

Definite Integrals using Orthogonality and Integral Transforms^{*}

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Abstract. We obtain definite integrals for products of associated Legendre functions with Bessel functions, associated Legendre functions, and Chebyshev polynomials of the first kind using orthogonality and integral transforms.

Key words: definite integrals; associated Legendre functions; Bessel functions; Chebyshev polynomials of the first kind

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1 Introduction

In [3] and [6] (see also [4]), we present some definite integral and infinite series addition theorems which arise from expanding fundamental solutions of elliptic equations on \mathbb{R}^d in axisymmetric coordinate systems which separate Laplace's equation. We utilize orthogonality and integral transforms to obtain new definite integrals from some of these addition theorems.

2 Definite integrals from integral transforms

2.1 Application of Hankel's transform

We use the following result where for $x \in (0, \infty)$ we define

$$F(r \pm 0) := \lim_{x \rightarrow r \pm} F(x);$$

see [15, p. 456]:

Theorem 1. *Let $F : (0, \infty) \rightarrow \mathbb{C}$ be such that*

$$\int_0^\infty \sqrt{x} |F(x)| dx < \infty, \tag{1}$$

and let $\nu \geq -\frac{1}{2}$. Then

$$\frac{1}{2}(F(r+0) + F(r-0)) = \int_0^\infty u J_\nu(ur) \int_0^\infty x F(x) J_\nu(ux) dx du \tag{2}$$

provided that the positive number r lies inside an interval in which $F(x)$ has finite variation.

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As an illustration for the method of integral transforms, we give the following example. According to [15, (13.22.2)] (see also [7, (6.612.3)]), we have for $\operatorname{Re} a > 0$, $b, c > 0$, $\operatorname{Re} \nu > -\frac{1}{2}$, then

$$\int_0^\infty e^{-ka} J_\nu(kb) J_\nu(kc) dk = \frac{1}{\pi\sqrt{bc}} Q_{\nu-1/2} \left(\frac{a^2 + b^2 + c^2}{2bc} \right), \quad (3)$$

where $J_\nu : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$, for order $\nu \in \mathbb{C}$ is the Bessel function of the first kind defined in [12, (10.2.2)] and $Q_\nu^\mu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C}$ for $\nu + \mu \notin -\mathbb{N}$ with degree ν and order μ , is the associated Legendre function of the second kind defined in [12, (14.3.7), § 14.21]. The Legendre function of the second kind $Q_\nu : \mathbb{C} \setminus (-\infty, 1] \rightarrow \mathbb{C}$ for $\nu \notin -\mathbb{N}$ is defined in terms of the zero-order associated Legendre function of the second kind $Q_\nu(z) := Q_\nu^0(z)$. If we apply Theorem 1 to the function $F : (0, \infty) \rightarrow \mathbb{C}$ defined by

$$F(k) := \frac{\pi\sqrt{c}}{k} e^{-ka} J_\nu(kc), \quad (4)$$

then condition (1) is satisfied. If we use (4) in (2) then we obtain the following result. If $\operatorname{Re} a > 0$, $c > 0$, $\operatorname{Re} \nu > -\frac{1}{2}$, then

$$\int_0^\infty J_\nu(kb) Q_{\nu-1/2} \left(\frac{a^2 + b^2 + c^2}{2bc} \right) \sqrt{b} db = \frac{\pi\sqrt{c}}{k} e^{-ka} J_\nu(kc),$$

which is actually given in [13, (2.18.8.11)].

Hardy [8, (33.16)] derives an interesting extension of (3) (see also [15, p. 389] and [2, p. 17]). We apply the Whipple formula [12, (14.9.17), § 14.21] to Hardy's extension to obtain

$$\begin{aligned} & \int_0^\infty k e^{-ka} J_\nu(kb) J_\nu(kc) dk \\ &= \frac{-2a}{\pi\sqrt{bc} (a^2 + (b+c)^2)^{1/2} (a^2 + (b-c)^2)^{1/2}} Q_{\nu-1/2}^1 \left(\frac{a^2 + b^2 + c^2}{2bc} \right), \end{aligned} \quad (5)$$

for $\operatorname{Re} a > 0$, $b, c > 0$, $\operatorname{Re} \nu > -1$. It is mentioned in [15, p. 389] that (5) can be derived from (3) by differentiation with respect to a . Using this integral and Theorem 1, we prove the following theorem.

Theorem 2. *Let $\operatorname{Re} a > 0$, $c > 0$, $\operatorname{Re} \nu > -1$. Then*

$$\int_0^\infty \frac{J_\nu(kb)}{(a^2 + (b+c)^2)^{1/2} (a^2 + (b-c)^2)^{1/2}} Q_{\nu-1/2}^1 \left(\frac{a^2 + b^2 + c^2}{2bc} \right) \sqrt{b} db = -\frac{\pi\sqrt{c}}{2a} e^{-ka} J_\nu(kc).$$

Proof. By applying Theorem 1 to the function $F : (0, \infty) \rightarrow \mathbb{C}$ defined by

$$F(k) := -\frac{\pi\sqrt{c}}{2a} e^{-ka} J_\nu(kc),$$

using (5), we obtain the desired result. ■

Now, we give another example of how an integral expansion for a fundamental solution of Laplace's equation on \mathbb{R}^3 in parabolic coordinates can be used to prove a new definite integral.

Theorem 3. *Let $m \in \mathbb{N}_0$, $\lambda' \in (0, \infty)$, $\mu, \mu' \in (0, \infty)$, $\mu \neq \mu'$, $k \in (0, \infty)$. Then*

$$\int_0^\infty Q_{m-1/2}(\chi) J_m(k\lambda) \sqrt{\lambda} d\lambda = 2\pi\sqrt{\lambda'\mu\mu'} J_m(k\lambda') I_m(k\mu_<) K_m(k\mu_>),$$

where

$$\chi = \frac{4\lambda^2\mu^2 + 4\lambda'^2\mu'^2 + (\lambda^2 - \lambda'^2 + \mu'^2 - \mu^2)^2}{8\lambda\lambda'\mu\mu'} > 1$$

and $\mu_{\leq} := \min_{\max} \{\mu, \mu'\}$.

Proof. We apply Theorem 1 to the function $F : (0, \infty) \rightarrow \mathbb{C}$ defined by

$$F(k) := 2\pi\sqrt{\lambda'\mu\mu'}J_m(k\lambda')I_m(k\mu_{<})K_m(k\mu_{>}),$$

where $I_\nu : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$, for $\nu \in \mathbb{C}$, is the modified Bessel function of the first kind defined in [12, (10.25.2)] and $K_\nu : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$, for $\nu \in \mathbb{C}$, is the modified Bessel function of the second kind defined in [12, (10.27.4)]. Again, we see that condition (1) is satisfied. We obtain the desired result from [6, (41)], namely, for $\lambda \in (0, \infty)$,

$$\int_0^\infty J_m(k\lambda)J_m(k\lambda')I_m(k\mu_{<})K_m(k\mu_{>})kdk = \frac{Q_{m-1/2}(\chi)}{2\pi\sqrt{\lambda\lambda'\mu\mu'}}. \quad \blacksquare$$

2.2 Application of Fourier cosine transform

Theorem 4. Let $a, b \in (0, \infty)$ with $b \leq a$, $k \in (0, \infty)$, $\operatorname{Re} \nu > -\frac{1}{2}$. Then

$$\int_0^\infty Q_{\nu-1/2}\left(\frac{a^2 + b^2 + z^2}{2ab}\right)\cos(kz)dz = \pi\sqrt{ab}I_\nu(kb)K_\nu(ka). \quad (6)$$

Proof. According to [7, (6.672.4)] we have the integral relation

$$\int_0^\infty I_\nu(kb)K_\nu(ka)\cos(kz)dk = \frac{1}{2\sqrt{ab}}Q_{\nu-1/2}\left(\frac{a^2 + b^2 + z^2}{2ab}\right),$$

where $a, b \in (0, \infty)$ with $b < a$, $z > 0$, $\operatorname{Re} \nu > -\frac{1}{2}$. We obtain the desired result from Theorem 1 with $F : (0, \infty) \rightarrow \mathbb{C}$ defined such that

$$F(k) := \pi\sqrt{\frac{ab}{k}}I_\nu(kb)K_\nu(ka)$$

and $\nu = -\frac{1}{2}$. Furthermore, if one makes the replacement $z \mapsto z/(\sqrt{2}a)$, $k \mapsto \sqrt{2}ka$ in [7, (7.162.6)], namely

$$\int_0^\infty Q_{\nu-1/2}(1+z^2)\cos(kz)dz = \frac{\pi}{\sqrt{2}}I_\nu\left(\frac{k}{\sqrt{2}}\right)K_\nu\left(\frac{k}{\sqrt{2}}\right),$$

where $k \in (0, \infty)$ and $\operatorname{Re} \nu > -\frac{1}{2}$, then we see that (6) holds also for any $a, b \in (0, \infty)$ with $a = b$. \blacksquare

3 Definite integrals from orthogonality relations

3.1 Degree orthogonality for associated Legendre functions with integer degree and order

We take advantage of the degree orthogonality relation for the Ferrers function of the first kind with integer degree and order, namely (cf. [7, (7.112.1)])

$$\int_0^\pi P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n,n'}, \quad (7)$$

where $m, n, n' \in \mathbb{N}_0$, and $m \leq n$, $m \leq n'$. We are using the associated Legendre function of the first kind (on-the-cut), $P_\nu^\mu : (-1, 1) \rightarrow \mathbb{C}$, for $\nu, \mu \in \mathbb{C}$, the Ferrers function of the first kind, which is defined in [12, (14.3.1)].

The following estimates for the Ferrers function of the first kind with integer degree and order will be useful. If $\theta \in [0, \pi]$ and $m, n \in \mathbb{N}_0$ then [14, § 5.3, (19)]

$$|P_n^m(\cos \theta)| \leq \frac{(m+n)!}{n!} \quad (8)$$

and if $\theta \in (0, \pi)$ then [10, p. 203]

$$|P_n^m(\cos \theta)| < 2 \frac{(n+m)!}{n!} (\pi n)^{-1/2} (\csc \theta)^{m+1/2}. \quad (9)$$

If $\mu \in \mathbb{C}$, $\xi > 0$ are fixed and $0 \leq \nu \rightarrow +\infty$, we also have the following asymptotic formulas for the associated Legendre functions

$$P_\nu^\mu(\cosh \xi) = (2\pi \sinh \xi)^{-1/2} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} e^{(\nu + \frac{1}{2})\xi} (1 + O(\nu^{-1})), \quad (10)$$

$$Q_\nu^\mu(\cosh \xi) = \left(\frac{\pi}{2 \sinh \xi} \right)^{1/2} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} e^{-(\nu + \frac{1}{2})\xi + i\pi\mu} (1 + O(\nu^{-1})), \quad (11)$$

$$P_\nu^\mu(i \sinh \xi) = (2\pi \cosh \xi)^{-1/2} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} e^{(\nu + \frac{1}{2})\xi + i\pi\nu/2} (1 + O(\nu^{-1})), \quad (12)$$

$$Q_\nu^\mu(i \sinh \xi) = \left(\frac{\pi}{2 \cosh \xi} \right)^{1/2} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} e^{-(\nu + \frac{1}{2})\xi - i\pi(\nu+1)/2 + i\pi\mu} (1 + O(\nu^{-1})). \quad (13)$$

These asymptotic formulae follow from representations of Legendre functions by Gauss hypergeometric functions; see [1, (8.1.1), (8.10.4–5)].

Theorem 5. *Let $n, m \in \mathbb{N}_0$, with $n \geq m$, $\nu \in \mathbb{C} \setminus \{2m, 2m+2, 2m+4, \dots\}$, $r, r' \in (0, \infty)$, $r \neq r'$, $\theta' \in (0, \pi)$. Then*

$$\begin{aligned} & \int_0^\pi (\chi^2 - 1)^{(\nu+1)/4} Q_{m-1/2}^{-(\nu+1)/2}(\chi) P_n^m(\cos \theta) (\sin \theta)^{(\nu+2)/2} d\theta \\ &= \frac{i\sqrt{\pi}}{2^{(\nu+1)/2} (\sin \theta')^{\nu/2}} \left(\frac{r_{>}^2 - r_{<}^2}{rr'} \right)^{(\nu+2)/2} Q_n^{-(\nu+2)/2} \left(\frac{r^2 + r'^2}{2rr'} \right) P_n^m(\cos \theta'), \end{aligned} \quad (14)$$

where

$$\chi = \frac{r^2 + r'^2 - 2rr' \cos \theta \cos \theta'}{2rr' \sin \theta \sin \theta'} \quad (15)$$

and $r_{\leq} := \min_{\max} \{r, r'\}$.

Proof. We start with the following addition theorem for the associated Legendre function of the second kind (see [3]), namely for $\theta \in (0, \pi)$,

$$\begin{aligned} (\chi^2 - 1)^{(\nu+1)/4} (\sin \theta)^{\nu/2} Q_{m-1/2}^{-(\nu+1)/2}(\chi) &= \frac{i\sqrt{\pi}}{2^{(\nu+3)/2}} (\sin \theta')^{-\nu/2} \left(\frac{r_{>}^2 - r_{<}^2}{rr'} \right)^{(\nu+2)/2} \\ &\times \sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} Q_n^{-(\nu+2)/2} \left(\frac{r^2 + r'^2}{2rr'} \right) P_n^m(\cos \theta) P_n^m(\cos \theta'), \end{aligned} \quad (16)$$

where $\chi > 1$ is given by (15). By (8) and (11) the infinite series is uniformly convergent for $\theta \in [0, \pi]$. Therefore, if we multiply both sides of (16) by $\sin \theta P_{n'}^m(\cos \theta)$, where $n' \in \mathbb{N}_0$ and integrate over $\theta \in (0, \pi)$ we obtain (14). \blacksquare

Corollary 1. Let $n, m \in \mathbb{N}_0$ with $n \geq m$, $r, r' \in (0, \infty)$, $r \neq r'$, $\theta' \in (0, \pi)$. Then

$$\int_0^\pi Q_{m-1/2}(\chi) P_n^m(\cos \theta) \sqrt{\sin \theta} d\theta = \frac{2\pi \sqrt{\sin \theta'}}{2n+1} P_n^m(\cos \theta') \left(\frac{r_{<}}{r_{>}} \right)^{n+1/2},$$

where $\chi > 1$ is given by (15).

Proof. Substitute $\nu = -1$ in (14) and use [12, (14.5.17)]. ■

Theorem 6. Let $m, n \in \mathbb{N}_0$ with $n \geq m$, $\sigma, \sigma' \in (0, \infty)$, $\theta' \in (0, \pi)$. Then

$$\begin{aligned} \int_0^\pi Q_{m-1/2}(\chi) P_n^m(\cos \theta) \sqrt{\sin \theta} d\theta &= 2\pi (-1)^m \frac{(n-m)!}{(n+m)!} \\ &\times \sqrt{\sinh \sigma \sinh \sigma' \sin \theta'} P_n^m(\cos \theta') P_n^m(\cosh \sigma_{<}) Q_n^m(\cosh \sigma_{>}), \end{aligned} \quad (17)$$

where

$$\chi = \frac{\cosh^2 \sigma + \cosh^2 \sigma' - \sin^2 \theta - \sin^2 \theta' - 2 \cosh \sigma \cosh \sigma' \cos \theta \cos \theta'}{2 \sinh \sigma \sinh \sigma' \sin \theta \sin \theta'} \quad (18)$$

and $\sigma_{\leq} := \min_{\max} \{\sigma, \sigma'\}$.

Proof. We start with the following addition theorem for the associated Legendre function of the second kind (see [6, (37)]), namely

$$\begin{aligned} Q_{m-1/2}(\chi) &= \pi (-1)^m \sqrt{\sinh \sigma \sinh \sigma' \sin \theta \sin \theta'} \sum_{n=m}^{\infty} (2n+1) \left[\frac{(n-m)!}{(n+m)!} \right]^2 \\ &\times P_n^m(\cos \theta) P_n^m(\cos \theta') P_n^m(\cosh \sigma_{<}) Q_n^m(\cosh \sigma_{>}), \end{aligned} \quad (19)$$

where $\chi \geq 1$ is defined by (18). Note that $\chi = 1$ only if $\sigma = \sigma'$ and $\theta = \theta'$, and in that case $Q_{m-1/2}(\chi)$ has a logarithmic singularity. If $\sigma \neq \sigma'$ then (8), (10), (11) show that the series in (19) is uniformly convergent for $\theta \in [0, \pi]$. Therefore, if we multiply both sides of (19) by $\sqrt{\sin \theta} P_n^m(\cos \theta)$ and integrate over $\theta \in [0, \pi]$, then by (7) we have obtained (17). If $\sigma = \sigma'$ then one may use (9), (10), (11) and the orthogonality relation (7) to show that the series in (19) (as a series of functions in the variable $\theta \in (0, \pi)$) converges in $L^2(0, \pi)$. That is

$$\sum_{n=m}^{\infty} \left\{ \frac{(n-m)!}{(n+m)!} \phi_n(\theta') P_n^m(\cosh \sigma_{<}) Q_n^m(\cosh \sigma_{>}) \right\}^2 < \infty, \quad (20)$$

where $\phi_n \in L^2(0, \pi)$ forms an orthonormal basis and is defined as

$$\phi_n(\theta) := \sqrt{\sin \theta} \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta),$$

for $n = m, m+1, \dots$. Then by the asymptotics of P_n^m and Q_n^m (cf. (10), (11))

$$P_n^m(\cosh \sigma_{<}) Q_n^m(\cosh \sigma_{>}) = O(n^{2m-1}) \quad \text{as } n \rightarrow \infty.$$

Also from the estimate of P_n^m (9), $\phi_n(\theta') = O(1)$. Therefore,

$$\frac{(n-m)!}{(n+m)!} \phi_n(\theta') P_n^m(\cosh \sigma_{<}) Q_n^m(\cosh \sigma_{>}) = O(n^{-1}),$$

and this implies (20) because $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Therefore, we again obtain (17). ■

Theorem 7. Let $m, n \in \mathbb{N}_0$ with $0 \leq m \leq n$, $\sigma, \sigma' \in (0, \infty)$, $\theta' \in [0, \pi]$. Then

$$\int_0^\pi Q_{m-1/2}(\chi) P_n^m(\cos \theta) \sqrt{\sin \theta} d\theta = 2\pi i (-1)^m \frac{(n-m)!}{(n+m)!} \\ \times \sqrt{\cosh \sigma \cosh \sigma' \sin \theta'} P_n^m(\cos \theta') P_n^m(i \sinh \sigma_<) Q_n^m(i \sinh \sigma_>), \quad (21)$$

where

$$\chi = \frac{\sinh^2 \sigma + \sinh^2 \sigma' + \sin^2 \theta + \sin^2 \theta' - 2 \sinh \sigma \sinh \sigma' \cos \theta \cos \theta'}{2 \cosh \sigma \cosh \sigma' \sin \theta \sin \theta'}. \quad (22)$$

Proof. We start with oblate spheroidal coordinates on \mathbb{R}^3 , namely

$$x = a \cosh \sigma \sin \theta \cos \phi, \quad y = a \cosh \sigma \sin \theta \sin \phi, \quad z = a \sinh \sigma \cos \theta,$$

where $a > 0$, $\sigma \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. The reciprocal distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$ expanded in terms of the separable harmonics in this coordinate system is given in [9, (41), p. 218], namely

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{i}{a} \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^n (-1)^m \epsilon_m \left[\frac{(n-m)!}{(n+m)!} \right]^2 \cos(m(\phi - \phi')) \\ \times P_n^m(\cos \theta) P_n^m(\cos \theta') P_n^m(i \sinh \sigma_<) Q_n^m(i \sinh \sigma_>),$$

where $\epsilon_m := 2 - \delta_{m,0}$ is the Neumann factor [11, p. 744] commonly occurring in Fourier cosine series, with $\sigma' \in [0, \infty)$, $\theta' \in [0, \pi]$, $\phi' \in [0, 2\pi)$. Note that the corresponding expression given in [6, § 5.2] is given incorrectly (see [4]). By reversing the order of summations in the above expression and comparing with the Fourier cosine expansion for the reciprocal distance between two points, namely

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{1}{\pi a \sqrt{\cosh \sigma \cosh \sigma' \sin \theta \sin \theta'}} \sum_{m=0}^{\infty} \epsilon_m \cos(m(\phi - \phi')) Q_{m-1/2}(\chi),$$

where $\chi > 1$ is given by (22), we obtain the following addition theorem for the associated Legendre function of the second kind

$$Q_{m-1/2}(\chi) = i\pi (-1)^m \sqrt{\cosh \sigma \cosh \sigma' \sin \theta \sin \theta'} \sum_{n=m}^{\infty} (2n+1) \left[\frac{(n-m)!}{(n+m)!} \right]^2 \\ \times P_n^m(\cos \theta) P_n^m(\cos \theta') P_n^m(i \sinh \sigma_<) Q_n^m(i \sinh \sigma_>). \quad (23)$$

If we multiply both sides of (23) by $\sqrt{\sin \theta} P_n^m(\cos \theta)$ and integrate over $\theta \in [0, \pi]$, then by (7) we have obtained (21). We justify the interchange of integral and infinite sum as before by using the asymptotic formulas (12), (13). ■

Theorem 8. Let $m, n \in \mathbb{N}_0$ with $0 \leq m \leq n$, $\sigma, \sigma' \in (0, \infty)$, $\theta' \in (0, \pi)$. Then

$$\int_0^\pi Q_{m-1/2}(\chi) P_n^m(\cos \theta) \sqrt{\sin \theta} d\theta = \frac{2\pi \sqrt{\sin \theta'}}{2n+1} P_n^m(\cos \theta') e^{-(n+1/2)(\sigma_> - \sigma_<)}, \quad (24)$$

where if we define $s = \cosh \sigma$, $s' = \cosh \sigma'$, $\tau = \cos \theta$, $\tau' = \cos \theta'$, then

$$\chi = \frac{\sin^2 \theta (s' - \tau')^2 + \sin^2 \theta' (s - \tau)^2 + [(s' - \tau') \sinh \sigma - (s - \tau) \sinh \sigma']^2}{2 \sin \theta \sin \theta' (s - \tau)(s' - \tau')}. \quad (25)$$

Proof. We start with bispherical coordinates on \mathbb{R}^3 , namely

$$x = \frac{a \sin \theta \cos \phi}{\cosh \sigma - \cos \theta}, \quad y = \frac{a \sin \theta \sin \phi}{\cosh \sigma - \cos \theta}, \quad z = \frac{a \sinh \sigma}{\cosh \sigma - \cos \theta},$$

where $a > 0$, $\sigma \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. The reciprocal distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$ expanded in terms of the separable harmonics in this coordinate system is given in [9, (9), p. 222], namely

$$\begin{aligned} \frac{1}{\|\mathbf{x} - \mathbf{x}'\|} &= \frac{1}{a} \sqrt{(\cosh \sigma - \cos \theta)(\cosh \sigma' - \cos \theta')} \sum_{n=0}^{\infty} e^{-(n+1/2)(\sigma_{>} - \sigma_{<})} \\ &\quad \times \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \theta') \cos(m(\phi - \phi')), \end{aligned}$$

where $\sigma' \in [0, \infty)$, $\theta' \in [0, \pi]$, $\phi' \in [0, 2\pi)$. By reversing the order of summations in the above expression and comparing with the Fourier cosine expansion for the reciprocal distance between two points, namely

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{\sqrt{(\cosh \sigma - \cos \theta)(\cosh \sigma' - \cos \theta')}}{\pi a \sqrt{\sin \theta \sin \theta'}} \sum_{m=0}^{\infty} \epsilon_m \cos(m(\phi - \phi')) Q_{m-1/2}(\chi),$$

where $\chi \geq 1$ is given by (25), we obtain the following addition theorem for the associated Legendre function of the second kind

$$Q_{m-1/2}(\chi) = \pi \sqrt{\sin \theta \sin \theta'} \sum_{n=m}^{\infty} \frac{(n-m)!}{(n+m)!} e^{-(n+1/2)(\sigma_{>} - \sigma_{<})} P_n^m(\cos \theta) P_n^m(\cos \theta'). \quad (26)$$

If $\sigma = \sigma'$ and $\theta = \theta'$, then $\chi = 1$, and $Q_{m-1/2}(\chi)$ has a logarithmic singularity. Note that the corresponding expression given in [6, § 6.1, (45)] is given incorrectly (see [4]). If we multiply both sides of (26) by $\sqrt{\sin \theta} P_n^m(\cos \theta)$ and integrate over $\theta \in [0, \pi]$, then by (7) we have obtained (24). We justify the interchange of integral and infinite sum in the same way as in the proof of Theorem 6. ■

3.2 Order orthogonality for associated Legendre functions with integer degree and order

In this subsection we take advantage of the order orthogonality relation for the Ferrers function of the first kind with integer degree and order (cf. [12, (14.17.8)])

$$\int_0^\pi P_n^m(\cos \theta) P_n^{m'}(\cos \theta) \frac{1}{\sin \theta} d\theta = \frac{1}{m} \frac{(n+m)!}{(n-m)!} \delta_{m,m'}, \quad (27)$$

with $m \geq 1$.

Theorem 9. Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ with $1 \leq m \leq n$, $\theta' \in [0, \pi]$, $\phi, \phi' \in [0, 2\pi)$. Then

$$\int_0^\pi P_n(\cos \gamma) P_n^m(\cos \theta) \frac{1}{\sin \theta} d\theta = \frac{2}{m} P_n^m(\cos \theta') \cos(m(\phi - \phi')),$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi').$$

Proof. We start with the addition theorem for spherical harmonics (cf. [12, (14.18.1)]), namely

$$P_n(\cos \gamma) = \sum_{m=-n}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \theta') e^{im(\phi-\phi')}, \quad (28)$$

where $P_n : \mathbb{C} \rightarrow \mathbb{C}$, for $n \in \mathbb{N}_0$, is the Legendre polynomial which can be defined in terms of the terminating Gauss hypergeometric series (see for instance [12, Chapters 15, 18]) as follows

$$P_n(z) := {}_2F_1 \left(\begin{matrix} -n, n+1 \\ 1 \end{matrix}; \frac{1-z}{2} \right).$$

We then take advantage of the order orthogonality relation for the Ferrers functions of the first kind with integer degree and order. If we multiply both sides of (28) by $(\sin \theta)^{-1} P_n^{m'}(\cos \theta)$ and integrate over $\theta \in (0, \pi)$, by using (27) we obtain the desired result. ■

Theorem 9, originating from (28), is the only example of a definite integral that we could find using the order orthogonality relation for the Ferrers functions of the first kind (27). Therefore we highly suspect that this result is previously known, and include it mainly for completeness sake. It would however be very interesting to find another example using this orthogonality relation.

3.3 Orthogonality for Chebyshev polynomials of the first kind

Here we take advantage of orthogonality from Chebyshev polynomials of the first kind (cf. [12, § 18.3])

$$\int_0^\pi T_m(\cos \theta) T_n(\cos \theta) d\theta = \frac{\pi}{\epsilon_n} \delta_{m,n}, \quad (29)$$

where $T_n : \mathbb{C} \rightarrow \mathbb{C}$, for $n \in \mathbb{N}_0$, is the Chebyshev polynomial of the first kind which can be defined in terms of the terminating Gauss hypergeometric series (see [12, Chapter 18])

$$T_n(z) = {}_2F_1 \left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix}; \frac{1-z}{2} \right).$$

The Chebyshev polynomials of the first kind satisfy the identity [12, (18.5.1)]

$$T_n(\cos \theta) = \cos(n\theta).$$

Theorem 10. *Let $m, n \in \mathbb{Z}$, $\sigma, \sigma' \in (0, \infty)$. Then*

$$\begin{aligned} \int_0^\pi Q_{m-1/2}(\chi) \cos(n\psi) d\psi &= \pi (-1)^m \sqrt{\sinh \sigma \sinh \sigma'} \\ &\times \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} P_{n-1/2}^m(\cosh \sigma_{<}) Q_{n-1/2}^m(\cosh \sigma_{>}), \end{aligned} \quad (30)$$

where

$$\chi = \coth \sigma \coth \sigma' - \operatorname{csch} \sigma \operatorname{csch} \sigma' \cos \psi. \quad (31)$$

Proof. We start with toroidal coordinates on \mathbb{R}^3 , namely

$$x = \frac{a \sinh \sigma \cos \phi}{\cosh \sigma - \cos \psi}, \quad y = \frac{a \sinh \sigma \sin \phi}{\cosh \sigma - \cos \psi}, \quad z = \frac{a \sin \psi}{\cosh \sigma - \cos \psi},$$

where $a > 0$, $\sigma \in (0, \infty)$, $\psi, \phi \in [0, 2\pi)$. The reciprocal distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$ is given algebraically by

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{1}{a} \sqrt{\frac{(\cosh \sigma - \cos \psi)(\cosh \sigma' - \cos \psi')}{2 \sinh \sigma \sinh \sigma'}} \times \left[\frac{\cosh \sigma \cosh \sigma' - \cos(\psi - \psi')}{\sinh \sigma \sinh \sigma'} - \cos(\phi - \phi') \right]^{-1/2},$$

where (σ', ψ', ϕ') are the toroidal coordinates corresponding to the point \mathbf{x}' . Using Heine's reciprocal square root identity (see for instance [5, (3.11)])

$$\frac{1}{\sqrt{z - x}} = \frac{\sqrt{2}}{\pi} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(z) T_m(x),$$

where $z > 1$ and $x \in [-1, 1]$, we can obtain a Fourier cosine series representation for the reciprocal distance between two points in toroidal coordinates on \mathbb{R}^3 , namely

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{1}{\pi a} \sqrt{\frac{(\cosh \sigma - \cos \psi)(\cosh \sigma' - \cos \psi')}{\sinh \sigma \sinh \sigma'}} \sum_{m=0}^{\infty} \epsilon_m \cos(m(\phi - \phi')) Q_{m-1/2}(\chi),$$

where $\chi > 1$ is given by (31). We can further expand the associated Legendre function of the second kind using the following addition theorem (cf. [7, (8.795.2)])

$$Q_{m-1/2}(\chi) = (-1)^m \sqrt{\sinh \sigma \sinh \sigma'} \sum_{n=0}^{\infty} \epsilon_n \cos(n(\psi - \psi')) \times \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} P_{n-1/2}^m(\cosh \sigma_{<}) Q_{n-1/2}^m(\cosh \sigma_{>}). \quad (32)$$

Note that with the above addition theorem, we have the expansion of the reciprocal distance between two points in terms of the separable harmonics in toroidal coordinates

$$\frac{1}{\|\mathbf{x} - \mathbf{x}'\|} = \frac{1}{\pi a} \sqrt{(\cosh \sigma - \cos \psi)(\cosh \sigma' - \cos \psi')} \sum_{m=0}^{\infty} (-1)^m \epsilon_m \cos(m(\phi - \phi')) \times \sum_{n=0}^{\infty} \epsilon_n \cos(n(\psi - \psi')) \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} P_{n-1/2}^m(\cosh \sigma_{<}) Q_{n-1/2}^m(\cosh \sigma_{>})$$

(see also [6, § 6.2] and [4]). If we relabel $\psi - \psi' \mapsto \psi$ and multiply both sides of (32) by $\cos(n\psi)$ and integrate over $\psi \in [0, \pi]$, then by (29) we have obtained (30). The interchange of infinite sum and integral is justified by (10), (11). ■

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References

- [1] Abramowitz M., Stegun I.A., Handbook of mathematical functions with formulas, graphs, and mathematical tables, *National Bureau of Standards Applied Mathematics Series*, Vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.
- [2] Askey R., Orthogonal polynomials and special functions, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975.
- [3] Cohl H.S., Fourier, Gegenbauer and Jacobi expansions for a power-law fundamental solution of the polyharmonic equation and polyspherical addition theorems, [arXiv:1209.6047](https://arxiv.org/abs/1209.6047).
- [4] Cohl H.S., Erratum: Developments in determining the gravitational potential using toroidal functions, *Astronom. Nachr.* **333** (2012), 784–785.
- [5] Cohl H.S., Dominici D.E., Generalized Heine’s identity for complex Fourier series of binomials, *Proc. R. Soc. Lond. Ser. A* **467** (2011), 333–345, [arXiv:0912.0126](https://arxiv.org/abs/0912.0126).
- [6] Cohl H.S., Tohline J.E., Rau A.R.P., Srivastava H.M., Developments in determining the gravitational potential using toroidal functions, *Astronom. Nachr.* **321** (2000), 363–372.
- [7] Gradshteyn I.S., Ryzhik I.M., Table of integrals, series, and products, seventh ed., Elsevier/Academic Press, Amsterdam, 2007.
- [8] Hardy G.H., Further researches in the theory of divergent series and integrals, *Trans. Cambridge Philos. Soc.* **21** (1908), 1–48.
- [9] MacRobert T.M., Spherical harmonics. An elementary treatise on harmonic functions with applications, 2nd ed., Methuen & Co. Ltd., London, 1947.
- [10] Magnus W., Oberhettinger F., Soni R.P., Formulas and theorems for the special functions of mathematical physics, 3rd ed., *Die Grundlehren der mathematischen Wissenschaften*, Bd. 52, Springer-Verlag, New York, 1966.
- [11] Morse P.M., Feshbach H., Methods of theoretical physics, Vols. 1, 2, McGraw-Hill Book Co. Inc., New York, 1953.
- [12] Olver F.W.J., Lozier D.W., Boisvert R.F., Clark C.W. (Editors), NIST handbook of mathematical functions, Cambridge University Press, Cambridge, 2010.
- [13] Prudnikov A.P., Brychkov Y.A., Marichev O.I., Integrals and series. Vol. 3. More special functions, Gordon and Breach Science Publishers, New York, 1990.
- [14] Schäfke F.W., Einführung in die Theorie der speziellen Funktionen der mathematischen Physik, *Die Grundlehren der mathematischen Wissenschaften*, Bd. 118, Springer-Verlag, Berlin, 1963.
- [15] Watson G.N., A treatise on the theory of Bessel functions, 2nd ed., *Cambridge Mathematical Library*, Cambridge University Press, Cambridge, 1944.