

A Vertex Operator Approach for Form Factors of Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -Symmetric Model and Its Application^{*}

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Abstract. A vertex operator approach for form factors of Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is constructed on the basis of bosonization of vertex operators in the $A_{n-1}^{(1)}$ model and vertex-face transformation. As simple application for $n = 2$, we obtain expressions for $2m$ -point form factors related to the σ^z and σ^x operators in the eight-vertex model.

Key words: vertex operator approach; form factors; Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model; integral formulae

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1 Introduction

In [1] and [2] we derived the integral formulae for correlation functions and form factors, respectively, of Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [3, 4] on the basis of vertex operator approach [5]. Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is an n -state generalization of Baxter's eight-vertex model [6], which has $(\mathbb{Z}/2\mathbb{Z})$ -symmetries. As for the eight-vertex model, the integral formulae for correlation functions and form factors were derived by Lashkevich and Pugai [7] and by Lashkevich [8], respectively.

It was found in [7] that the correlation functions of the eight-vertex model can be obtained by using the free field realization of the vertex operators in the eight-vertex SOS model [9], with insertion of the nonlocal operator Λ , called 'the tail operator'. The vertex operator approach for higher spin generalization of the eight-vertex model was presented in [10]. The vertex operator approach for higher rank generalization was presented in [1]. The expression of the spontaneous polarization of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [11] was also reproduced in [1], on the basis of vertex operator approach. Concerning form factors, the bosonization scheme for the eight-vertex model was constructed in [8]. The higher rank generalization of [8] was presented in [2]. It was shown in [12, 13] that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ relevant to the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model provides the Drinfeld realization of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by a Heisenberg algebra.

The present paper is organized as follows. In Section 2 we review the basic definitions of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [3], the corresponding dual face model $A_{n-1}^{(1)}$ model [14], and the vertex-face correspondence. In Section 3 we summarize the vertex operator algebras relevant to the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the $A_{n-1}^{(1)}$ model [1, 2]. In Section 4 we construct the free field representations of the tail operators, in terms of those of the basic operators for the type I [15]

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and the type II [16] vertex operators in the $A_{n-1}^{(1)}$ model. Note that in the present paper we use a different convention from the one used in [1, 2]. In Section 5 we calculate $2m$ -point form factors of the σ^z -operator and σ^x -operator in the eight-vertex model, as simple application for $n = 2$. In Section 6 we give some concluding remarks. Useful operator product expansion (OPE) formulae and commutation relations for basic bosons are given in Appendix A.

2 Basic definitions

The present section aims to formulate the problem, thereby fixing the notation.

2.1 Theta functions

The Jacobi theta function with two pseudo-periods 1 and τ ($\text{Im } \tau > 0$) are defined as follows:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v; \tau) := \sum_{m \in \mathbb{Z}} \exp \left\{ \pi \sqrt{-1} (m+a) [(m+a)\tau + 2(v+b)] \right\},$$

for $a, b \in \mathbb{R}$. Let $n \in \mathbb{Z}_{\geq 2}$ and $r \in \mathbb{R}_{>1}$, and also fix the parameter x such that $0 < x < 1$. We will use the abbreviations,

$$\begin{aligned} [v] &:= x^{\frac{v^2}{r}-v} \Theta_{x^{2r}}(x^{2v}), & [v]' &:= [v]|_{r \mapsto r-1}, & [v]_1 &:= [v]|_{r \mapsto 1}, \\ \{v\} &:= x^{\frac{v^2}{r}-v} \Theta_{x^{2r}}(-x^{2v}), & \{v\}' &:= \{v\}|_{r \mapsto r-1}, & \{v\}_1 &:= \{v\}|_{r \mapsto 1}, \end{aligned}$$

where

$$\begin{aligned} \Theta_q(z) &= (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2} (-z)^m, \\ (z; q_1, \dots, q_m)_\infty &= \prod_{i_1, \dots, i_m \geq 0} (1 - zq_1^{i_1} \cdots q_m^{i_m}). \end{aligned}$$

Note that

$$\begin{aligned} \vartheta \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \left(\frac{v}{r}, \frac{\pi \sqrt{-1}}{\epsilon r} \right) &= \sqrt{\frac{\epsilon r}{\pi}} \exp \left(-\frac{\epsilon r}{4} \right) [v], \\ \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \left(\frac{v}{r}, \frac{\pi \sqrt{-1}}{\epsilon r} \right) &= \sqrt{\frac{\epsilon r}{\pi}} \exp \left(-\frac{\epsilon r}{4} \right) \{v\}, \end{aligned}$$

where $x = e^{-\epsilon}$ ($\epsilon > 0$).

For later conveniences we also introduce the following symbols:

$$r_j(v) = z^{\frac{r-1}{r} \frac{n-j}{n}} \frac{g_j(z^{-1})}{g_j(z)}, \quad g_j(z) = \frac{\{x^{2n+2r-j-1}z\} \{x^{j+1}z\}}{\{x^{2n-j+1}z\} \{x^{2r+j-1}z\}}, \quad (2.1)$$

$$r_j^*(v) = z^{\frac{r}{r-1} \frac{n-j}{n}} \frac{g_j^*(z^{-1})}{g_j^*(z)}, \quad g_j^*(z) = \frac{\{x^{2n+2r-j-1}z\}' \{x^{j-1}z\}'}{\{x^{2n-j-1}z\}' \{x^{2r+j-1}z\}'}, \quad (2.2)$$

$$\chi_j(v) = (-z)^{-\frac{j(n-j)}{n}} \frac{\rho_j(z^{-1})}{\rho_j(z)}, \quad \rho_j(z) = \frac{(x^{2j+1}z; x^2, x^{2n})_\infty (x^{2n-2j+1}z; x^2, x^{2n})_\infty}{(xz; x^2, x^{2n})_\infty (x^{2n+1}z; x^2, x^{2n})_\infty}, \quad (2.3)$$

where $z = x^{2v}$, $1 \leq j \leq n$ and

$$\{z\} = (z; x^{2r}, x^{2n})_\infty, \quad \{z\}' = (z; x^{2r-2}, x^{2n})_\infty.$$

In particular we denote $\chi(v) = \chi_1(v)$. These factors will appear in the commutation relations among the type I and type II vertex operators.

The integral kernel for the type I and the type II vertex operators will be given as the products of the following elliptic functions:

$$\begin{aligned} f(v, w) &= \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]}, & h(v) &= \frac{[v - 1]}{[v + 1]}, \\ f^*(v, w) &= \frac{[v - \frac{1}{2} + w]'}{[v + \frac{1}{2}]'}, & h^*(v) &= \frac{[v + 1]'}{[v - 1]'} \end{aligned}$$

2.2 Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model

Let $V = \mathbb{C}^n$ and $\{\varepsilon_\mu\}_{0 \leq \mu \leq n-1}$ be the standard orthonormal basis with the inner product $\langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu}$. Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [3] is a vertex model on a two-dimensional square lattice \mathcal{L} such that the state variables take the values of $(\mathbb{Z}/n\mathbb{Z})$ -spin. The model is $(\mathbb{Z}/n\mathbb{Z})$ -symmetric in a sense that the R -matrix satisfies the following conditions:

- (i) $R(v)_{jl}^{ik} = 0$, unless $i + k = j + l \pmod{n}$,
- (ii) $R(v)_{j+pl+p}^{i+pk+p} = R(v)_{jl}^{ik}$, $\forall i, j, k, l, p \in \mathbb{Z}/n\mathbb{Z}$.

The definition of the R -matrix in the principal regime can be found in [2]. The present R -matrix has three parameters v , ϵ and r , which lie in the following region:

$$\epsilon > 0, \quad r > 1, \quad 0 < v < 1.$$

2.3 The $A_{n-1}^{(1)}$ model

The dual face model of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is called the $A_{n-1}^{(1)}$ model. This is a face model on a two-dimensional square lattice \mathcal{L}^* , the dual lattice of \mathcal{L} , such that the state variables take the values of the dual space of Cartan subalgebra \mathfrak{h}^* of $A_{n-1}^{(1)}$:

$$\mathfrak{h}^* = \bigoplus_{\mu=0}^{n-1} \mathbb{C}\omega_\mu,$$

where

$$\omega_\mu := \sum_{\nu=0}^{\mu-1} \bar{\varepsilon}_\nu, \quad \bar{\varepsilon}_\mu = \varepsilon_\mu - \frac{1}{n} \sum_{\mu=0}^{n-1} \varepsilon_\mu.$$

The weight lattice P and the root lattice Q of $A_{n-1}^{(1)}$ are usually defined. For $a \in \mathfrak{h}^*$, we set

$$a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu, \quad \bar{a}_\mu = \langle a + \rho, \varepsilon_\mu \rangle = \langle a + \rho, \bar{\varepsilon}_\mu \rangle, \quad \rho = \sum_{\mu=1}^{n-1} \omega_\mu.$$

An ordered pair $(a, b) \in \mathfrak{h}^{*2}$ is called admissible if $b = a + \bar{\varepsilon}_\mu$, for a certain μ ($0 \leq \mu \leq n-1$). For $(a, b, c, d) \in \mathfrak{h}^{*4}$, let $W \left[\begin{array}{c|c} c & d \\ b & a \end{array} \middle| v \right]$ be the Boltzmann weight of the $A_{n-1}^{(1)}$ model for the state configuration $\left[\begin{array}{c|c} c & d \\ b & a \end{array} \right]$ round a face. Here the four states a , b , c and d are ordered clockwise

from the SE corner. In this model $W \left[\begin{array}{cc|c} c & d & v \\ b & a & \end{array} \right] = 0$ unless the four pairs $(a, b), (a, d), (b, c)$ and (d, c) are admissible. Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter v . The explicit expressions of W can be found in [2]. We consider the so-called Regime III in the model, i.e., $0 < v < 1$.

2.4 Vertex-face correspondence

Let $t(v)_{a-\bar{\varepsilon}_\mu}^a$ be the intertwining vectors in \mathbb{C}^n , whose elements are expressed in terms of theta functions. As for the definitions see [2]. Then $t(v)_{a-\bar{\varepsilon}_\mu}^a$'s relate the R -matrix of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model in the principal regime and Boltzmann weights W of the $A_{n-1}^{(1)}$ model in the regime III

$$R(v_1 - v_2)t(v_1)_a^d \otimes t(v_2)_d^c = \sum_b t(v_1)_b^c \otimes t(v_2)_a^b W \left[\begin{array}{cc|c} c & d & v_1 - v_2 \\ b & a & \end{array} \right]. \quad (2.4)$$

Let us introduce the dual intertwining vectors satisfying

$$\sum_{\mu=0}^{n-1} t_\mu^*(v)_a^{a'} t^\mu(v)_{a''}^a = \delta_{a''}^{a'}, \quad \sum_{\nu=0}^{n-1} t^\nu(v)_{a-\bar{\varepsilon}_\nu}^a t_{\mu'}^*(v)_a^{a-\bar{\varepsilon}_\nu} = \delta_{\mu'}^\mu. \quad (2.5)$$

From (2.4) and (2.5), we have

$$t^*(v_1)_c^b \otimes t^*(v_2)_b^a R(v_1 - v_2) = \sum_d W \left[\begin{array}{cc|c} c & d & v_1 - v_2 \\ b & a & \end{array} \right] t^*(v_1)_d^a \otimes t^*(v_2)_c^d.$$

For fixed $r > 1$, let

$$S(v) = -R(v)|_{r \rightarrow r-1}, \quad W' \left[\begin{array}{cc|c} c & d & v \\ b & a & \end{array} \right] = -W \left[\begin{array}{cc|c} c & d & v \\ b & a & \end{array} \right] \Big|_{r \rightarrow r-1},$$

and $t^*(v)_a^b$ is the dual intertwining vector of $t'(v)_b^a$. Here,

$$t'(v)_b^a := f'(v)t(v; \epsilon, r-1)_b^a,$$

with

$$f'(v) = \frac{x^{-\frac{v^2}{n(r-1)} - \frac{(r+n-2)v}{n(r-1)} - \frac{(n-1)(3r+n-5)}{6n(r-1)}}}{\sqrt{-(x^{2r-2}; x^{2r-2})_\infty}} \times \frac{(x^2 z^{-1}; x^{2n}, x^{2r-2})_\infty (x^{2r+2n-2}z; x^{2n}, x^{2r-2})_\infty}{(z^{-1}; x^{2n}, x^{2r-2})_\infty (x^{2r+2n-4}z; x^{2n}, x^{2r-2})_\infty}, \quad (2.6)$$

and $z = x^{2v}$. Then we have

$$t^*(v_1)_c^b \otimes t^*(v_2)_b^a S(v_1 - v_2) = \sum_d W' \left[\begin{array}{cc|c} c & d & v_1 - v_2 \\ b & a & \end{array} \right] t^*(v_1)_d^a \otimes t^*(v_2)_c^d.$$

3 Vertex operator algebra

3.1 Vertex operators for the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model

Let $\mathcal{H}^{(i)}$ be the \mathbb{C} -vector space spanned by the half-infinite pure tensor vectors of the forms

$$\varepsilon_{\mu_1} \otimes \varepsilon_{\mu_2} \otimes \varepsilon_{\mu_3} \otimes \cdots \quad \text{with } \mu_j \in \mathbb{Z}/n\mathbb{Z}, \mu_j = i + 1 - j \pmod{n} \text{ for } j \gg 0.$$

The type I vertex operator $\Phi^\mu(v)$ can be defined as a half-infinite transfer matrix. The operator $\Phi^\mu(v)$ is an intertwiner from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(i+1)}$, satisfying the following commutation relation:

$$\Phi^\mu(v_1)\Phi^\nu(v_2) = \sum_{\mu', \nu'} R(v_1 - v_2)_{\mu'\nu'}^{\mu\nu} \Phi^{\nu'}(v_2)\Phi^{\mu'}(v_1).$$

When we consider an operator related to 'creation-annihilation' process, we need another type of vertex operators, the type II vertex operators that satisfy the following commutation relations:

$$\begin{aligned} \Psi_\nu^*(v_2)\Psi_\mu^*(v_1) &= \sum_{\mu', \nu'} \Psi_{\mu'}^*(v_1)\Psi_{\nu'}^*(v_2)S(v_1 - v_2)_{\mu\nu}^{\mu'\nu'}, \\ \Phi^\mu(v_1)\Psi_\nu^*(v_2) &= \chi(v_1 - v_2)\Psi_\nu^*(v_2)\Phi^\mu(v_1). \end{aligned}$$

Let

$$\rho^{(i)} = x^{2nH_{\text{CTM}}} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)},$$

where H_{CTM} is the CTM Hamiltonian defined as follows:

$$\begin{aligned} H_{\text{CTM}}(\mu_1, \mu_2, \mu_3, \dots) &= \frac{1}{n} \sum_{j=1}^{\infty} j H_v(\mu_j, \mu_{j+1}), \\ H_v(\mu, \nu) &= \begin{cases} \mu - \nu - 1 & \text{if } 0 \leq \nu < \mu \leq n - 1, \\ n - 1 + \mu - \nu & \text{if } 0 \leq \mu \leq \nu \leq n - 1. \end{cases} \end{aligned} \quad (3.1)$$

Then we have the homogeneity relations

$$\Phi^\mu(v)\rho^{(i)} = \rho^{(i+1)}\Phi^\mu(v - n), \quad \Psi_\mu^*(v)\rho^{(i)} = \rho^{(i+1)}\Psi_\mu^*(v - n).$$

3.2 Vertex operators for the $A_{n-1}^{(1)}$ model

For $k = a + \rho, l = \xi + \rho$ and $0 \leq i \leq n - 1$, let $\mathcal{H}_{l,k}^{(i)}$ be the space of admissible paths (a_0, a_1, a_2, \dots) such that

$$\begin{aligned} a_0 &= a, & a_j - a_{j+1} &\in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{n-1}\} \quad \text{for } j = 0, 1, 2, 3, \dots, \\ a_j &= \xi + \omega_{i+1-j} \quad \text{for } j \gg 0. \end{aligned}$$

The type I vertex operator $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ can be defined as a half-infinite transfer matrix. The operator $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ is an intertwiner from $\mathcal{H}_{l,k}^{(i)}$ to $\mathcal{H}_{l,k+\bar{\varepsilon}_\mu}^{(i+1)}$, satisfying the following commutation relation:

$$\Phi(v_1)_b^c \Phi(v_2)_a^b = \sum_d W \left[\begin{array}{cc|c} c & d & \\ b & a & \end{array} \middle| v_1 - v_2 \right] \Phi(v_2)_d^c \Phi(v_1)_a^d.$$

The free field realization of $\Phi(v_2)_a^b$ was constructed in [15]. See Section 4.2.

The type II vertex operators should satisfy the following commutation relations:

$$\begin{aligned} \Psi^*(v_2)_{\xi_d}^{\xi_c} \Psi^*(v_1)_{\xi_a}^{\xi_d} &= \sum_{\xi_b} \Psi^*(v_1)_{\xi_b}^{\xi_c} \Psi^*(v_2)_{\xi_a}^{\xi_b} W' \left[\begin{array}{cc} \xi_c & \xi_d \\ \xi_b & \xi_a \end{array} \middle| v_1 - v_2 \right], \\ \Phi(v_1)_a^{a'} \Psi^*(v_2)_{\xi}^{\xi'} &= \chi(v_1 - v_2) \Psi^*(v_2)_{\xi}^{\xi'} \Phi(v_1)_a^{a'}. \end{aligned}$$

Let

$$\rho_{l,k}^{(i)} = G_a x^{2nH_{l,k}^{(i)}}, \quad G_a = \prod_{0 \leq \mu < \nu \leq n-1} [a_{\mu\nu}],$$

where $H_{l,k}^{(i)}$ is the CTM Hamiltonian of $A_{n-1}^{(1)}$ model in regime III is given as follows:

$$H_{l,k}^{(i)}(a_0, a_1, a_2, \dots) = \frac{1}{n} \sum_{j=1}^{\infty} j H_f(a_{j-1}, a_j, a_{j+1}),$$

$$H_f(a + \bar{\varepsilon}_\mu + \bar{\varepsilon}_\nu, a + \bar{\varepsilon}_\mu, a) = H_v(\nu, \mu),$$

and $H_v(\nu, \mu)$ is the same one as (3.1). Then we have the homogeneity relations

$$\Phi(v)_a^{a'} \frac{\rho_{a+\rho,l}^{(i)}}{G_a} = \frac{\rho_{a'+\rho,l}^{(i+1)}}{G_{a'}} \Phi(v-n)_a^{a'}, \quad \Psi^*(v)_{\xi}^{\xi'} \rho_{k,\xi+\rho}^{(i)} = \rho_{k,\xi'+\rho}^{(i+1)} \Psi^*(v-n)_{\xi}^{\xi'}.$$

The free field realization of $\Psi^*(v)_{\xi}^{\xi'}$ was constructed in [16]. See Section 4.3.

3.3 Tail operators and commutation relations

In [1] we introduced the intertwining operators between $\mathcal{H}^{(i)}$ and $\mathcal{H}_{l,k}^{(i)}$ ($k = l + \omega_i \pmod{Q}$):

$$\begin{aligned} T(u)_{\xi a_0}^{\xi a_0} &= \prod_{j=0}^{\infty} t^{\mu_j} (-u)_{a_{j+1}}^{a_j} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}_{l,k}^{(i)}, \\ T(u)_{\xi a_0} &= \prod_{j=0}^{\infty} t_{\mu_j}^* (-u)_{a_j}^{a_{j+1}} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}^{(i)}, \end{aligned}$$

which satisfy

$$\rho^{(i)} = \left(\frac{(x^{2r-2}; x^{2r-2})_{\infty}}{(x^{2r}; x^{2r})_{\infty}} \right)^{(n-1)(n-2)/2} \frac{1}{G'_{\xi}} \sum_{\substack{k \equiv l + \omega_i \\ (\text{mod } Q)}} T(u)_{a\xi} \rho_{l,k}^{(i)} T(u)^{a\xi}. \quad (3.2)$$

In order to obtain the form factors of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model, we need the free field representations of the tail operator which is offdiagonal with respect to the boundary conditions:

$$\Lambda(u)_{\xi a}^{\xi' a'} = T(u)_{\xi' a'}^{\xi' a'} T(u)_{\xi a} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}_{l',k'}^{(i)}, \quad (3.3)$$

where $k = a + \rho$, $l = \xi + \rho$, $k' = a' + \rho$, and $l' = \xi' + \rho$. Let

$$L \left[\begin{array}{cc} a'_0 & a'_1 \\ a_0 & a_1 \end{array} \middle| u \right] := \sum_{\mu=0}^{n-1} t_{\mu}^* (-u)_{a_0}^{a_1} t^{\mu} (-u)_{a'_1}^{a'_0}.$$

Then we have

$$\Lambda(u)_{\xi a_0}^{\xi' a'_0} = \prod_{j=0}^{\infty} L \left[\begin{array}{cc|c} a'_j & a'_{j+1} & u \end{array} \right].$$

From the invertibility of the intertwining vector and its dual vector, we have

$$\Lambda(u_0)_{\xi a}^{\xi' a} = \delta_{\xi \xi'}. \quad (3.4)$$

Note that the tail operator (3.3) satisfies the following intertwining relations [1, 2]:

$$\Lambda(u)_{\xi_b^c}^{\xi' c} \Phi(v)_a^b = \sum_d L \left[\begin{array}{cc|c} c & d & u-v \end{array} \right] \Phi(v)_d^c \Lambda(u)_{\xi_a}^{\xi' d}, \quad (3.5)$$

$$\Psi^*(v)_{\xi_d^c}^{\xi c} \Lambda(u)_{\xi_a a'}^{\xi d a'} = \sum_{\xi_b} L' \left[\begin{array}{cc|c} \xi_c & \xi_d & u + \Delta u - v \end{array} \right] \Lambda(u)_{\xi_b a}^{\xi c a'} \Psi^*(v)_{\xi_a}^{\xi b}, \quad (3.6)$$

where

$$L' \left[\begin{array}{cc|c} \xi_c & \xi_d & u \end{array} \right] = L \left[\begin{array}{cc|c} \xi_c & \xi_d & u \end{array} \right] \Big|_{r \mapsto r-1}.$$

We should find a representation of $\Lambda(u)_{\xi_a}^{\xi' a'}$ and fix the constant Δu that solves (3.5) and (3.6).

4 Free field realization

4.1 Bosons

In [17, 18] the bosons B_m^j ($1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\}$) relevant to elliptic algebra were introduced. For $\alpha, \beta \in \mathfrak{h}^*$ we denote the zero mode operators by P_α, Q_β . Concerning commutation relations among these operators see [17, 18, 2].

We will deal with the bosonic Fock spaces $\mathcal{F}_{l,k}$, ($l, k \in \mathfrak{h}^*$) generated by B_{-m}^j ($m > 0$) over the vacuum vectors $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}\{\{B_{-1}^j, B_{-2}^j, \dots\}_{1 \leq j \leq n} | l, k\rangle\},$$

where

$$|l, k\rangle = \exp(\sqrt{-1}(\beta_1 Q_k + \beta_2 Q_l)) |0, 0\rangle,$$

and

$$t^2 - \beta_0 t - 1 = (t - \beta_1)(t - \beta_2), \quad \beta_0 = \frac{1}{\sqrt{r(r-1)}}, \quad \beta_1 < \beta_2.$$

4.2 Type I vertex operators

Let us define the basic operators for $j = 1, \dots, n-1$

$$U_{-\alpha_j}(v) = z^{\frac{r-1}{r}} : \exp \left(-\beta_1 (\sqrt{-1} Q_{\alpha_j} + P_{\alpha_j} \log z) + \sum_{m \neq 0} \frac{B_m^j - B_m^{j+1}}{m} (x^j z)^{-m} \right) :,$$

$$U_{\omega_j}(v) = z^{\frac{r-1}{2r} \frac{j(n-j)}{n}} : \exp \left(\beta_1 (\sqrt{-1} Q_{\omega_j} + P_{\omega_j} \log z) - \sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^j x^{(j-2k+1)m} B_m^k z^{-m} \right) :,$$

where $\beta_1 = -\sqrt{\frac{r-1}{r}}$ and $z = x^{2v}$ as usual. The normal product operation places P_α 's to the right of Q_β 's, as well as B_m 's ($m > 0$) to the right of B_{-m} 's. For some useful OPE formulae and commutation relations, see Appendix A.

In what follows we set

$$\pi_\mu = \sqrt{r(r-1)}P_{\bar{\varepsilon}_\mu}, \quad \pi_{\mu\nu} = \pi_\mu - \pi_\nu = rL_{\mu\nu} - (r-1)K_{\mu\nu}.$$

The operators $K_{\mu\nu}$, $L_{\mu\nu}$ and $\pi_{\mu\nu}$ act on $\mathcal{F}_{l,k}$ as scalars $\langle \varepsilon_\mu - \varepsilon_\nu, k \rangle$, $\langle \varepsilon_\mu - \varepsilon_\nu, l \rangle$ and $\langle \varepsilon_\mu - \varepsilon_\nu, rl - (r-1)k \rangle$, respectively. In what follows we often use the symbols

$$G_K = \prod_{0 \leq \mu < \nu \leq n-1} [K_{\mu\nu}], \quad G'_L = \prod_{0 \leq \mu < \nu \leq n-1} [L_{\mu\nu}'].$$

For $0 \leq \mu \leq n-1$ the type I vertex operator $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ can be expressed in terms of $U_{\omega_j}(v)$ and $U_{-\alpha_j}(v)$ on the bosonic Fock space $\mathcal{F}_{l,a+\rho}$. The explicit expression of $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$ can be found in [15].

4.3 Type II vertex operators

Let us define the basic operators for $j = 1, \dots, n-1$

$$V_{-\alpha_j}(v) = (-z)^{\frac{r}{r-1}} : \exp \left(-\beta_2 (\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j} \log(-z)) - \sum_{m \neq 0} \frac{A_m^j - A_m^{j+1}}{m} (x^j z)^{-m} \right) :,$$

$$V_{\omega_j}(v) = (-z)^{\frac{r}{2(r-1)} \frac{j(n-j)}{n}}$$

$$\times : \exp \left(\beta_2 (\sqrt{-1}Q_{\omega_j} + P_{\omega_j} \log(-z)) + \sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^j x^{(j-2k+1)m} A_m^k z^{-m} \right) :,$$

where $\beta_2 = \sqrt{\frac{r}{r-1}}$ and $z = x^{2v}$, and $A_m^j = \frac{[rm]_x}{[(r-1)m]_x} B_m^j$. For some useful OPE formulae and commutation relations, see Appendix A.

For $0 \leq \mu \leq n-1$ the type II vertex operator $\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}$ can be expressed in terms of $V_{\omega_j}(v)$ and $V_{-\alpha_j}(v)$ on the bosonic Fock space $\mathcal{F}_{\xi+\rho,k}$. The explicit expression of $\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}$ can be found in [16].

4.4 Free field realization of tail operators

In order to construct free field realization of the tail operators, we also need another type of basic operators:

$$W_{-\alpha_j}(v) = ((-1)^r z)^{\frac{1}{r(r-1)}}$$

$$\times : \exp \left(-\beta_0 (\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j} \log(-1)^r z) - \sum_{m \neq 0} \frac{O_m^j - O_m^{j+1}}{m} (x^j z)^{-m} \right) :,$$

where $\beta_0 = \beta_1 + \beta_2 = \frac{1}{\sqrt{r(r-1)}}$, $(-1)^r := \exp(\pi\sqrt{-1}r)$ and $O_m^j = \frac{[m]_x}{[(r-1)m]_x} B_m^j$. Concerning useful OPE formulae and commutation relations, see Appendix A.

We cite the results on the free field realization of tail operators. In [1] we obtained the free field representation of $\Lambda(u)_{\xi_a}^{\xi_a'}$ satisfying (3.5) for $\xi' = \xi$:

$$\Lambda(u)_{\xi_a}^{\xi_a'} = G_K \oint \prod_{j=\mu+1}^{\nu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots U_{-\alpha_\nu}(v_\nu)$$

$$\times \prod_{j=\mu}^{\nu-1} (-1)^{L_{j\nu}-K_{j\nu}} f(v_{j+1}-v_j, \pi_{j\nu}) G_K^{-1}, \quad (4.1)$$

where $v_\mu = u$ and $\mu < \nu$. In [2] we obtained the free field representation of $\Lambda(v)_{\xi+\bar{\varepsilon}_\mu a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}a+\bar{\varepsilon}_{n-1}}$ satisfying (3.6) as follows:

$$\begin{aligned} \Lambda(u)_{\xi+\bar{\varepsilon}_\mu a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}a+\bar{\varepsilon}_{n-1}} &= \frac{(-1)^{n-\mu} [a_{n-2n-1}] [\xi_{\mu n-1} - 1]'}{(x^{-1} - x)(x^{2r}; x^{2r})_\infty^3} \frac{[\xi_{\mu n-1} - 1]'}{[1]'} G_K G_L'^{-1} \\ &\times \oint_{C'} \prod_{j=\mu+1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}(u - \frac{r-1}{2}) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\times \prod_{j=\mu+1}^{n-2} (-1)^{L_{\mu j}-K_{\mu j}} f^*(v_j - v_{j+1}, \pi_{\mu j}) G_K^{-1} G_L', \end{aligned} \quad (4.2)$$

for $0 \leq \mu \leq n-2$ with $\Delta u = -\frac{n-1}{2}$ and $v_{n-1} = u$. Concerning other types of tail operators $\Lambda(u)_{\xi a}^{\xi a'}$, the expressions of the free field representation can be found in [1, 2].

4.5 Free field realization of CTM Hamiltonian

Let

$$\begin{aligned} H_F &= \sum_{m=1}^{\infty} \frac{[rm]_x}{[(r-1)m]_x} \sum_{j=1}^{n-1} \sum_{k=1}^j x^{(2k-2j-1)m} B_{-m}^k (B_m^j - B_m^{j+1}) + \frac{1}{2} \sum_{j=1}^{n-1} P_{\omega_j} P_{\alpha_j} \\ &= \sum_{m=1}^{\infty} \frac{[rm]_x}{[(r-1)m]_x} \sum_{j=1}^{n-1} \sum_{k=1}^j x^{(2j-2k-1)m} (B_{-m}^j - B_{-m}^{j+1}) B_m^k + \frac{1}{2} \sum_{j=1}^{n-1} P_{\omega_j} P_{\alpha_j} \end{aligned} \quad (4.3)$$

be the CTM Hamiltonian on the Fock space $\mathcal{F}_{l,k}$ [19]. Then we have the homogeneity relation

$$\phi_\mu(z) q^{H_F} = q^{H_F} \phi_\mu(q^{-1}z),$$

and the trace formula

$$\mathrm{tr}_{\mathcal{F}_{l,k}} (x^{2nH_F} G_a) = \frac{x^{n|\beta_1 k + \beta_2 l|^2}}{(x^{2n}; x^{2n})_\infty^{n-1}} G_a.$$

Let $\rho_{l,k}^{(i)} = G_a x^{2nH_F}$. Then the relation (3.2) holds. We thus indentify H_F with free field representations of $H_{l,k}^{(i)}$, the CTM Hamiltonian of $A_{n-1}^{(1)}$ model in regime III.

5 Form factors for $n = 2$

In this section we would like to find explicit expressions of form factors for $n = 2$ case, i.e., the eight-vertex model form factors. Here, we adopt the convention that the components 0 and 1 for $n = 2$ are denoted by + and -. Form factors of the eight-vertex model are defined as matrix elements of some local operators. For simplicity, we choose σ^z as a local operator:

$$\sigma^z = E_{++}^{(1)} - E_{--}^{(1)},$$

where $E_{\mu\mu'}^{(j)}$ is the matrix unit on the j -th site. The free field representation of σ^z is given by

$$\widehat{\sigma^z} = \sum_{\varepsilon=\pm} \varepsilon \Phi_\varepsilon^*(u) \Phi^\varepsilon(u).$$

Here, $\Phi_\varepsilon^*(u)$ is the dual type I vertex operator, whose free filed representation can be found in [1, 2].

The corresponding form factors with $2m$ ‘charged’ particles are given by

$$F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} = \frac{1}{\chi^{(i)}} \text{Tr}_{\mathcal{H}^{(i)}} \left(\Psi_{\nu_1}^*(u_1) \cdots \Psi_{\nu_{2m}}^*(u_{2m}) \widehat{\sigma^z} \rho^{(i)} \right), \quad (5.1)$$

where

$$\chi^{(i)} = \text{Tr}_{\mathcal{H}^{(i)}} \rho^{(i)} = \frac{(x^4; x^4)_\infty}{(x^2; x^2)_\infty}.$$

In this section we denote the spectral parameters by $z_j = x^{2u_j}$, and denote integral variables by $w_a = x^{2v_a}$.

By using the vertex-face transformation, we can rewrite (5.1) as follows:

$$\begin{aligned} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} &= \frac{1}{\chi^{(i)}} \sum_{l_1, \dots, l_{2m}} t_{\nu_1}^* \left(u_1 - u_0 + \frac{1}{2}\right)_{l_1}^{l_1} \cdots t_{\nu_{2m}}^* \left(u_{2m} - u_0 + \frac{1}{2}\right)_{l_{2m-1}}^{l_{2m-1}} \\ &\times \sum_{k \equiv l+i \pmod{2}} \sum_{\varepsilon = \pm} \varepsilon \sum_{k_1 = k \pm 1} \sum_{k_2 = k_1 \pm 1} t_\varepsilon^*(u - u_0)_{k_1}^k t^\varepsilon(u - u_0)_{k_2}^{k_1} \\ &\times \text{Tr}_{\mathcal{H}_{l_i, k}^{(i)}} \left(\Psi^*(u_1)_{l_1}^{l_1} \cdots \Psi^*(u_{2m})_{l_{2m}}^{l_{2m-1}} \Phi^*(u)_{k_1}^k \Phi(u)_{k_2}^{k_1} \Lambda(u_0)_{l k}^{l_{2m} k_2} \frac{[k] x^{4H_F}}{[l]'} \right), \end{aligned}$$

where H_F is the CTM Hamiltonian defined by (4.3).

Let

$$\begin{aligned} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{l_1 \dots l_{2m}} &= \frac{1}{\chi^{(i)}} \sum_{k \equiv l+i \pmod{2}} \sum_{\varepsilon = \pm} \varepsilon \sum_{k_1 = k \pm 1} \sum_{k_2 = k_1 \pm 1} t_\varepsilon^*(u - u_0)_{k_1}^k t^\varepsilon(u - u_0)_{k_2}^{k_1} \\ &\times \text{Tr}_{\mathcal{H}_{l_i, k}^{(i)}} \left(\Psi^*(u_1)_{l_1}^{l_1} \cdots \Psi^*(u_{2m})_{l_{2m}}^{l_{2m-1}} \Phi^*(u)_{k_1}^k \Phi(u)_{k_2}^{k_1} \Lambda(u_0)_{l k}^{l_{2m} k_2} \frac{[k] x^{4H_F}}{[l]'} \right). \end{aligned} \quad (5.2)$$

Then we have

$$\begin{aligned} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{l_1 \dots l_{2m}} &= \sum_{\nu_1, \dots, \nu_{2m}} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} \\ &\times t^{\nu_1} \left(u_1 - u_0 + \frac{1}{2}\right)_{l_1}^{l_1} \cdots t^{\nu_{2m}} \left(u_{2m} - u_0 + \frac{1}{2}\right)_{l_{2m}}^{l_{2m-1}}. \end{aligned}$$

For simplicity, let $l_j = l - j$ for $1 \leq j \leq 2m$. Then from the relation (3.4), $\Lambda(u_0)_{l k}^{l_{2m} k_2}$ vanishes unless $k_2 = k - 2$. Thus, the sum over k_1 and k_2 on (5.2) reduces to only one non-vanishing term. Furthermore, we note the formula

$$\sum_{\varepsilon = \pm} \varepsilon t_\varepsilon^*(u - u_0)_{k-1}^k t^\varepsilon(u - u_0)_{k-2}^{k-1} = (-1)^{1-i} \frac{\{0\} \{u - u_0 - 1 + k\}}{[u - u_0][k - 1]}.$$

Here, we use $k - l \equiv i \pmod{2}$. The sum with respect to k for the trace over the zero-modes parts can be calculated as follows:

$$\begin{aligned} &\sum_{k \equiv l+i \pmod{2}} \{u - u_0 - 1 + k\} \prod_{j=1}^{2m} (-z_j)^{\frac{rl}{2(r-1)} - \frac{k}{2}} (x^{-1}z)^{-l + \frac{(r-1)k}{r}} \prod_{a=1}^{m-1} (-w_a)^{-\frac{rl}{r-1} + k} \\ &\times \left((-1)^r x^{-r+1} z_0 \right)^{-\frac{l}{r-1} + \frac{k}{r}} x^{\frac{rl^2}{r-1} - 2kl + \frac{(r-1)k^2}{r}} \end{aligned}$$

$$\begin{aligned}
&= x^{\frac{i^2}{r-1} + l} \left(2 + \frac{r}{r-1} \sum_{j=1}^{2n} u_j - 2u - \frac{2r}{r-1} \sum_{a=1}^{m-1} v_a - \frac{2u_0}{r-1} \right) x^{\frac{1}{r}(u-u_0-1)^2 - (u-u_0-1)} \sum_{k \equiv l+i \pmod{2}} x^{(k-l)^2} \\
&\quad \times \sum_{n \in \mathbb{Z}} x^{rn(n-1)} x^{2(u-u_0-1+k)n} x^k \left(2 \sum_{a=1}^{m-1} v_a + 2u - \sum_{j=1}^{2m} u_j - 3 \right) \\
&= \frac{(-1)^{1-i}}{2} x^{\frac{1}{r}(u-u_0-1)^2 - \frac{1}{r-1} \left(u_0 + \sum_{a=1}^{m-1} v_a - \frac{1}{2} \sum_{j=1}^{2m} u_j \right)^2 - \left(\sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j - 1 \right)^2} \\
&\quad \times Z_m^{(i)}(l, u, u_0, u_j, v_a),
\end{aligned}$$

where

$$\begin{aligned}
Z_m^{(i)}(l, u, u_0, u_j, v_a) &= \left[l - u_0 - \sum_{a=1}^{m-1} v_a + \frac{1}{2} \sum_{j=1}^{2m} u_j \right]' \left[\sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j \right]_1 \\
&\quad + (-1)^{1-i} \left\{ l - u_0 - \sum_{a=1}^{m-1} v_a + \frac{1}{2} \sum_{j=1}^{2m} u_j \right\}' \left\{ \sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j \right\}_1.
\end{aligned}$$

Thus, $F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{l-1 \dots l-2m}$ can be obtained as follows:

$$\begin{aligned}
&(-1)^{m-1} \beta_m^{-1} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{l-1 \dots l-2m} \\
&= \prod_{j < j'} (-z_j)^{\frac{r}{2(r-1)}} F_{\psi^* \psi^*}(z_{j'}/z_j) \prod_{j=1}^{2m} \frac{(-z_j)^{-\frac{1}{r-1}} x^{\frac{(u_j - u_0 + 1/2)^2}{4(r-1)} + \frac{r(u_j - u_0 + 1/2)}{2(r-1)} + \frac{1}{4}}}{x^{-1} z(x z_j / z; x^4)_\infty (x^3 z / z_j; x^4)_\infty} f'(u_j - u_0 + \frac{1}{2}) \\
&\quad \times \oint_C \prod_{a=1}^{m-1} \frac{dw_a}{2\pi \sqrt{-1} w_a} Z_m^{(i)}(l, u, u_0, u_j, v_a) \prod_{a < b} (-w_b)^{\frac{2r}{r-1}} [v_a - v_b]' [v_a - v_b]_1 x^{-\frac{r}{r-1}(v_a - v_b - 1)^2} \\
&\quad \times \prod_{a=1}^{m-1} x^{-2} z^2 x^{-(v_a - u)^2 + v_a - u} [v_a - u]_1 (-w_a)^{\frac{2}{r-1}} x^{-\frac{1}{r-1}(u_0 - v_a - 1)^2 + u_0 - v_a - 1} [v_a - u_0 + l - m]' \\
&\quad \times \prod_{a=1}^{m-1} \prod_{j=1}^{2m} (-z_j)^{-\frac{r}{r-1}} \frac{(x^{2r-1} w_a / z_j; x^4, x^{2r-2})_\infty (x^{2r+3} z_j / w_a; x^4, x^{2r-2})_\infty}{(x^{-1} w_a / z_j; x^4, x^{2r-2})_\infty (x^3 z_j / w_a; x^4, x^{2r-2})_\infty} \\
&\quad \times \frac{(x^{-1} z)^{\frac{2}{r}} x^{-\frac{r+2}{r}(u_0 - u) - \frac{1}{r} - \frac{1}{r-1} \left(u_0 + \sum_{a=1}^{m-1} v_a - \frac{1}{2} \sum_{j=1}^{2m} u_j \right)^2 - \left(\sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j - 1 \right)^2}}{2}, \tag{5.3}
\end{aligned}$$

where $f'(v)$ is defined by (2.6) for $n = 2$, a scalar function $F_{\psi^* \psi^*}(z)$ and a scalar β_m are

$$\begin{aligned}
F_{\psi^* \psi^*}(z) &= \frac{(z; x^4, x^4, x^{2r-2})_\infty (x^4 z^{-1}; x^4, x^4, x^{2r-2})_\infty}{(x^2 z; x^4, x^4, x^{2r-2})_\infty (x^6 z^{-1}; x^4, x^4, x^{2r-2})_\infty} \\
&\quad \times \frac{(x^{2r+2} z; x^4, x^4, x^{2r-2})_\infty (x^{2r+6} z^{-1}; x^4, x^4, x^{2r-2})_\infty}{(x^{2r} z; x^4, x^4, x^{2r-2})_\infty (x^{2r+4} z^{-1}; x^4, x^4, x^{2r-2})_\infty},
\end{aligned}$$

and

$$\begin{aligned}
\beta_m &= \frac{x^{-\frac{r-1}{4r}} \{0\} [m-1]! (x^{-2} z)^{\frac{r-1}{2r}} (x^2, x^4)_\infty^2 (x^2; x^{2r})_\infty (x^{2r+1}; x^{2r-2})_\infty}{(m-1)! [1]^m (x^{-1} - x) g_1(x^2) (x^{2r}; x^{2r})_\infty^2 (x^{2r+1}; x^{2r})_\infty} \\
&\quad \times \frac{(x^2; x^2)_\infty^{m-1} (x^{2r}; x^{2r-2})_\infty^{m-1} (x^4; x^4, x^4, x^{2r-2})_\infty^m (x^{2r+6}; x^4, x^4, x^{2r-2})_\infty^m}{(x^6; x^4, x^4, x^{2r-2})_\infty^m (x^{2r+4}; x^4, x^4, x^{2r-2})_\infty^m},
\end{aligned}$$

with

$$[m]!' = \prod_{p=1}^m [p]'$$

On (5.3), the integral contour C should be chosen such that all integral variables w_a lie in the convergence domain $x^3|z_j| < |w_a| < x|z_j|$.

Gathering phase factors on (5.3), we have $e^{-\pi\sqrt{-1}\frac{3mr}{2(r-1)}}$. Redefining $f'(v)$ by a scalar factor, we thus obtain the equality:

$$\begin{aligned} & \sum_{\nu_1, \dots, \nu_{2m}} F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} \prod_{j=1}^{2m} \vartheta \left[\begin{matrix} 0 \\ b_{\nu_j} \end{matrix} \right] \left(\frac{u_j - u_0 + \frac{1}{2} + l - j + 1}{2(r-1)}, \frac{\pi\sqrt{-1}}{2\epsilon(r-1)} \right) \\ &= \beta_m \prod_{j < j'} z_j^{\frac{r}{2(r-1)}} F_{\psi^* \psi^*}(z_{j'}/z_j) \prod_{j=1}^{2m} \frac{z_j^{-\frac{1}{r-1}}}{x^{-1}z(xz_j/z; x^4)_\infty (x^3z/z_j; x^4)_\infty} \\ &\times \oint_C \prod_{a=1}^{m-1} \frac{dw_a}{2\pi\sqrt{-1}w_a} Z_m^{(i)}(l, u, u_0, u_j, v_a) \prod_{a < b} w_b^{\frac{2r}{r-1}} [v_a - v_b]' [v_a - v_b]_1 x^{-\frac{r}{r-1}(v_a - v_b - 1)^2} \\ &\times \prod_{a=1}^{m-1} x^{-2} z^2 x^{-(v_a - u)^2 + v_a - u} [v_a - u]_1 w_a^{\frac{2}{r-1}} x^{-\frac{1}{r-1}(u_0 - v_a - 1)^2 + u_0 - v_a - 1} [v_a - u_0 + l - m]' \\ &\times \prod_{a=1}^{m-1} \prod_{j=1}^{2m} z_j^{-\frac{r}{r-1}} \frac{(x^{2r-1}w_a/z_j; x^4, x^{2r-2})_\infty (x^{2r+3}z_j/w_a; x^4, x^{2r-2})_\infty}{(x^{-1}w_a/z_j; x^4, x^{2r-2})_\infty (x^3z_j/w_a; x^4, x^{2r-2})_\infty} \\ &\times \frac{(x^{-1}z)^{\frac{2}{r}} x^{-\frac{r+2}{r}(u_0 - u) - \frac{1}{r} - \frac{1}{r-1} \left(u_0 + \sum_{a=1}^{m-1} v_a - \frac{1}{2} \sum_{j=1}^{2m} u_j \right)^2 - \left(\sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j - 1 \right)^2}}{2}, \end{aligned} \quad (5.4)$$

where

$$b_\nu = \begin{cases} 0 & (\nu = +), \\ \frac{1}{2} & (\nu = -). \end{cases}$$

By comparing the transformation properties with respect to l for both sides on (5.4), we conclude that $F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}}$ are independent of l , and also that

$$F_m^{(i)}(\sigma^z; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} = 0 \quad \text{unless} \quad \frac{1}{2} \sum_{j=1}^{2m} \nu_j \equiv 0 \pmod{2},$$

as expected.

When $m = 1$, we have

$$\begin{aligned} F^{(i)}(\sigma^z; u_1, u_2)_{\nu_1 \nu_2} &= \delta_{\nu_1 + \nu_2, 0} C^{(z)} z_1^{\frac{r}{2(r-1)}} \prod_{j=1}^2 \frac{z_j^{-\frac{1}{r-1}}}{x^{-1}z(xz_j/z; x^4)_\infty (x^3z/z_j; x^4)_\infty} \\ &\times \frac{(x^{-1}z)^{\frac{2}{r}}}{4} x^{-\frac{r+2}{r}(u_0 - u) - \frac{1}{r} - \frac{1}{r-1}(u_0 - (u_1 + u_2)/2)^2 - (u - (u_1 + u_2)/2 - 1)^2} \\ &\times F_{\psi^* \psi^*}(z_2/z_1) \left(\nu_1 \frac{[u - \frac{u_1 + u_2}{2}]_1}{[\frac{u_2 - u_1 - 1}{2}]'} + (-1)^{1-i} \frac{\{u - \frac{u_1 + u_2}{2}\}_1}{\{\frac{u_2 - u_1 - 1}{2}\}'}, \right), \end{aligned} \quad (5.5)$$

where $C^{(z)}$ is a constant. This is a same result obtained by Lashkevich in [8], up to a scalar factor¹.

Next, let us choose σ^x as a local operator:

$$\sigma^z = E_{+-}^{(1)} + E_{-+}^{(1)}.$$

Then the relation for $F_m^{(i)}(\sigma^x; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}}$ reduces to (5.4) with $Z_m^{(i)}(l, u, u_0, u_j, v_a)$ replaced by

$$\begin{aligned} X_m^{(i)}(l, u, u_0, u_j, v_a) &= \left[\left[l - u_0 - \sum_{a=1}^{m-1} v_a + \frac{1}{2} \sum_{j=1}^{2m} u_j \right] \right]' \left\{ \left\{ \sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j \right\} \right\}_1 \\ &+ (-1)^{1-i} \left\{ \left\{ l - u_0 - \sum_{a=1}^{m-1} v_a + \frac{1}{2} \sum_{j=1}^{2m} u_j \right\} \right\}' \left[\left[\sum_{a=1}^{m-1} v_a + u - \frac{1}{2} \sum_{j=1}^{2m} u_j \right] \right]_1, \end{aligned}$$

and with $\{0\}$ in β_m replaced by $\llbracket 0 \rrbracket$, respectively. Here,

$$\begin{aligned} \llbracket v \rrbracket &:= x^{\frac{v^2}{r}} \Theta_{x^{2r}}(x^{2v+r}), & \llbracket v \rrbracket' &:= \llbracket v \rrbracket|_{r \mapsto r-1}, & \llbracket v \rrbracket_1 &:= \llbracket v \rrbracket|_{r \mapsto 1}, \\ \{\{v\}\} &:= x^{\frac{v^2}{r}} \Theta_{x^{2r}}(-x^{2v+r}), & \{\{v\}\}' &:= \{\{v\}\}|_{r \mapsto r-1}, & \{\{v\}\}_1 &:= \{\{v\}\}|_{r \mapsto 1}. \end{aligned}$$

The transformation properties with respect to l implies that $F_m^{(i)}(\sigma^x; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}}$ are independent of l , and also that

$$F_m^{(i)}(\sigma^x; u_1, \dots, u_{2m})_{\nu_1 \dots \nu_{2m}} = 0 \quad \text{unless} \quad \frac{1}{2} \sum_{j=1}^{2m} \nu_j \equiv 1 \pmod{2},$$

as expected. Furthermore, 2-point form factors for σ^x -operator can be obtained as follows:

$$\begin{aligned} F^{(i)}(\sigma^x; u_1, u_2)_{\nu_1 \nu_2} &= \delta_{\nu_1 \nu_2} C^{(x)} z_1^{\frac{r}{2(r-1)}} \prod_{j=1}^2 \frac{z_j^{-\frac{1}{r-1}}}{x^{-1} z(x z_j / z; x^4)_\infty (x^3 z / z_j; x^4)_\infty} \\ &\times \frac{(x^{-1} z)^{\frac{2}{r}}}{4} x^{-\frac{r+2}{r}(u_0-u) - \frac{1}{r} - \frac{1}{r-1}(u_0 - (u_1+u_2)/2)^2 - (u - (u_1+u_2)/2 - 1)^2} \\ &\times F_{\psi^* \psi^*}(z_2/z_1) \left(\nu_1 \frac{\{\{u - \frac{u_1+u_2}{2}\}\}_1}{\llbracket \frac{u_2-u_1-1}{2} \rrbracket'} + (-1)^{1-i} \frac{\llbracket u - \frac{u_1+u_2}{2} \rrbracket_1}{\{\{ \frac{u_2-u_1-1}{2} \}\}' } \right), \end{aligned} \quad (5.6)$$

where $C^{(x)}$ is a constant. The expressions (5.5) and (5.6) are essentially same as the results obtained by Lukyanov and Terras [20]².

6 Concluding remarks

In this paper we present a vertex operator approach for form factors of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model. For that purpose we constructed the free field representations of the tail operators $\Lambda_{\xi a}^{\xi' a'}$,

¹This scalar factor results from the difference between the present normalization of the type II vertex operators and that used in [8].

²Strictly speaking, we consider the parameterization of the coupling constants $|J_x| > J_x > J_y$ while Lukyanov and Terras [20] considered that of $J_x > J_y > |J_z|$. Thus, the present results (5.5) and (5.6) correspond to their results of the 2-point form factors for σ^x -operator and σ^y -operator, respectively. Furthermore, we note that their rapidity θ_j can be obtained from our spectral parameter u_j by a constant shift. After such substitution, we claim that our results (5.5) and (5.6) agree with their corresponding results in [20].

the nonlocal operators which relate the physical quantities of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the $A_{n-1}^{(1)}$ model. As a result, we can obtain the integral formulae for form factors of the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model, in principle.

Our approach is based on some assumptions. We assumed that the vertex operator algebra defined by (3.2) and (3.5), (3.6) correctly describes the intertwining relation between the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the $A_{n-1}^{(1)}$ model. We also assumed that the free field representations (4.1), (4.2) provide relevant representations of the vertex operator algebra.

As a consistency check of our bosonization scheme, we presented the integral formulae for form factors which are related to the σ^z -operator and σ^x -operator in the eight-vertex model, i.e., the $(\mathbb{Z}/2\mathbb{Z})$ -symmetric model. The expressions (5.3) and (5.4) for σ^z form factors and σ^x analogues remind us of the determinant structure of sine-Gordon form factors found by Smirnov [21]. In Smirnov's approach form factors in integrable models can be obtained by solving matrix Riemann-Hilbert problems. We wish to find form factors formulae in the eight-vertex model on the basis of Smirnov's approach in a separate paper.

A OPE formulae and commutation relations

In this paper we use some different definitions of the basic bosons from the one used in [2]. Accordingly, some formulae listed in Appendix B of [2] should be changed. Here we list such formulae. Concerning unchanged formulae see [2]. In what follows we denote $z = x^{2v}$, $z' = x^{2v'}$.

First, useful OPE formulae are:

$$\begin{aligned}
V_{\omega_1}(v)V_{\omega_j}(v') &= (-z)^{\frac{r}{r-1}\frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_1}(v)V_{\omega_j}(v') :, \\
V_{\omega_j}(v)V_{\omega_1}(v') &= (-z)^{\frac{r}{r-1}\frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_j}(v)V_{\omega_1}(v') :, \\
V_{\omega_j}(v)V_{-\alpha_j}(v') &= (-z)^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{\omega_j}(v)V_{-\alpha_j}(v') :, \\
V_{-\alpha_j}(v)V_{\omega_j}(v') &= (-z)^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{\omega_j}(v') :, \\
V_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') &= (-z)^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') :, \\
V_{-\alpha_j}(v)V_{-\alpha_j}(v') &= (-z)^{\frac{2r}{r-1}} \left(1 - \frac{z'}{z}\right) \frac{(x^{-2}z'/z; x^{2r-2})_{\infty}}{(x^{2r}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{-\alpha_j}(v') :, \\
V_{\omega_j}(v)U_{\omega_j}(v') &= (-z)^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : V_{\omega_1}(v)U_{\omega_j}(v') :, \\
U_{\omega_j}(v)V_{\omega_j}(v') &= z^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : U_{\omega_j}(v)V_{\omega_j}(v') :, \\
V_{\omega_j}(v)U_{-\alpha_j}(v') &= -z \left(1 - \frac{z'}{z}\right) : V_{\omega_j}(v)U_{-\alpha_j}(v') := U_{-\alpha_j}(v')V_{\omega_j}(v), \\
U_{\omega_j}(v)V_{-\alpha_j}(v') &= z \left(1 - \frac{z'}{z}\right) : U_{\omega_j}(v)V_{-\alpha_j}(v') := V_{-\alpha_j}(v')U_{\omega_j}(v), \\
V_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') &= -z \left(1 - \frac{z'}{z}\right) : V_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') := U_{-\alpha_{j\pm 1}}(v')V_{-\alpha_j}(v), \\
V_{-\alpha_j}(v)U_{-\alpha_j}(v') &= \frac{: V_{-\alpha_j}(v)U_{-\alpha_j}(v') :}{z^2 \left(1 - \frac{xz'}{z}\right) \left(1 - \frac{x^{-1}z'}{z}\right)}, \tag{A.1} \\
U_{-\alpha_j}(v)V_{-\alpha_j}(v') &= \frac{: U_{-\alpha_j}(v)V_{-\alpha_j}(v') :}{z^2 \left(1 - \frac{xz'}{z}\right) \left(1 - \frac{x^{-1}z'}{z}\right)}, \tag{A.2}
\end{aligned}$$

where $g_j^*(z)$ and $\rho_j(z)$ are defined by (2.2) and (2.3), respectively. From (A.1) and (A.2), we obtain the following commutation relations:

$$[V_{-\alpha_j}(v), U_{-\alpha_j}(v')] = \frac{\delta(\frac{z}{xz'}) - \delta(\frac{z'}{xz})}{(x - x^{-1})zz'} : V_{-\alpha_j}(v)U_{-\alpha_j}(v') :,$$

where the δ -function is defined by the following formal power series

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

Finally, we list the OPE formulae for $W_{-\alpha_j}(v)$ and other basic operators:

$$\begin{aligned} W_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') &= -(-z)^{-\frac{1}{r-1}} \frac{(x^r z'/z; x^{2r-2})_\infty}{(x^{r-2} z'/z; x^{2r-2})_\infty} : W_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') :, \\ V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') &= (-z)^{-\frac{1}{r-1}} \frac{(x^r z'/z; x^{2r-2})_\infty}{(x^{r-2} z'/z; x^{2r-2})_\infty} : V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') :, \\ V_{\omega_j}(v)W_{-\alpha_j}(v') &= (-z)^{-\frac{1}{r-1}} \frac{(x^r z'/z; x^{2r-2})_\infty}{(x^{r-2} z'/z; x^{2r-2})_\infty} : V_{\omega_j}(v)W_{-\alpha_j}(v') :, \\ W_{-\alpha_j}(v)V_{\omega_j}(v') &= -(-z)^{-\frac{1}{r-1}} \frac{(x^r z'/z; x^{2r-2})_\infty}{(x^{r-2} z'/z; x^{2r-2})_\infty} : W_{-\alpha_j}(v)V_{\omega_j}(v') :, \end{aligned}$$

From these, we obtain

$$\begin{aligned} W_{-\alpha_j}\left(v + \frac{r}{2}\right)V_{-\alpha_{j\pm 1}}(v) &= 0 = V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}\left(v - \frac{r}{2}\right), \\ W_{-\alpha_j}\left(v + \frac{r}{2}\right)V_{\omega_j}(v) &= 0 = V_{\omega_j}(v)W_{-\alpha_j}\left(v - \frac{r}{2}\right). \end{aligned}$$

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