q^{(k\ell)}$ commute only when i, j, k, ℓ are all different and hence the largest set of commuting intermediate Casimir operators has two elements. Note that the intermediate Casimir operators are linearly related to the total Casimir operator as per the relation

$$Q = \sum_{1 \leq i < j \leq 4} Q^{(ij)} - 2 \sum_{1 \leq i \leq 4} Q^{(i)}.$$

In considering the total Casimir operator (2.4), one can take the value of the square radius r^2 to be fixed since the operator

$$2K_0 + K_+ + K_- = r^2,$$

commutes with Q and all the intermediate Casimir operators $Q^{(ij)}$. We shall take $r^2 = 1$, thus considering the Hamiltonian (1.1) on the unit 3-sphere.

2.2 The $9j$ symbols

In general, the representation $\bigotimes_{i=1}^4 V^{(\nu_i)}$ is not irreducible, but it is known to be completely reducible in representations of the positive-discrete series. In this context, the $9j$ symbols arise as the overlap coefficients between natural bases associated to two different decomposition schemes.

- In the first scheme, one first decomposes $V^{(\nu_1)} \otimes V^{(\nu_2)}$ and $V^{(\nu_3)} \otimes V^{(\nu_4)}$ in irreducible components $V^{(\nu_{12})}$, $V^{(\nu_{34})}$ and then decomposes $V^{(\nu_{12})} \otimes V^{(\nu_{34})}$ in irreducible components $V^{(\nu)}$ for each occurring values of (ν_{12}, ν_{34}) . The natural (orthonormal) basis vectors for this scheme are denoted $|\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle$ and defined by

$$\begin{aligned} Q^{(12)} |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle &= \nu_{12}(\nu_{12} - 1) |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle, \\ Q^{(34)} |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle &= \nu_{34}(\nu_{34} - 1) |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle, \\ Q |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle &= \nu(\nu - 1) |\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle, \end{aligned} \quad (2.6)$$

where $\vec{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4)$.

- In the second scheme, one first decomposes $V^{(\nu_1)} \otimes V^{(\nu_3)}$ and $V^{(\nu_2)} \otimes V^{(\nu_4)}$ in irreducible components $V^{(\nu_{13})}$, $V^{(\nu_{24})}$ and then decomposes $V^{(\nu_{13})} \otimes V^{(\nu_{24})}$ in irreducible components $V^{(\nu)}$ for each occurring values of (ν_{13}, ν_{24}) . The natural (orthonormal) basis vectors for this scheme are denoted $|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle$ and defined by

$$\begin{aligned} Q^{(13)}|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle &= \nu_{13}(\nu_{13} - 1)|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle, \\ Q^{(24)}|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle &= \nu_{24}(\nu_{24} - 1)|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle, \\ Q|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle &= \nu(\nu - 1)|\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle. \end{aligned} \quad (2.7)$$

The $9j$ symbols are defined as the overlap coefficients between these two bases, i.e.

$$|\vec{\nu}; \nu_{12}, \nu_{34}; \nu\rangle = \sum_{\nu_{13}, \nu_{24}} \begin{Bmatrix} \nu_1 & \nu_2 & \nu_{12} \\ \nu_3 & \nu_4 & \nu_{34} \\ \nu_{13} & \nu_{24} & \nu \end{Bmatrix} |\vec{\nu}; \nu_{13}, \nu_{24}; \nu\rangle.$$

For the $9j$ symbols to be non-vanishing, one must have

$$\begin{aligned} \nu_{12} &= \nu_1 + \nu_2 + m, & \nu_{34} &= \nu_3 + \nu_4 + n, & \nu_{13} &= \nu_1 + \nu_3 + x, & \nu_{24} &= \nu_2 + \nu_4 + y, \\ \nu &= \nu_1 + \nu_2 + \nu_3 + \nu_4 + N, \end{aligned} \quad (2.8)$$

where m, n, x, y and N are non-negative integers such that $m + n \leq N$ and $x + y \leq N$.

In view of the coordinate realization stemming from the previous subsection, the bases (2.6) and (2.7) can be constructed explicitly by solving the corresponding eigenvalue equations: these bases correspond to the diagonalization of the Hamiltonian (1.1) together with the pairs of commuting intermediate Casimir operators (symmetries) $(Q^{(12)}, Q^{(34)})$ or $(Q^{(13)}, Q^{(24)})$. In view of the conditions (2.2), (2.8) and for notational convenience, the basis corresponding to the scheme (2.6) shall be simply denoted by $|m, n\rangle_N$, the basis corresponding to (2.7) by $|x, y\rangle_N$ and the $9j$ coefficients will be written as

$$|m, n\rangle_N = \sum_{\substack{x, y \\ x+y \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} |x, y\rangle_N, \quad (2.9)$$

or equivalently as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = {}_N \langle x, y | m, n \rangle_N.$$

The $9j$ coefficients are taken to be real. Since they are transition coefficients between two orthonormal bases, it follows from elementary linear algebra that

$$\sum_{\substack{x, y \\ x+y \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \begin{Bmatrix} \alpha_1 & \alpha_2 & m' \\ \alpha_3 & \alpha_4 & n' \\ x & y & N \end{Bmatrix} = \delta_{mm'} \delta_{nn'}, \quad (2.10)$$

and similarly

$$\sum_{\substack{m, n \\ m+n \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \begin{Bmatrix} \alpha_1 & \alpha_2 & m' \\ \alpha_3 & \alpha_4 & n' \\ x' & y' & N \end{Bmatrix} = \delta_{xx'} \delta_{yy'}.$$

2.3 The canonical bases by separation of variables

Let us now obtain the explicit realizations for the bases $|m, n\rangle_N$ and $|x, y\rangle_N$ corresponding to the coupling schemes (2.6) and (2.7). As shall be seen, these bases correspond to the separation of variables in the equation $H\Upsilon = \Lambda\Upsilon$ using different cylindrical coordinate systems. Note that this eigenvalue equation has been studied by Kalnins, Miller and Tratnik in [15].

2.3.1 The basis for $\{Q^{(12)}, Q^{(34)}\}$

To obtain the coordinate realization of the basis corresponding to the first coupling scheme (2.6), we look for functions $\Psi_{m,n;N}$ on the 3-sphere that satisfy

$$Q^{(12)}\Psi_{m,n;N} = \lambda_m^{(12)}\Psi_{m,n;N}, \quad Q^{(34)}\Psi_{m,n;N} = \lambda_n^{(34)}\Psi_{m,n;N}, \quad Q\Psi_{m,n;N} = \Lambda_N\Psi_{m,n;N},$$

with eigenvalues

$$\begin{aligned} \lambda_m^{(12)} &= (m + \alpha_1/2 + \alpha_2/2)(m + \alpha_1/2 + \alpha_2/2 + 1), \\ \lambda_n^{(34)} &= (n + \alpha_3/2 + \alpha_4/2)(n + \alpha_3/2 + \alpha_4/2 + 1), \\ \Lambda_N &= (N + |\alpha|/2 + 1)(N + |\alpha|/2 + 2), \end{aligned}$$

where $|\alpha| = \sum_i \alpha_i$. The expressions for the spectra follow directly from the fact that the operators are intermediate Casimir operators in the addition of $\mathfrak{su}(1, 1)$ representations of the positive-discrete series [32]. Consider the set of cylindrical coordinates $\{\theta, \phi_1, \phi_2\}$ defined by

$$s_1 = \cos \theta \cos \phi_1, \quad s_2 = \cos \theta \sin \phi_1, \quad s_3 = \sin \theta \cos \phi_2, \quad s_4 = \sin \theta \sin \phi_2. \quad (2.11)$$

In these coordinates, one finds from (2.5) that the operators $Q^{(12)}, Q^{(34)}$ read

$$\begin{aligned} Q^{(12)} &= \frac{1}{4} \left(-\partial_{\phi_1}^2 + a_1 \operatorname{tg}^2 \phi_1 + \frac{a_2}{\operatorname{tg}^2 \phi_1} + (a_1 + a_2 - 1) \right), \\ Q^{(34)} &= \frac{1}{4} \left(-\partial_{\phi_2}^2 + a_3 \operatorname{tg}^2 \phi_2 + \frac{a_4}{\operatorname{tg}^2 \phi_2} + (a_3 + a_4 - 1) \right), \end{aligned}$$

and that Q takes the form

$$\begin{aligned} Q &= \frac{1}{4} \left[-\partial_{\theta}^2 + \left(\operatorname{tg} \theta - \frac{1}{\operatorname{tg} \theta} \right) \partial_{\theta} \right. \\ &\quad \left. + \frac{1}{\cos^2 \theta} \left(-\partial_{\phi_1}^2 + \frac{a_1}{\cos^2 \phi_1} + \frac{a_2}{\sin^2 \phi_1} \right) + \frac{1}{\sin^2 \theta} \left(-\partial_{\phi_2}^2 + \frac{a_3}{\cos^2 \phi_2} + \frac{a_4}{\sin^2 \phi_2} \right) \right]. \end{aligned}$$

It is directly seen from the above expressions that $\Psi_{m,n;N}$ will separate in the coordinates (2.11). Using standard techniques, one finds that the wavefunctions have the expression

$$\begin{aligned} \langle \theta, \phi_1, \phi_2 | m, n \rangle_N &= \Psi_{m,n;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(\theta, \phi_1, \phi_2) = \eta_m^{(\alpha_2, \alpha_1)} \eta_n^{(\alpha_4, \alpha_3)} \\ &\quad \times \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} (\cos \theta \cos \phi_1)^{\alpha_1+1/2} (\cos \theta \sin \phi_1)^{\alpha_2+1/2} \\ &\quad \times (\sin \theta \cos \phi_2)^{\alpha_3+1/2} (\sin \theta \sin \phi_2)^{\alpha_4+1/2} \cos^{2m} \theta \sin^{2n} \theta P_m^{(\alpha_2, \alpha_1)}(\cos 2\phi_1) \\ &\quad \times P_n^{(\alpha_4, \alpha_3)}(\cos 2\phi_2) P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)}(\cos 2\theta), \end{aligned} \quad (2.12)$$

where $P_n^{(\alpha, \beta)}(x)$ are the classical Jacobi polynomials (see Appendix A). The normalization factor

$$\eta_n^{(\alpha, \beta)} = \sqrt{\frac{2 \Gamma(m+1) \Gamma(m+\alpha+\beta+1) \Gamma(2m+\alpha+\beta+2)}{\Gamma(m+\alpha+1) \Gamma(m+\beta+1) \Gamma(2m+\alpha+\beta+1)}}, \quad (2.13)$$

ensures that the following orthonormality condition holds:

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} {}_N \langle m', n' | \theta, \phi_1, \phi_2 \rangle \langle \theta, \phi_1, \phi_2 | m, n \rangle_N d\Omega = \delta_{mm'} \delta_{nn'} \delta_{NN'}, \quad (2.14)$$

where $d\Omega = \cos \theta \sin \theta d\theta d\phi_1 d\phi_2$. In Cartesian coordinates, $\Psi_{m,n;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}$ assumes the form

$$\begin{aligned} \langle s_1, s_2, s_3, s_4 | m, n \rangle_N &= \Psi_{m,n;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(s_1, s_2, s_3, s_4) = \eta_m^{(\alpha_2, \alpha_1)} \eta_n^{(\alpha_4, \alpha_3)} \\ &\times \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} \left(\prod_{i=1}^4 s_i^{\alpha_i+1/2} \right) (s_1^2 + s_2^2)^m (s_3^2 + s_4^2)^n P_m^{(\alpha_2, \alpha_1)} \left(\frac{s_1^2 - s_2^2}{s_1^2 + s_2^2} \right) \\ &\times P_n^{(\alpha_4, \alpha_3)} \left(\frac{s_3^2 - s_4^2}{s_3^2 + s_4^2} \right) P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} (s_1^2 + s_2^2 - s_3^2 - s_4^2). \end{aligned} \quad (2.15)$$

The wavefunctions $\Psi_{m,n;N}$ thus provide a concrete realization in the position representation of the basis state $|m, n\rangle_N$ corresponding to the first coupling scheme. A different realization of this state is given by Lievens and Van der Jeugt in [19], who examined realizations of coupled vectors in the coherent state representation for general tensor products.

2.3.2 The basis for $\{Q^{(13)}, Q^{(24)}\}$

To obtain the coordinate realization of the basis corresponding to the second coupling scheme (2.7), we look for functions $\Xi_{x,y;N}$ on the 3-sphere that satisfy

$$Q^{(13)} \Xi_{x,y;N} = \lambda_x^{(13)} \Xi_{x,y;N}, \quad Q^{(24)} \Xi_{x,y;N} = \lambda_y^{(24)} \Xi_{x,y;N}, \quad Q \Xi_{x,y;N} = \Lambda_N \Xi_{x,y;N},$$

where

$$\begin{aligned} \lambda_x^{(13)} &= (x + \alpha_1/2 + \alpha_3/2)(x + \alpha_1/2 + \alpha_3/2 + 1), \\ \lambda_y^{(24)} &= (y + \alpha_2/2 + \alpha_4/2)(y + \alpha_2/2 + \alpha_4/2 + 1), \\ \Lambda_N &= (N + |\alpha|/2 + 1)(N + |\alpha|/2 + 2), \end{aligned}$$

and $|\alpha| = \sum_{i=1}^4 \alpha_i$. Consider the set of cylindrical coordinates $\{\vartheta, \varphi_1, \varphi_2\}$ defined by

$$s_1 = \cos \vartheta \cos \varphi_1, \quad s_2 = \sin \vartheta \cos \varphi_2, \quad s_3 = \cos \vartheta \sin \varphi_1, \quad s_4 = \sin \vartheta \sin \varphi_2. \quad (2.16)$$

In these coordinates, the operators $Q^{(13)}, Q^{(24)}$ have the expressions

$$\begin{aligned} Q^{(13)} &= \frac{1}{4} \left(-\partial_{\varphi_1}^2 + a_1 \operatorname{tg}^2 \varphi_1 + \frac{a_3}{\operatorname{tg}^2 \varphi_1} + (a_1 + a_3 - 1) \right), \\ Q^{(24)} &= \frac{1}{4} \left(-\partial_{\varphi_2}^2 + a_2 \operatorname{tg}^2 \varphi_2 + \frac{a_4}{\operatorname{tg}^2 \varphi_2} + (a_2 + a_4 - 1) \right), \end{aligned}$$

and the total Casimir operator Q reads

$$\begin{aligned} Q &= \frac{1}{4} \left[-\partial_{\vartheta}^2 + \left(\operatorname{tg} \vartheta + \frac{1}{\operatorname{tg} \vartheta} \right) \partial_{\vartheta} \right. \\ &\quad \left. + \frac{1}{\cos^2 \vartheta} \left(-\partial_{\varphi_1}^2 + \frac{a_1}{\cos^2 \varphi_1} + \frac{a_3}{\sin^2 \varphi_1} \right) + \frac{1}{\sin^2 \vartheta} \left(-\partial_{\varphi_2}^2 + \frac{a_2}{\cos^2 \varphi_2} + \frac{a_4}{\sin^2 \varphi_2} \right) \right]. \end{aligned}$$

It is clear from the above that the functions $\Xi_{x,y;N}$ will separate in the coordinates (2.16). The wavefunctions $\Xi_{x,y;N}$ have the expression

$$\begin{aligned} \langle \vartheta, \varphi_1, \varphi_2 | x, y \rangle_N &= \Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(\vartheta, \varphi_1, \varphi_2) = \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \\ &\times \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} (\cos \vartheta \cos \varphi_1)^{\alpha_1+1/2} (\sin \vartheta \cos \varphi_2)^{\alpha_2+1/2} \\ &\times (\cos \vartheta \sin \varphi_1)^{\alpha_3+1/2} (\sin \vartheta \sin \varphi_2)^{\alpha_4+1/2} \cos^{2x} \vartheta \sin^{2y} \vartheta P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) \\ &\times P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)}(\cos 2\vartheta), \end{aligned}$$

where $\eta_n^{(\alpha, \beta)}$ is given by (2.13) and where $P_n^{(\alpha, \beta)}(x)$ are again the classical Jacobi polynomials. The wavefunctions obey the orthonormality condition

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} {}_N \langle x', y' | \vartheta, \varphi_1, \varphi_2 \rangle \langle \vartheta, \varphi_1, \varphi_2 | x, y \rangle_N d\Omega = \delta_{xx'} \delta_{yy'} \delta_{NN'}, \quad (2.17)$$

where $d\Omega = \cos \vartheta \sin \vartheta d\vartheta d\varphi_1 d\varphi_2$. In Cartesian coordinates, one has

$$\begin{aligned} \langle s_1, s_2, s_3, s_4 | x, y \rangle_N &= \Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(s_1, s_2, s_3, s_4) = \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \\ &\times \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} \left(\prod_{i=1}^4 s_i^{\alpha_i+1/2} \right) (s_1^2 + s_3^2)^x (s_2^2 + s_4^2)^y P_x^{(\alpha_3, \alpha_1)} \left(\frac{s_1^2 - s_3^2}{s_1^2 + s_3^2} \right) \\ &\times P_y^{(\alpha_4, \alpha_2)} \left(\frac{s_2^2 - s_4^2}{s_2^2 + s_4^2} \right) P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} (s_1^2 + s_3^2 - s_2^2 - s_4^2). \end{aligned} \quad (2.18)$$

Note that (2.18) can be obtained directly from (2.15) by permuting the indices 2 and 3. The wavefunctions $\Xi_{x,y;N}$ thus provide a concrete realization of the basis states $|x, y\rangle_N$ corresponding to the coupling scheme (2.7) in the position representation.

2.4 $9j$ symbols as overlap coefficients, integral representation and symmetries

In view of (2.9), the $9j$ coefficients for the positive-discrete series of $\mathfrak{su}(1, 1)$ representations can be expressed as the expansion coefficients between the wavefunctions $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ at a given point, i.e.

$$\Psi_{m,n;N} = \sum_{\substack{x,y \\ x+y \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \Xi_{x,y;N}. \quad (2.19)$$

The orthogonality relation (2.17) immediately yields the integral formula

$$\begin{aligned} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} &= \eta_m^{(\alpha_2, \alpha_1)} \eta_n^{(\alpha_4, \alpha_3)} \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} \\ &\times \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} \int_{S_+^3} \prod_{i=1}^4 (s_i^2)^{\alpha_i+1/2} ds_i (s_1^2 + s_2^2)^m (s_3^2 + s_4^2)^n \\ &\times (s_1^2 + s_3^2)^x (s_2^2 + s_4^2)^y P_m^{(\alpha_2, \alpha_1)} \left(\frac{s_1^2 - s_2^2}{s_1^2 + s_2^2} \right) P_n^{(\alpha_4, \alpha_3)} \left(\frac{s_3^2 - s_4^2}{s_3^2 + s_4^2} \right) P_x^{(\alpha_3, \alpha_1)} \left(\frac{s_1^2 - s_3^2}{s_1^2 + s_3^2} \right) \\ &\times P_y^{(\alpha_4, \alpha_2)} \left(\frac{s_2^2 - s_4^2}{s_2^2 + s_4^2} \right) P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} (s_1^2 + s_2^2 - s_3^2 - s_4^2) \end{aligned}$$

$$\times P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)}(s_1^2 + s_3^2 - s_2^2 - s_4^2), \quad (2.20)$$

where S_+^3 stands for the totally positive octant of the 3-sphere described by $\sum_{i=1}^4 s_i^2 = 1$ with $s_i \geq 0$. The integral expression (2.20) looks rather complicated and shall be simplified in the next section. However, the formula (2.20) and the elementary properties of the Jacobi polynomials can be used to efficiently obtain the symmetry relations satisfied by the $9j$ symbols (2.9). As a first example, one can read off directly from (2.20) the symmetry relation

$$\begin{pmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_3 & x \\ \alpha_2 & \alpha_4 & y \\ m & n & N \end{pmatrix}. \quad (2.21)$$

which we shall refer to as the ‘‘duality property’’ of $9j$ symbols. As a second example, using the well-known identity $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$, one finds that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{pmatrix} = (-1)^{N+m+n-x-y} \begin{pmatrix} \alpha_2 & \alpha_1 & m \\ \alpha_4 & \alpha_3 & n \\ y & x & N \end{pmatrix} = (-1)^{N+x+y-m-n} \begin{pmatrix} \alpha_3 & \alpha_4 & n \\ \alpha_1 & \alpha_2 & m \\ x & y & N \end{pmatrix}. \quad (2.22)$$

A number of other symmetries can be derived by combining the above. Let us note that the formula (2.20) can also be found from the results of [18].

3 Double integral formula and the vacuum $9j$ coefficients

In this section, a double integral formula for the $9j$ symbols is obtained by extending the wavefunctions to the complex three-sphere and taking the complex radius to zero. The formula is then used to compute the vacuum $9j$ coefficients explicitly.

3.1 Extension of the wavefunctions

The wavefunctions $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ can easily be extended to the complex three-sphere of radius r^2 using their expressions in Cartesian coordinates. The extended wavefunctions $\tilde{\Psi}_{m,n;N}$, $\tilde{\Xi}_{x,y;N}$ have the expressions

$$\begin{aligned} \tilde{\Psi}_{m,n;N} &= \eta_m^{(\alpha_2, \alpha_1)} \eta_n^{(\alpha_4, \alpha_3)} \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} \left(\prod_{i=1}^4 s_i^{\alpha_i+1/2} \right) \\ &\quad \times (s_1^2 + s_2^2)^m (s_3^2 + s_4^2)^n (s_1^2 + s_2^2 + s_3^2 + s_4^2)^{N-m-n} P_m^{(\alpha_2, \alpha_1)} \left(\frac{s_1^2 - s_2^2}{s_1^2 + s_2^2} \right) \\ &\quad \times P_n^{(\alpha_4, \alpha_3)} \left(\frac{s_3^2 - s_4^2}{s_3^2 + s_4^2} \right) P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} \left(\frac{s_1^2 + s_2^2 - s_3^2 - s_4^2}{s_1^2 + s_2^2 + s_3^2 + s_4^2} \right), \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \tilde{\Xi}_{x,y;N} &= \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} \left(\prod_{i=1}^4 s_i^{\alpha_i+1/2} \right) \\ &\quad \times (s_1^2 + s_3^2)^x (s_2^2 + s_4^2)^y (s_1^2 + s_2^2 + s_3^2 + s_4^2)^{N-x-y} P_x^{(\alpha_3, \alpha_1)} \left(\frac{s_1^2 - s_3^2}{s_1^2 + s_3^2} \right) \\ &\quad \times P_y^{(\alpha_4, \alpha_2)} \left(\frac{s_2^2 - s_4^2}{s_2^2 + s_4^2} \right) P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} \left(\frac{s_1^2 + s_3^2 - s_2^2 - s_4^2}{s_1^2 + s_2^2 + s_3^2 + s_4^2} \right), \end{aligned} \quad (3.2)$$

with $s_i \in \mathbb{C}$ for $i = 1, \dots, 4$. The expressions (3.1) and (3.2) correspond to the bases constructed by Lievens and Van der Jeugt in [18] in their examination of $3nj$ symbols for $\mathfrak{su}(1, 1)$. The basis vectors (3.1) and (3.2) also resemble the harmonic functions on S^3 of Dunkl and Xu [7], but do not correspond to the same separation of variables.

When the coordinates satisfy $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$, the wavefunctions (3.1) and (3.2) coincide with (2.15) and (2.18), respectively. When $r^2 \neq 1$, $\tilde{\Psi}_{m,n;N}$ and $\tilde{\Xi}_{x,y;N}$ differ from $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ by a constant factor of $r^{N+|\alpha|+2}$. Since the parameters N and α_i are fixed, the expansion (2.19) is not affected by this common multiplicative factor and one can write

$$\tilde{\Psi}_{m,n;N}(s_1, s_2, s_3, s_4) = \sum_{\substack{x,y \\ x+y \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \tilde{\Xi}_{x,y;N}(s_1, s_2, s_3, s_4), \quad (3.3)$$

for a given point (s_1, s_2, s_3, s_4) satisfying $s_1^2 + s_2^2 + s_3^2 + s_4^2 = r^2$. Let us now impose the condition

$$s_1^2 + s_2^2 = -(s_3^2 + s_4^2),$$

which corresponds to taking the radius of the complex three-sphere to zero. Upon introducing the new variables u and v defined by

$$u = \frac{s_1^2 - s_3^2}{s_1^2 + s_3^2}, \quad v = \frac{s_3^2 + 2s_4^2 + s_1^2}{s_1^2 + s_3^2},$$

and using the identity

$$(x+y)^m P_m^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_m}{m!} x^m {}_2F_1\left[\begin{matrix} -m, -\beta-m \\ \alpha+1 \end{matrix} \middle| -\frac{y}{x} \right],$$

in (3.1) and (3.2), one finds that the expansion (3.3) reduces to

$$\begin{aligned} c_{m,n;N} & \left(\frac{u-v}{2}\right)^N P_m^{(\alpha_2,\alpha_1)}\left(\frac{u+v+2}{u-v}\right) P_n^{(\alpha_4,\alpha_3)}\left(\frac{2-u-v}{v-u}\right) \\ & = \sum_{\substack{x,y \\ x+y \leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} d_{x,y;N} P_x^{(\alpha_3,\alpha_1)}(u) P_y^{(\alpha_4,\alpha_2)}(v), \end{aligned}$$

where the coefficients $c_{m,n;N}$ and $d_{x,y;N}$ read

$$\begin{aligned} c_{m,n;N} & = \eta_m^{(\alpha_2,\alpha_1)} \eta_n^{(\alpha_4,\alpha_3)} \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1, 2m+\alpha_1+\alpha_2+1)} \frac{(-1)^n (N+m+n+|\alpha|+3)_{N-m-n}}{(N-m-n)!}, \\ d_{x,y;N} & = \eta_x^{(\alpha_3,\alpha_1)} \eta_y^{(\alpha_4,\alpha_2)} \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1, 2x+\alpha_1+\alpha_3+1)} \frac{(-1)^y (N+x+y+|\alpha|+3)_{N-x-y}}{(N-x-y)!}. \end{aligned}$$

Here $(a)_n$ stands for the Pochhammer symbol defined by

$$(a)_n = (a)(a+1) \cdots (a+n-1), \quad (a)_0 = 1.$$

The orthogonality relation (A.1) for the Jacobi polynomials then leads to the integral representation

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \left[\frac{c_{m,n;N}}{d_{x,y;N}} \frac{2^{-N}}{h_x^{(\alpha_3,\alpha_1)} h_y^{(\alpha_4,\alpha_2)}} \right]$$

$$\begin{aligned} & \times \int_{-1}^1 \int_{-1}^1 du dv (1-u)^{\alpha_3} (1+u)^{\alpha_1} (1-v)^{\alpha_4} (1+v)^{\alpha_2} \\ & \times P_x^{(\alpha_3, \alpha_1)}(u) \left[P_m^{(\alpha_2, \alpha_1)} \left(\frac{u+v+2}{u-v} \right) (u-v)^N P_n^{(\alpha_4, \alpha_3)} \left(\frac{2-u-v}{v-u} \right) \right] P_y^{(\alpha_4, \alpha_2)}(v), \end{aligned} \quad (3.4)$$

where $h_n^{(\alpha, \beta)}$ is given by (A.2). The integral formula (3.4) coincides with the one found by Granovskii and Zhedanov [10] using a related approach. The formula (3.4) is one of the most simple expressions for $9j$ symbols. Given the wealth of results on the asymptotic behavior of Jacobi polynomials, one can expect the formula (3.4) to be useful in the examination of the asymptotic behavior of the $9j$ symbols, an active field [5, 34] of interest in particular for the study of spin networks related to quantum gravity [11].

3.2 The vacuum $9j$ coefficients

The integral expression (3.4) will now be used to obtain the explicit expression for the ‘‘vacuum’’ $9j$ coefficients, which correspond to the special case $m = n = 0$. These shall be used in the next section to further characterize the $9j$ symbols. Upon using the binomial expansion, the formula (3.4) gives the following expression for the vacuum $9j$ coefficients:

$$\begin{aligned} \left\{ \begin{array}{ccc} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{array} \right\} &= \left[\frac{c_{0,0;N}}{d_{x,y;N}} \frac{2^{-N}}{h_x^{(\alpha_3, \alpha_1)} h_y^{(\alpha_4, \alpha_2)}} \right] \sum_{k=0}^N \binom{N}{k} (-1)^{N-k} \int_{-1}^1 \int_{-1}^1 du dv \\ & \times (1-u)^{\alpha_3} (1+u)^{\alpha_1} P_x^{(\alpha_3, \alpha_1)}(u) u^k (1-v)^{\alpha_4} (1+v)^{\alpha_2} P_y^{(\alpha_4, \alpha_2)}(v) v^{N-k}, \end{aligned} \quad (3.5)$$

where $\binom{N}{k}$ is the binomial coefficient. To evaluate the integrals, one can use the expansion of the power function in series of Jacobi polynomials [3] which reads

$$x^k = \sum_{j=0}^k \left\{ \frac{2^j k!}{(k-j)!} \frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(2j+\alpha+\beta+1)} {}_2F_1 \left[\begin{array}{c} j-k, j+\alpha+1 \\ 2j+\alpha+\beta+2 \end{array} \middle| 2 \right] \right\} P_j^{(\alpha, \beta)}(x),$$

where ${}_pF_q$ stands for the generalized hypergeometric function [3]. Upon inserting the above expansion in (3.5) and using the orthogonality relation (A.1) for the Jacobi polynomials, one finds

$$\begin{aligned} \left\{ \begin{array}{ccc} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{array} \right\} &= \left[\frac{\eta_0^{(\alpha_2, \alpha_1)} \eta_0^{(\alpha_4, \alpha_3)} \eta_N^{(\alpha_3 + \alpha_4 + 1, \alpha_1 + \alpha_2 + 1)}}{\eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \eta_{N-x-y}^{(2y + \alpha_2 + \alpha_4 + 1, 2x + \alpha_1 + \alpha_3 + 1)}} \right] \\ & \times \left[\frac{(-1)^{N+x+y}}{2^{N-x-y}} \frac{(N+|\alpha|+3)_N}{(N+x+y+|\alpha|+3)_{N-x-y}} \right] \left[\frac{\Gamma(x+\alpha_1+\alpha_3+1) \Gamma(y+\alpha_2+\alpha_4+1)}{\Gamma(2x+\alpha_1+\alpha_3+1) \Gamma(2y+\alpha_2+\alpha_4+1)} \right] \\ & \times \sum_{k=0}^{N-x-y} \binom{N-x-y}{k} (-1)^k \\ & \times {}_2F_1 \left[\begin{array}{c} -k, x+\alpha_3+1 \\ 2x+\alpha_1+\alpha_3+2 \end{array} \middle| 2 \right] {}_2F_1 \left[\begin{array}{c} -(N-x-y-k), y+\alpha_4+1 \\ 2y+\alpha_2+\alpha_4+2 \end{array} \middle| 2 \right]. \end{aligned}$$

The summation in the above relation can be evaluated by means of the formula

$$\sum_{\ell=0}^M \frac{(-N)_\ell}{\ell!} {}_2F_1 \left[\begin{array}{c} -\ell, a_1 \\ b_1 \end{array} \middle| x \right] {}_2F_1 \left[\begin{array}{c} \ell - N, a_2 \\ b_2 \end{array} \middle| x \right] = x^N \frac{(a_1)_N}{(b_1)_N} {}_3F_2 \left[\begin{array}{c} -N, a_2, 1 - b_1 - N \\ b_2, 1 - a_1 - N \end{array} \middle| 1 \right].$$

Then using identity $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, the following expression is obtained:

$$\begin{aligned} \begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} &= \binom{N}{x, y}^{1/2} [(\alpha_1 + 1)_x (\alpha_2 + 1)_y (\alpha_3 + 1)_x (\alpha_4 + 1)_y]^{1/2} \\ &\times \left[\frac{(N + |\alpha| + 3)_{x+y}}{(\alpha_1 + \alpha_2 + 2)_N (\alpha_3 + \alpha_4 + 2)_N} \right]^{1/2} \left[\frac{(\alpha_1 + \alpha_3 + 1)_x (\alpha_1 + \alpha_3 + 2)_{N+x-y}}{(\alpha_1 + \alpha_3 + 1)_{2x} (\alpha_1 + \alpha_3 + 2)_{2x}} \right]^{1/2} \\ &\times \left[\frac{(\alpha_2 + \alpha_4 + 1)_y (\alpha_2 + \alpha_4 + 2)_{2y}}{(\alpha_2 + \alpha_4 + 1)_{2y} (\alpha_2 + \alpha_4 + 2)_{N-x+y}} \right]^{1/2} (y + \alpha_4 + 1)_{N-x-y} \\ &\times {}_3F_2 \left[\begin{matrix} -(N - x - y), -(N - x + y + \alpha_2 + \alpha_4 + 1), x + \alpha_3 + 1 \\ -(N - x + \alpha_4), 2x + \alpha_1 + \alpha_3 + 2 \end{matrix} \middle| 1 \right], \end{aligned} \quad (3.6)$$

where $\binom{N}{x, y} = \frac{N!}{x!y!(N-x-y)!}$ stands for the trinomial coefficients. The analogous formula for the $9j$ coefficients of $\mathfrak{su}(2)$ has been given by Hoare and Rahman [12]. The duality formula (2.21) can be used to obtain a similar expression for the case where $x = y = 0$.

4 Raising, lowering operators and contiguity relations

In this section, raising and lowering operators are introduced and are called upon to obtain the relations between contiguous $9j$ symbols by direct computation. These relations are used to show that the $9j$ symbols can be expressed as the product of the vacuum $9j$ coefficients and a rational function of the variables x, y .

4.1 Raising, lowering operators and factorization

Let $A_{\pm}^{(\alpha_1, \alpha_2)}$ be defined as

$$A_{\pm}^{(\alpha_1, \alpha_2)} = \frac{1}{2} \left[\pm \partial_{\phi_1} - \operatorname{tg} \phi_1 (\alpha_1 + 1/2) + \frac{1}{\operatorname{tg} \phi_1} (\alpha_2 + 1/2) \right], \quad (4.1)$$

and let $B_{\pm}^{(\alpha_3, \alpha_4)}$ have the expression

$$B_{\pm}^{(\alpha_3, \alpha_4)} = \frac{1}{2} \left[\pm \partial_{\phi_2} - \operatorname{tg} \phi_2 (\alpha_3 + 1/2) + \frac{1}{\operatorname{tg} \phi_2} (\alpha_4 + 1/2) \right], \quad (4.2)$$

where the coordinates (2.11) have been used. It is directly checked that with respect to the scalar product in (2.14), one has $(A_{\pm}^{(\alpha_1, \alpha_2)})^{\dagger} = A_{\mp}^{(\alpha_1, \alpha_2)}$ and $(B_{\pm}^{(\alpha_3, \alpha_4)})^{\dagger} = B_{\mp}^{(\alpha_3, \alpha_4)}$, where x^{\dagger} stands for the adjoint of x . With the help of the relations (A.4) and (A.3), it is easily verified that one has on the one hand

$$\begin{aligned} &A_+^{(\alpha_1, \alpha_2)} \Psi_{m, n; N}^{(\alpha_1+1, \alpha_2+1, \alpha_3, \alpha_4)}(\theta, \phi_1, \phi_2) \\ &= \sqrt{(m+1)(m+\alpha_1+\alpha_2+2)} \Psi_{m+1, n; N+1}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(\theta, \phi_1, \phi_2), \\ &A_-^{(\alpha_1, \alpha_2)} \Psi_{m, n; N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(\theta, \phi_1, \phi_2) \\ &= \sqrt{m(m+\alpha_1+\alpha_2+1)} \Psi_{m-1, n; N-1}^{(\alpha_1+1, \alpha_2+1, \alpha_3, \alpha_4)}(\theta, \phi_1, \phi_2), \end{aligned} \quad (4.3)$$

and on the other hand

$$B_+^{(\alpha_3, \alpha_4)} \Psi_{m, n; N}^{(\alpha_1, \alpha_2, \alpha_3+1, \alpha_4+1)}(\theta, \phi_1, \phi_2)$$

$$\begin{aligned}
&= \sqrt{(n+1)(n+\alpha_3+\alpha_4+2)} \Psi_{m,n+1;N+1}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta, \phi_1, \phi_2), \\
B_-^{(\alpha_3,\alpha_4)} \Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta, \phi_1, \phi_2) \\
&= \sqrt{n(n+\alpha_3+\alpha_4+1)} \Psi_{m,n-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(\theta, \phi_1, \phi_2),
\end{aligned}$$

where $\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ is given by (2.12). The operators (4.1) and (4.2) provide a factorization of the intermediate Casimir operators $Q^{(12)}$ and $Q^{(34)}$, respectively. Indeed, it is directly checked that

$$\begin{aligned}
A_+^{(\alpha_1,\alpha_2)} A_-^{(\alpha_1,\alpha_2)} &= Q^{(12)} - (\alpha_1/2 + \alpha_2/2)(\alpha_1/2 + \alpha_2/2 + 1), \\
B_+^{(\alpha_3,\alpha_4)} B_-^{(\alpha_3,\alpha_4)} &= Q^{(34)} - (\alpha_3/2 + \alpha_4/2)(\alpha_3/2 + \alpha_4/2 + 1).
\end{aligned} \tag{4.4}$$

4.2 Contiguity relations

The raising/lowering operators (4.1) and (4.2) can be used to obtain the relations satisfied by contiguous $9j$ symbols. To facilitate the computations, let us make explicit the dependence of the canonical basis vectors $|m, n\rangle_N$, $|x, y\rangle_N$ on the parameters α_i by writing

$$|m, n\rangle_N \equiv |\alpha_1, \alpha_2, \alpha_3, \alpha_4; m, n\rangle_N, \quad |x, y\rangle_N \equiv |\alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y\rangle_N.$$

With this notation the $9j$ symbols are written as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = {}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | \alpha_1, \alpha_2, \alpha_3, \alpha_4; m, n \rangle_N.$$

To obtain the first contiguity relation for $9j$ symbols, one considers the matrix element

$${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1,\alpha_2)} | \alpha_1 + 1, \alpha_2 + 1, \alpha_3, \alpha_4; m, n \rangle_{N-1}.$$

By acting with $A_+^{(\alpha_1,\alpha_2)}$ on $|\alpha_1 + 1, \alpha_2 + 1, \alpha_3, \alpha_4; m, n\rangle_{N-1}$ using (4.3), one finds

$$\begin{aligned}
&\sqrt{(m+1)(m+\alpha_1+\alpha_2+2)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \\
&= {}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1,\alpha_2)} | \alpha_1 + 1, \alpha_2 + 1, \alpha_3, \alpha_4; m, n \rangle_{N-1}.
\end{aligned} \tag{4.5}$$

To obtain the desired relation, one must determine ${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1,\alpha_2)}$ or equivalently

$$(A_+^{(\alpha_1,\alpha_2)})^\dagger | \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y \rangle_N = A_-^{(\alpha_1,\alpha_2)} | \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y \rangle_N,$$

where the reality of the basis functions $\Xi_{x,y;N}$ has been used. This can be done directly by writing $A_-^{(\alpha_1,\alpha_2)}$ in the coordinates $\{\vartheta, \varphi_1, \varphi_2\}$ defined in (2.16), acting with this operator on the wavefunctions $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\vartheta, \varphi_1, \varphi_2)$ and using the properties of the Jacobi polynomials. Since this step represents no fundamental difficulties, the details of the computation are relegated to Appendix B. One finds that

$$\begin{aligned}
&A_-^{(\alpha_1,\alpha_2)} \Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \\
&= \sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}
\end{aligned}$$

$$\begin{aligned}
& \times \Xi_{x,y;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\
& + \sqrt{\frac{x(x+\alpha_3)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \Xi_{x-1,y;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\
& - \sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)y(y+\alpha_4)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \Xi_{x,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\
& - \sqrt{\frac{x(x+\alpha_3)y(y+\alpha_4)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \Xi_{x-1,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}, \quad (4.6)
\end{aligned}$$

where the shorthand notation $\alpha_{ij} = \alpha_i + \alpha_j$ was used. Combining (4.2) with (4.6), one finds the contiguity relation

$$\begin{aligned}
& \sqrt{(m+1)(m+\alpha_{12}+2)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \\
& = \sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N-1 \end{Bmatrix} \\
& + \sqrt{\frac{x(x+\alpha_3)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y & N-1 \end{Bmatrix} \\
& - \sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)y(y+\alpha_4)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y-1 & N-1 \end{Bmatrix} \\
& - \sqrt{\frac{x(x+\alpha_3)y(y+\alpha_4)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y-1 & N-1 \end{Bmatrix}. \quad (4.7)
\end{aligned}$$

To obtain the second contiguity relation, we could consider the matrix element

$${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | B_+^{(\alpha_3, \alpha_4)} | \alpha_1, \alpha_2, \alpha_3+1, \alpha_4+1; m, n \rangle_{N-1},$$

and proceed similarly by direct computation. However, it is easier to use the symmetry relation (2.22) to permute the first two rows of the relation (4.7) and then take $\alpha_1 \leftrightarrow \alpha_3$, $\alpha_2 \leftrightarrow \alpha_4$,

$m \leftrightarrow n$. This directly leads to the second contiguity relation

$$\begin{aligned}
& \sqrt{(n+1)(n+\alpha_{34}+2)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} \\
&= \sqrt{\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)(y+\alpha_4+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x & y & N-1 \end{Bmatrix} \\
&- \sqrt{\frac{x(x+\alpha_1)(y+\alpha_4+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x-1 & y & N-1 \end{Bmatrix} \\
&+ \sqrt{\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)y(y+\alpha_2)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x & y-1 & N-1 \end{Bmatrix} \\
&- \sqrt{\frac{x(x+\alpha_1)y(y+\alpha_2)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x-1 & y-1 & N-1 \end{Bmatrix}. \tag{4.8}
\end{aligned}$$

A third contiguity relation can be found by considering the matrix element

$${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_-^{(\alpha_1-1, \alpha_2-1)} | m, n; \alpha_1-1, \alpha_2-1, \alpha_3, \alpha_4 \rangle_{N+1}.$$

Upon using the action (4.3), one has

$$\begin{aligned}
& \sqrt{m(m+\alpha_1+\alpha_2-1)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \\
&= {}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_-^{(\alpha_1-1, \alpha_2-1)} | m, n; \alpha_1-1, \alpha_2-1, \alpha_3, \alpha_4 \rangle_N.
\end{aligned}$$

To obtain the relation, one needs to compute ${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_-^{(\alpha_1-1, \alpha_2-1)}$ or equivalently

$$(A_-^{(\alpha_1-1, \alpha_2-1)})^\dagger | \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y \rangle_N = A_+^{(\alpha_1-1, \alpha_2-1)} | \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y \rangle_N. \tag{4.9}$$

Following the calculations of Appendix C, one arrives at

$$\begin{aligned}
& A_+^{(\alpha_1-1, \alpha_2-1)} \Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \\
&= \sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+\alpha_2)(y+\alpha_{24})(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}
\end{aligned}$$

$$\begin{aligned}
& \times \Xi_{x,y;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)} \\
& + \sqrt{\frac{(x+1)(x+\alpha_3+1)(y+\alpha_2)(y+\alpha_{24})(N-x+y+\alpha_{24}+1)(N+x-y+\alpha_{13}+2)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \Xi_{x+1,y;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)} \\
& - \sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+1)(y+\alpha_4+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \Xi_{x,y+1;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)} \\
& - \sqrt{\frac{(x+1)(x+\alpha_3+1)(y+1)(y+\alpha_4+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \Xi_{x+1,y+1;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)}. \tag{4.10}
\end{aligned}$$

Combining the above relation with (4.9), there comes

$$\begin{aligned}
& \sqrt{m(m+\alpha_{12}-1)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \\
& = \sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+\alpha_2)(y+\alpha_{24})(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \begin{Bmatrix} \alpha_1-1 & \alpha_2-1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N+1 \end{Bmatrix} \\
& + \sqrt{\frac{(x+1)(x+\alpha_3+1)(y+\alpha_2)(y+\alpha_{24})(N-x+y+\alpha_{24}+1)(N+x-y+\alpha_{13}+2)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \\
& \times \begin{Bmatrix} \alpha_1-1 & \alpha_2-1 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y & N+1 \end{Bmatrix} \\
& - \sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+1)(y+\alpha_4+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \begin{Bmatrix} \alpha_1-1 & \alpha_2-1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y+1 & N+1 \end{Bmatrix} \\
& - \sqrt{\frac{(x+1)(x+\alpha_3+1)(y+1)(y+\alpha_4+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \\
& \times \begin{Bmatrix} \alpha_1-1 & \alpha_2-1 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y+1 & N+1 \end{Bmatrix}. \tag{4.11}
\end{aligned}$$

Upon applying the symmetry relation (2.22) on (4.11) and then performing the substitutions $\alpha_1 \leftrightarrow \alpha_3$, $\alpha_2 \leftrightarrow \alpha_4$ and $m \leftrightarrow n$, one finds a fourth contiguity relation

$$\sqrt{n(n+\alpha_{34}-1)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n-1 \\ x & y & N \end{Bmatrix}$$

$$\begin{aligned}
&= \sqrt{\frac{(x + \alpha_3)(x + \alpha_{13})(y + \alpha_4)(y + \alpha_{24})(N - x - y + 1)(N + x + y + |\alpha| + 2)}{(2x + \alpha_{13})(2x + \alpha_{13} + 1)(2y + \alpha_{24})(2y + \alpha_{24} + 1)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 - 1 & \alpha_4 - 1 & n \\ x & y & N + 1 \end{Bmatrix} \\
&- \sqrt{\frac{(x + 1)(x + \alpha_1 + 1)(y + \alpha_4)(y + \alpha_{24})(N - x + y + \alpha_{24} + 1)(N + x - y + \alpha_{13} + 2)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24})(2y + \alpha_{24} + 1)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 - 1 & \alpha_4 - 1 & n \\ x + 1 & y & N + 1 \end{Bmatrix} \\
&+ \sqrt{\frac{(x + \alpha_3)(x + \alpha_{13})(y + 1)(y + \alpha_2 + 1)(N + x - y + \alpha_{13} + 1)(N - x + y + \alpha_{24} + 2)}{(2x + \alpha_{13})(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)(2y + \alpha_{24} + 2)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 - 1 & \alpha_4 - 1 & n \\ x & y + 1 & N + 1 \end{Bmatrix} \\
&- \sqrt{\frac{(x + 1)(x + \alpha_1 + 1)(y + 1)(y + \alpha_2 + 1)(N - x - y)(N + x + y + |\alpha| + 3)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)(2y + \alpha_{24} + 2)}} \\
&\times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 - 1 & \alpha_4 - 1 & n \\ x + 1 & y + 1 & N + 1 \end{Bmatrix}. \tag{4.12}
\end{aligned}$$

The relations (4.7), (4.8), (4.11) and (4.12) are usually obtained by writing the $9j$ symbols in terms of Clebsch–Gordan coefficients (given in terms of the Hahn polynomials) and using the properties of the latter. In our presentation however, these relations emerge from a direct computation involving Jacobi polynomials.

4.3 $9j$ symbols and rational functions

It will now be shown that the $9j$ symbols of $\mathfrak{su}(1, 1)$ can be expressed as the product of the vacuum coefficients and a rational function. To this end, let us write the $9j$ symbols as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} R_{m,n;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(x, y),$$

where $R_{0,0;N}(x, y) \equiv 1$, $R_{-1,n;N}(x, y) = R_{m,-1;N}(x, y) = R_{m,n;-1}(x, y) = 0$. Since the vacuum $9j$ coefficients are known explicitly, the contiguity relations (4.7), (4.8) can be used to generate the functions $R_{m,n;N}(x, y)$. Upon taking

$$G_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} = {}_3F_2 \left[\begin{matrix} -(N - x - y), -(N - x + y + \alpha_2 + \alpha_4 + 1), x + \alpha_3 + 1 \\ -(N - x + \alpha_4), 2x + \alpha_1 + \alpha_3 + 2 \end{matrix} \middle| 1 \right],$$

using the expression (3.6) for the vacuum coefficients, the relations (4.7) and (4.8) become

$$\begin{aligned}
&\sqrt{\frac{(m + 1)(m + \alpha_{12} + 2)N(N + \alpha_{12} + 2)(N + |\alpha| + 3)(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_{12} + 2)(\alpha_{12} + 3)(N + \alpha_{34} + 1)}} \\
&\times R_{m+1,n;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}(x, y) = \frac{G_{x,y;N-1}^{(\alpha_1+1, \alpha_2+1, \alpha_3, \alpha_4)}}{G_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}} R_{m,n;N-1}^{(\alpha_1+1, \alpha_2+1, \alpha_3, \alpha_4)}(x, y)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{(x + \alpha_1 + 1)(x + \alpha_{13} + 1)(y + \alpha_2 + 1)(y + \alpha_{24} + 1)(N - x - y)(N + x + y + |\alpha| + 3)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)(N - x + \alpha_4)} \right] \\
& + \left[\frac{x(y + \alpha_2 + 1)(y + \alpha_2 + \alpha_4 + 1)}{(2y + \alpha_{24} + 1)} \right] \frac{G_{x-1,y;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}(x-1, y) \\
& - \left[\frac{(x + \alpha_1 + 1)(x + \alpha_{13} + 1)(N + x - y + \alpha_{13} + 2)y(y + \alpha_4)(N - x + y + \alpha_{24} + 1)}{(N - x + \alpha_4)(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)} \right] \\
& \times \frac{G_{x,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}(x, y-1) \\
& - \left[\frac{xy(y + \alpha_4)}{(2y + \alpha_{24} + 1)} \right] \frac{G_{x-1,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}(x-1, y-1), \tag{4.13}
\end{aligned}$$

and

$$\begin{aligned}
& \sqrt{\frac{(n+1)(n+\alpha_{34}+2)N(N+\alpha_{34}+2)(N+|\alpha|+3)(\alpha_3+1)(\alpha_4+1)}{(\alpha_{34}+2)(\alpha_{34}+3)(N+\alpha_{12}+1)}} R_{m,n+1;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \\
& = \frac{G_{x,y;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(x, y) \\
& \times \left[\frac{(x + \alpha_3 + 1)(x + \alpha_{13} + 1)(y + \alpha_{24} + 1)(N - x - y)(N + x + y + |\alpha| + 3)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)} \right] \\
& - \left[\frac{x(y + \alpha_{24} + 1)(N - x + \alpha_4 + 1)}{(2y + \alpha_{24} + 1)} \right] \frac{G_{x-1,y;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(x-1, y) \\
& + \left[\frac{(x + \alpha_3 + 1)(x + \alpha_{13} + 1)(y)(N + x - y + \alpha_{13} + 2)(N - x + y + \alpha_{24} + 1)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)} \right] \\
& \times \frac{G_{x,y-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(x, y-1) \\
& - \left[\frac{xy(N - x + \alpha_4 + 1)}{(2y + \alpha_{24} + 1)} \right] \frac{G_{x-1,y-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}}{G_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}} R_{m,n;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(x-1, y-1). \tag{4.14}
\end{aligned}$$

From (4.13) and (4.14), one can generate the functions $R_{m,n;N}(x, y)$ recursively. Writing the first few cases, one sees that the $R_{m,n}(x, y)$ are rational functions of the variables x, y . This is in contradiction with the assertion of [12], where the functions $R_{m,n}(x, y)$ are claimed to be polynomials in the variables x, y . In view of the orthogonality relation (2.10), the rational functions $R_{m,n}(x, y)$ satisfy the orthogonality relation

$$\sum_{\substack{x,y \\ x+y \leq N}} t_{x,y;N} R_{m,n;N}(x, y) R_{m',n'}(x, y) = \delta_{mm'} \delta_{nn'},$$

where the weight function is of the form

$$t_{x,y;N} = \left\{ \begin{array}{ccc} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{array} \right\}^2.$$

It is possible to express the $9j$ symbols of $\mathfrak{su}(1, 1)$ in terms of polynomials in the two variables x, y as was done by Van der Jeugt in [31]. However the involved family of polynomials

$P_{m,n;N}(x, y)$ is of degree $(N - m, N - n)$ the variables $x(x + \alpha_{13} + 1)$ and $y(y + \alpha_{24} + 1)$ and hence do not include polynomials whose total degree is less than N .

5 Difference equations and recurrence relations

In this section, it is shown that the factorization property of the intermediate Casimir operators and the contiguity relations can be used to exhibit difference equations and recurrence relations for the $9j$ symbols.

A first difference equation can be obtained by considering the matrix element

$${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1, \alpha_2)} A_-^{(\alpha_1, \alpha_2)} | \alpha_1, \alpha_2, \alpha_3, \alpha_4; m, n \rangle_N.$$

Using (4.3), one has on the one hand

$$\begin{aligned} & {}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1, \alpha_2)} A_-^{(\alpha_1, \alpha_2)} | \alpha_1, \alpha_2, \alpha_3, \alpha_4; m, n \rangle_N \\ &= m(m + \alpha_1 + \alpha_2 + 1) \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}. \end{aligned}$$

Using on the other hand (4.6) and (4.10) to compute ${}_N \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4; x, y | A_+^{(\alpha_1, \alpha_2)} A_-^{(\alpha_1, \alpha_2)} \rangle$, one arrives at the difference equation

$$\begin{aligned} m(m + \alpha_{12} + 1) \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} &= E_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y-1 & N \end{Bmatrix} \\ &+ E_{x+1,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y+1 & N \end{Bmatrix} + D_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y-1 & N \end{Bmatrix} \\ &+ D_{x,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y+1 & N \end{Bmatrix} + C_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y & N \end{Bmatrix} \\ &+ C_{x+1,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y & N \end{Bmatrix} + B_{x+1,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y-1 & N \end{Bmatrix} \\ &+ B_{x,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y+1 & N \end{Bmatrix} + A_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}. \end{aligned} \tag{5.1}$$

The coefficients are given by

$$\begin{aligned} E_{x,y} &= -\sqrt{(N + x + y + |\alpha| + 1)(N + x + y + |\alpha| + 2)} \\ &\times \sqrt{(N - x - y + 1)(N - x - y + 2)} \sqrt{\frac{x(x + \alpha_1)(x + \alpha_3)(x + \alpha_{13})}{(2x + \alpha_{13} - 1)(2x + \alpha_{13})^2(2x + \alpha_{13} + 1)}} \\ &\times \sqrt{\frac{y(y + \alpha_2)(y + \alpha_4)(y + \alpha_{24})}{(2y + \alpha_{24} - 1)(2y + \alpha_{24})^2(2y + \alpha_{24} + 1)}}, \\ D_{x,y} &= -\sqrt{(N + x - y + \alpha_{13} + 2)(N - x + y + \alpha_{24} + 1)} \\ &\times \sqrt{(N - x - y + 1)(N + x + y + |\alpha| + 2)} \sqrt{\frac{y(y + \alpha_2)(y + \alpha_4)(y + \alpha_{24})}{(2y + \alpha_{24} - 1)(2y + \alpha_{24})^2(2y + \alpha_{24} + 1)}} \end{aligned}$$

$$\begin{aligned}
& \left[\frac{x(x + \alpha_3)}{(2x + \alpha_{13})(2x + \alpha_{13} + 1)} + \frac{(x + \alpha_1 + 1)(x + \alpha_{13} + 1)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)} \right], \\
C_{x,y} &= \sqrt{(N + x - y + \alpha_{13} + 1)(N - x + y + \alpha_{24} + 2)} \\
& \times \sqrt{(N - x - y + 1)(N + x + y + |\alpha| + 2)} \sqrt{\frac{x(x + \alpha_1)(x + \alpha_3)(x + \alpha_{13})}{(2x + \alpha_{13} - 1)(2x + \alpha_{13})^2(2x + \alpha_{13} + 1)}} \\
& \left[\frac{y(y + \alpha_4)}{(2y + \alpha_{24})(2y + \alpha_{24} + 1)} + \frac{(y + \alpha_2 + 1)(y + \alpha_{24} + 1)}{(2y + \alpha_{24} + 1)(2y + \alpha_{24} + 2)} \right], \\
B_{x,y} &= -\sqrt{(N + x - y + \alpha_{13} + 1)(N + x - y + \alpha_{13} + 2)(N - x + y + \alpha_{24} + 1)} \\
& \sqrt{(N - x + y + \alpha_{24} + 2)} \sqrt{\frac{x(x + \alpha_1)(x + \alpha_3)(x + \alpha_{13})}{(2x + \alpha_{13} - 1)(2x + \alpha_{13})^2(2x + \alpha_{13} + 1)}} \\
& \times \sqrt{\frac{y(y + \alpha_2)(y + \alpha_4)(y + \alpha_{24})}{(2y + \alpha_{24} - 1)(2y + \alpha_{24})^2(2y + \alpha_{24} + 1)}}, \\
A_{x,y} &= \left[\frac{(x + \alpha_1 + 1)(x + \alpha_{13} + 1)y(y + \alpha_4)(N + x - y + \alpha_{13} + 2)(N - x + y + \alpha_{24} + 1)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24})(2y + \alpha_{24} + 1)} \right. \\
& + \frac{x(x + \alpha_3)(y + \alpha_2 + 1)(y + \alpha_{24} + 1)(N + x - y + \alpha_{13} + 1)(N - x + y + \alpha_{24} + 2)}{(2x + \alpha_{13})(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)(2y + \alpha_{24} + 2)} \\
& + \frac{(x + \alpha_1 + 1)(x + \alpha_{13} + 1)(y + \alpha_2 + 1)(y + \alpha_{24} + 1)(N - x - y)(N + x + y + |\alpha| + 3)}{(2x + \alpha_{13} + 1)(2x + \alpha_{13} + 2)(2y + \alpha_{24} + 1)(2y + \alpha_{24} + 2)} \\
& \left. + \frac{x(x + \alpha_3)y(y + \alpha_4)(N - x - y + 1)(N + x + y + |\alpha| + 2)}{(2x + \alpha_{13})(2x + \alpha_{13} + 1)(2y + \alpha_{24})(2y + \alpha_{24} + 1)} \right].
\end{aligned}$$

A second difference equation is found with the help of the symmetry relation (2.22). It reads

$$\begin{aligned}
n(n + \alpha_{34} + 1) & \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \tilde{E}_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x - 1 & y - 1 & N \end{Bmatrix} \\
& + \tilde{E}_{x+1,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x + 1 & y + 1 & N \end{Bmatrix} - \tilde{D}_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y - 1 & N \end{Bmatrix} - \tilde{D}_{x,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y + 1 & N \end{Bmatrix} \\
& - \tilde{C}_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x - 1 & y & N \end{Bmatrix} - \tilde{C}_{x+1,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x + 1 & y & N \end{Bmatrix} + \tilde{B}_{x+1,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x + 1 & y - 1 & N \end{Bmatrix} \\
& + \tilde{B}_{x,y+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x - 1 & y + 1 & N \end{Bmatrix} + \tilde{A}_{x,y} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}, \tag{5.2}
\end{aligned}$$

where the coefficients $\tilde{E}_{x,y}$, $\tilde{D}_{x,y}$, ..., etc. are obtained from $E_{x,y}$, $D_{x,y}$, ... by taking $\alpha_1 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$. Given the factorization property (4.4), the r.h.s. of equations (5.1), (5.2) give the action of the intermediate Casimir operators $Q^{(12)}$, $Q^{(34)}$ on the basis where $Q^{(13)}$, $Q^{(24)}$ are diagonal. Using the duality relation (2.21), it possible to write recurrence relations for the $9j$ symbols which give the action of the intermediate Casimir operators $Q^{(13)}$, $Q^{(24)}$ on the basis where $Q^{(12)}$, $Q^{(34)}$ are diagonal. These relations read

$$x(x + \alpha_{13} + 1) \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \hat{E}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m - 1 \\ \alpha_3 & \alpha_4 & n - 1 \\ x & y & N \end{Bmatrix}$$

$$\begin{aligned}
& + \widehat{E}_{m+1,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n+1 \\ x & yN & \end{Bmatrix} + \widehat{D}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n-1 \\ x & y & N \end{Bmatrix} + \widehat{D}_{m,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} \\
& + \widehat{C}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} + \widehat{C}_{m+1,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} + \widehat{B}_{m+1,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n-1 \\ x & yN & \end{Bmatrix} \\
& + \widehat{B}_{m,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} + \widehat{A}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}, \tag{5.3}
\end{aligned}$$

where $\widehat{E}_{m,n}, \widehat{D}_{m,n}, \dots$ are obtained from $E_{m,n}, D_{m,n}, \dots$ by taking $\alpha_2 \leftrightarrow \alpha_3$. The second recurrence relation is

$$\begin{aligned}
& y(y + \alpha_{24} + 1) \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \check{E}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n-1 \\ x & y & N \end{Bmatrix} \\
& + \check{E}_{m+1,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n+1 \\ x & yN & \end{Bmatrix} - \check{D}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n-1 \\ x & y & N \end{Bmatrix} - \check{D}_{m,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} \\
& - \check{C}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} - \check{C}_{m+1,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} + \check{B}_{m+1,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n-1 \\ x & yN & \end{Bmatrix} \\
& + \check{B}_{m,n+1} \begin{Bmatrix} \alpha_1 & \alpha_2 & m-1 \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} + \check{A}_{m,n} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}, \tag{5.4}
\end{aligned}$$

where $\check{E}_{m,n}, \check{D}_{m,n}$, etc. are obtained from $E_{m,n}, D_{m,n}$, etc, by effecting the permutation $\sigma = (1243)$ on the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Writing once again the $9j$ symbols as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} R_{m,n}(x, y),$$

and defining

$$\mathbb{R}_0(x, y) = (1), \quad \mathbb{R}_1(x, y) = \begin{pmatrix} R_{1,0}(x, y) \\ R_{0,1}(x, y) \end{pmatrix}, \quad \mathbb{R}_2(x, y) = \begin{pmatrix} R_{2,0}(x, y) \\ R_{1,1}(x, y) \\ R_{0,2}(x, y) \end{pmatrix}, \quad \dots$$

the recurrence relations (5.3) and (5.4) can be written in matrix form as follows

$$\begin{aligned}
x(x + \alpha_{13} + 1)\mathbb{R}_n(x, y) &= q_{n+2}^{(1)}\mathbb{R}_{n+2}(x, y) + r_{n+1}^{(1)}\mathbb{R}_{n+1}(x, y) \\
&+ s_n^{(1)}\mathbb{R}_n(x, y) + r_n^{(1)}\mathbb{R}_{n-1}(x, y) + q_n^{(1)}\mathbb{R}_{n-2}(x, y), \tag{5.5}
\end{aligned}$$

$$\begin{aligned}
y(y + \alpha_{24} + 1)\mathbb{R}_n(x, y) &= q_{n+2}^{(2)}\mathbb{R}_{n+2}(x, y) + r_{n+1}^{(2)}\mathbb{R}_{n+1}(x, y) \\
&+ s_n^{(2)}\mathbb{R}_n(x, y) + r_n^{(2)}\mathbb{R}_{n-1}(x, y) + q_n^{(2)}\mathbb{R}_{n-2}(x, y), \tag{5.6}
\end{aligned}$$

where the matrices $q_n^{(i)}, r_n^{(i)}$ and $s_n^{(i)}$ are easily found from the coefficients in (5.3) and (5.4). It is apparent from (5.5) and (5.6) that the vector functions $\mathbb{R}_m(x, y)$ satisfy a five term recurrence relation. In view of the multivariate extension of Favard's theorem [7], this confirms that the functions $\mathbb{R}_m(x, y)$ are not orthogonal polynomials.

6 Conclusion

In this paper, we have used the connection between the addition of four $\mathfrak{su}(1, 1)$ representations of the positive discrete series and the generic superintegrable model on the 3-sphere to study the $9j$ coefficients in the position representation. We constructed the canonical basis vectors of the $9j$ problem explicitly and related them to the separation of variables in cylindrical coordinates. Moreover, we have obtained by direct computation the contiguity relations, the difference equations and the recurrence relations satisfied by the $9j$ symbols. The properties of the $9j$ coefficients as bivariate functions have thus been clarified.

The present work suggests many avenues for further investigations. For example Lievens and Van der Jeugt [18] have constructed explicitly the coupled basis vectors arising in the tensor product of an arbitrary number of $\mathfrak{su}(1, 1)$ representations in the coherent state representation. Given this result, it would be of interest to give the realization of these vectors in the position representation by examining the generic superintegrable system on the n -sphere. Another interesting question is that of the orthogonal polynomials in two variables connected with the $9j$ problem. With the observations of the present work and those made by Van der Jeugt in [30], one must conclude that the study of $9j$ symbols do not naturally lead to families of bivariate orthogonal polynomials that would be two-variable extensions of the Racah polynomials. However, the results obtained by Kalnins, Miller and Post [14] and the connection between the generic model on the three-sphere and the $9j$ problem exhibited here suggest that an algebraic interpretation for the bivariate extension of the Racah polynomials, as defined by Tratnik [29], could be given in the framework of the addition of four $\mathfrak{su}(1, 1)$ algebras by investigating the overlap coefficients between bases which are different from the canonical ones. We plan to follow up on this.

A Properties of Jacobi polynomials

The Jacobi polynomials, denoted by $P_n^{(\alpha, \beta)}(z)$, are defined as follows [16]:

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} \middle| \frac{1 - z}{2} \right],$$

where ${}_pF_q$ stands for the generalized hypergeometric function [3]. The polynomials satisfy

$$\int_{-1}^1 (1 - z)^\alpha (1 + z)^\beta P_n^{(\alpha, \beta)}(z) P_m^{(\alpha, \beta)}(z) dz = h_n^{(\alpha, \beta)} \delta_{nm}, \quad (\text{A.1})$$

where the normalization coefficient is

$$h_n^{(\alpha, \beta)} = 2^{\alpha + \beta + 1} \frac{\Gamma(2n + \alpha + \beta + 1) \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(2n + \alpha + \beta + 2) \Gamma(n + \alpha + \beta + 1) \Gamma(n + 1)}. \quad (\text{A.2})$$

The derivatives of the Jacobi polynomials give [20]

$$\partial_z P_n^{(\alpha, \beta)}(z) = \left[\frac{n + \alpha + \beta + 1}{2} \right] P_{n-1}^{(\alpha+1, \beta+1)}(z), \quad (\text{A.3})$$

$$\partial_z \left((1 - z)^\alpha (1 + z)^\beta P_n^{(\alpha, \beta)}(z) \right) = -2(n + 1)(1 - z)^{\alpha-1} (1 + z)^{\beta-1} P_{n+1}^{(\alpha-1, \beta-1)}(z). \quad (\text{A.4})$$

One has

$$P_n^{(\alpha, \beta)}(z) = \left(\frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} \right) P_n^{(\alpha, \beta+1)}(z) + \left(\frac{n + \alpha}{2n + \alpha + \beta + 1} \right) P_{n-1}^{(\alpha, \beta+1)}(z). \quad (\text{A.5})$$

and

$$\begin{aligned} & \left(\frac{1-z}{2}\right) P_{n-1}^{(\alpha,\beta)}(z) \\ &= \left(\frac{n+\alpha-1}{2n+\alpha+\beta-1}\right) P_{n-1}^{(\alpha_1-1,\beta)}(z) - \left(\frac{n}{2n+\alpha+\beta-1}\right) P_n^{(\alpha-1,\beta)}(z). \end{aligned} \quad (\text{A.6})$$

Since $P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z)$, one has also

$$\begin{aligned} & \left(\frac{1+z}{2}\right) P_n^{(\alpha,\beta)}(z) \\ &= \left(\frac{n+\beta}{2n+\alpha+\beta+1}\right) P_n^{(\alpha,\beta-1)}(z) + \left(\frac{n+1}{2n+\alpha+\beta+1}\right) P_{n+1}^{(\alpha,\beta-1)}(z), \end{aligned} \quad (\text{A.7})$$

and

$$P_n^{(\alpha,\beta)}(z) = \left(\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\right) P_n^{(\alpha+1,\beta)}(z) - \left(\frac{n+\beta}{2n+\alpha+\beta+1}\right) P_{n-1}^{(\alpha+1,\beta)}(z). \quad (\text{A.8})$$

B Action of $A_-^{(\alpha_1,\alpha_2)}$ on $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

In Cartesian coordinates, the operator $A_-^{(\alpha_1,\alpha_2)}$ reads

$$A_-^{(\alpha_1,\alpha_2)} = -\frac{1}{2}(s_1\partial_{s_2} - s_2\partial_{s_1}) + \frac{s_1}{2s_2}(\alpha_2 + 1/2) - \frac{s_2}{2s_1}(\alpha_1 + 1/2).$$

The action of $A_-^{(\alpha_1,\alpha_2)}$ on the wavefunctions $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ can be written as

$$\begin{aligned} & \mathcal{F}\eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)} \\ & \times \left[\mathcal{F}^{-1} A_-^{(\alpha_1,\alpha_2)} \mathcal{F} \right] \left[P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2) \right], \end{aligned}$$

where

$$\mathcal{F} = (s_1^2 + s_3^2)^x (s_2^2 + s_4^2)^y \prod_{i=1}^4 s_i^{\alpha_i+1/2}.$$

One has

$$\left[\mathcal{F}^{-1} A_-^{(\alpha_1,\alpha_2)} \mathcal{F} \right] = -\frac{1}{2}(s_1\partial_{s_2} - s_2\partial_{s_1}) + x \frac{s_1 s_2}{s_1^2 + s_3^2} - y \frac{s_1 s_2}{s_2^2 + s_4^2}.$$

In the cylindrical coordinates (2.16), the operator reads

$$\begin{aligned} \left[\mathcal{F}^{-1} A_-^{(\alpha_1,\alpha_2)} \mathcal{F} \right] &= x \left[\text{tg } \vartheta \cos \varphi_1 \cos \varphi_2 \right] - y \left[\frac{\cos \varphi_1 \cos \varphi_2}{\text{tg } \vartheta} \right] \\ & - \frac{1}{2} \left[\cos \varphi_1 \cos \varphi_2 \partial_\vartheta + \text{tg } \vartheta \sin \varphi_1 \cos \varphi_2 \partial_{\varphi_1} - \frac{\cos \varphi_1 \sin \varphi_2}{\text{tg } \vartheta} \partial_{\varphi_2} \right]. \end{aligned}$$

Using the relation (A.3), one finds

$$A_-^{(\alpha_1,\alpha_2)} \Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = v_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3)} \left[(N+x+y+|\alpha|+3) (\cos^2 \vartheta)^x (\sin^2 \vartheta)^y \right]$$

$$\begin{aligned}
& \times P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) \\
& + (x + \alpha_{13} + 1)(\cos^2 \vartheta)^{x-1} (\sin^2 \vartheta)^y \sin^2 \varphi_1 \\
& \times P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) P_{x-1}^{(\alpha_3+1, \alpha_1+1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) \\
& - (y + \alpha_{24} + 1)(\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y-1} \sin^2 \varphi_2 \\
& \times P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_{y-1}^{(\alpha_4+1, \alpha_2+1)}(\cos 2\varphi_2) \\
& + \left[x(\cos^2 \vartheta)^{x-1} (\sin^2 \vartheta)^y - y(\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y-1} \right] \\
& \times P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) \Big],
\end{aligned}$$

where

$$v_{x,y,N}^{(\alpha_1, \alpha_2, \alpha_3)} = \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \eta_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)} (s_1)^{\alpha_1+3/2} (s_2)^{\alpha_2+3/2} (s_3)^{\alpha_3+1/2} (s_4)^{\alpha_4+1/2}.$$

The identities (A.5) and (A.6) can then be used to write the result in a form involving only terms of the type $P_k^{(\alpha_3, \alpha_1+1)}$ and $P_{k'}^{(\alpha_4, \alpha_2+1)}$. Regrouping the terms, one finds

$$\begin{aligned}
A_-^{(\alpha_1, \alpha_2)} \Xi_{x,y,N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} &= v_{x,y,N}^{(\alpha_1, \alpha_2, \alpha_3)} \left[\left\{ \frac{(x + \alpha_{13} + 1)(y + \alpha_{24} + 1)(N + x + y + |\alpha| + 3)}{(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)} \right\} \right. \\
& \times (\cos^2 \vartheta)^x (\sin^2 \vartheta)^y P_x^{(\alpha_3, \alpha_1+1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2+1)}(\cos 2\varphi_2) P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
& + \left\{ \frac{(x + \alpha_3)(y + \alpha_{24} + 1)(\cos^2 \vartheta)^{x-1} (\sin^2 \vartheta)^y}{(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)} \right\} P_{x-1}^{(\alpha_3, \alpha_1+1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2+1)}(\cos 2\varphi_2) \\
& \times \left((N + x + y + |\alpha| + 3) \cos^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \right. \\
& + (2x + \alpha_{13} + 1) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \\
& + \left\{ \frac{(y + \alpha_4)(x + \alpha_{13} + 1)(\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y-1}}{(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)} \right\} P_x^{(\alpha_3, \alpha_1+1)}(\cos 2\varphi_1) P_{y-1}^{(\alpha_4, \alpha_2+1)}(\cos 2\varphi_2) \\
& \times \left((N + x + y + |\alpha| + 3) \sin^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \right. \\
& - (2y + \alpha_{24} + 1) P_{n-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \\
& + \left\{ \frac{(y + \alpha_4)(x + \alpha_3)(\cos^2 \vartheta)^{x-1} (\sin^2 \vartheta)^{y-1}}{(2x + \alpha_{13} + 1)(2y + \alpha_{24} + 1)} \right\} P_{x-1}^{(\alpha_3, \alpha_1+1)}(\cos 2\varphi_1) P_{y-1}^{(\alpha_4, \alpha_2+1)}(\cos 2\varphi_2) \\
& \times \left((N + x + y + |\alpha| + 3) \cos^2 \vartheta \sin^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \right. \\
& + (2x + \alpha_{13} + 1) \sin^2 \vartheta P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& \left. \left. - (2y + \alpha_{24} + 1) \cos^2 \vartheta P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \right) \right].
\end{aligned}$$

The terms between parentheses in the above expression are easily evaluated and found to be

$$\begin{aligned}
& (N + x + y + |\alpha| + 3) \cos^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
& \quad + (2x + \alpha_{13} + 1) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& = (N + x - y + \alpha_{13} + 1) P_{N-x-y}^{(2y+\alpha_{24}+2, 2x+\alpha_{13})}(\cos 2\vartheta), \\
& (N + x + y + |\alpha| + 3) \sin^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
& \quad - (2y + \alpha_{24} + 1) P_{n-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta)
\end{aligned}$$

$$\begin{aligned}
&= -(N - x + y + \alpha_{24} + 1)P_{N-x-y}^{(2y+\alpha_{24}, 2x+\alpha_{13}+2)}(\cos 2\vartheta), \\
&(N + x + y + |\alpha| + 3) \cos^2 \vartheta \sin^2 \vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&\quad + (2x + \alpha_{13} + 1) \sin^2 \vartheta P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\quad - (2y + \alpha_{24} + 1) \cos^2 \vartheta P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&= -(N - x - y + 1)P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta).
\end{aligned}$$

Adjusting the normalization factors then yields the result (4.6).

C Action of $A_+^{(\alpha_1-1, \alpha_2-1)}$ on $\Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}$

In Cartesian coordinates, the operator $A_+^{(\alpha_1-1, \alpha_2-1)}$ reads

$$A_+^{(\alpha_1-1, \alpha_2-1)} = \frac{1}{2}(s_1\partial_{s_2} - s_2\partial_{s_1}) + \frac{s_1}{2s_2}(\alpha_2 - 1/2) - \frac{s_2}{2s_1}(\alpha_1 - 1/2).$$

The action of $A_+^{(\alpha_1-1, \alpha_2-1)}$ on the wavefunctions $\Xi_{x,y;N}$ can be expressed as

$$\begin{aligned}
&\mathcal{G}^{-1}\eta_x^{(\alpha_3, \alpha_1)}\eta_y^{(\alpha_4, \alpha_2)}\eta_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)} \\
&\quad \times \left[\mathcal{G}A_+^{(\alpha_1-1, \alpha_2-1)}\mathcal{G}^{-1} \right] \left[(\sin^2 \varphi_1)^{\alpha_3} (\cos^2 \varphi_1)^{\alpha_1} P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) \right] \\
&\quad \times \left[(\sin^2 \varphi_2)^{\alpha_4} (\cos^2 \varphi_2)^{\alpha_2} P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) \right] \\
&\quad \times \left[(\sin^2 \vartheta)^{2x+\alpha_{24}+1} (\cos^2 \vartheta)^{2x+\alpha_{13}+1} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \right],
\end{aligned}$$

where

$$\mathcal{G} = (s_1^2 + s_3^2)^{x+1} (s_2^2 + s_4^2)^{y+1} \prod_{i=1}^4 s_i^{\alpha_i-1/2}.$$

One has

$$[\mathcal{G}A_+^{(\alpha_1-1, \alpha_2-1)}\mathcal{G}^{-1}] = \frac{1}{2}(s_2\partial_{s_1} - s_1\partial_{s_2}) + (x+1)\frac{s_1s_2}{s_1^2 + s_3^2} - (y+1)\frac{s_1s_2}{s_2^2 + s_4^2}.$$

In the cylindrical coordinates (2.16), this operator reads

$$\begin{aligned}
[\mathcal{G}A_+^{(\alpha_1-1, \alpha_2-1)}\mathcal{G}^{-1}] &= \frac{1}{2} \left[\cos \varphi_1 \cos \varphi_2 \partial_\vartheta + \operatorname{tg} \vartheta \sin \varphi_1 \cos \varphi_2 \partial_{\varphi_1} \right. \\
&\quad \left. - \frac{\cos \varphi_1 \sin \varphi_2}{\operatorname{tg} \vartheta} \partial_{\varphi_2} \right] + (x+1)[\operatorname{tg} \vartheta \cos \varphi_1 \cos \varphi_2] - (y+1)[\operatorname{ctg} \vartheta \cos \varphi_1 \cos \varphi_2].
\end{aligned}$$

Using the identity (A.4), one finds

$$\begin{aligned}
A_+^{(\alpha_1-1, \alpha_2-1)}\Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} &= \kappa_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3)} \left[(N - x - y + 1) \cos^2 \varphi_1 \cos^2 \varphi_2 \right. \\
&\quad \times (\cos^2 \vartheta)^x (\sin^2 \vartheta)^y P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \\
&\quad + (x+1) \cos^2 \varphi_2 (\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y+1} \\
&\quad \times P_{x+1}^{(\alpha_3-1, \alpha_1-1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\quad \left. - (y+1) \cos^2 \varphi_1 (\cos^2 \vartheta)^{x+1} (\sin^2 \vartheta)^y \right]
\end{aligned}$$

$$\begin{aligned}
& \times P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_{y+1}^{(\alpha_4-1, \alpha_2-1)}(\cos 2\varphi_2) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& + (x+1) \cos^2 \varphi_1 \cos^2 \varphi_2 (\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y+1} \\
& \times P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& - (y+1) \cos^2 \varphi_1 \cos^2 \varphi_2 (\cos^2 \vartheta)^{x+1} (\sin^2 \vartheta)^y \\
& \times P_x^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2)}(\cos 2\varphi_2) P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big],
\end{aligned}$$

where

$$\kappa_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3)} = \eta_x^{(\alpha_3, \alpha_1)} \eta_y^{(\alpha_4, \alpha_2)} \eta_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)} (s_1)^{\alpha_1-1/2} (s_2)^{\alpha_2-1/2} (s_3)^{\alpha_3+1/2} (s_4)^{\alpha_4+1/2}.$$

Then using (A.7) and (A.8), one finds

$$\begin{aligned}
A_+^{(\alpha_1-1, \alpha_2-1)} \Xi_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} &= \kappa_{x,y;N}^{(\alpha_1, \alpha_2, \alpha_3)} \left[\left\{ \frac{(x+\alpha_1)(y+\alpha_2)(N-x-y+1)}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} \right. \\
& \times (\cos^2 \vartheta)^x (\sin^2 \vartheta)^y P_x^{(\alpha_3, \alpha_1-1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2-1)}(\cos 2\varphi_2) P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \\
& + \left\{ \frac{(x+1)(y+\alpha_2)(\cos^2 \vartheta)^{x+1} (\sin^2 \vartheta)^y}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_{x+1}^{(\alpha_3, \alpha_1-1)}(\cos 2\varphi_1) P_y^{(\alpha_4, \alpha_2-1)}(\cos 2\varphi_2) \\
& \times \left([2x+\alpha_{13}+1] \frac{\sin^2 \vartheta}{\cos^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \right. \\
& + [N-x-y+1] \frac{1}{\cos^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \Big) \\
& + \left\{ \frac{(x+\alpha_1)(y+1)(\cos^2 \vartheta)^x (\sin^2 \vartheta)^{y+1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_x^{(\alpha_3, \alpha_1-1)}(\cos 2\varphi_1) P_{y+1}^{(\alpha_4, \alpha_2-1)}(\cos 2\varphi_2) \\
& \times \left([N-x-y+1] \frac{1}{\sin^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \right. \\
& - [2y+\alpha_{24}+1] \frac{\cos^2 \vartheta}{\sin^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \\
& + \left\{ \frac{(x+1)(y+1)(\cos^2 \vartheta)^{x+1} (\sin^2 \vartheta)^{y+1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_{x+1}^{(\alpha_3, \alpha_1)}(\cos 2\varphi_1) P_{y+1}^{(\alpha_4, \alpha_2-2)}(\cos 2\varphi_2) \\
& \times \left([N-x-y+1] \frac{1}{\cos^2 \vartheta \sin^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \right. \\
& - [2y+\alpha_{24}+1] \frac{1}{\sin^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& \left. \left. + [2x+\alpha_{13}+1] \frac{1}{\cos^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \right) \right].
\end{aligned}$$

The term between the parentheses are easily determined to be the following

$$\begin{aligned}
& [2x+\alpha_{13}+1] \frac{\sin^2 \vartheta}{\cos^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
& + [N-x-y+1] \frac{1}{\cos^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \\
& = (N-x+y+\alpha_{24}+1) P_{N-x-y}^{(2y+\alpha_{24}, 2x+\alpha_{13}+2)}(\cos 2\vartheta), \\
& [N-x-y+1] \frac{1}{\sin^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \\
& - [2y+\alpha_{24}+1] \frac{\cos^2 \vartheta}{\sin^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta)
\end{aligned}$$

$$\begin{aligned}
&= -(N + x - y + \alpha_{13} + 1)P_{N-x-y}^{(2y+\alpha_{24}+2, 2x+\alpha_{13})}(\cos 2\vartheta), \\
&[N - x - y + 1] \frac{1}{\cos^2 \vartheta \sin^2 \vartheta} P_{N-x-y+1}^{(2y+\alpha_{24}, 2x+\alpha_{13})}(\cos 2\vartheta) \\
&\quad - [2y + \alpha_{24} + 1] \frac{1}{\sin^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\quad + [2x + \alpha_{13} + 1] \frac{1}{\cos^2 \vartheta} P_{N-x-y}^{(2y+\alpha_{24}+1, 2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&= -(N + x + y + |\alpha| + 3)P_{N-x-y-1}^{(2y+\alpha_{24}+2, 2x+\alpha_{13}+2)}(\cos 2\vartheta).
\end{aligned}$$

Adjusting the normalization factors then yields the result (4.10).

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