CENTRALISER DIMENSION AND UNIVERSAL CLASSES OF GROUPS

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Abstract. In this paper we establish results that will be required for the study of the algebraic geometry of partially commutative groups. We define classes of groups axiomatized by sentences determined by a graph. Among the classes which arise this way we find CSA and CT groups. We study the centralisers of a group, with particular attention to the height of the lattice of centralisers, which we call the centraliser dimension of the group. The behaviour of centraliser dimension under several common group operations is described. Groups with centraliser dimension 2 are studied in detail. It is shown that CT-groups are precisely those with centraliser dimension 2 and trivial centre.

1. Introduction

The purpose of this paper is to lay foundations for the study of equations over groups and in particular over free partially commutative groups. We construct universal and existential sentences based on graphs and relate these to groups. The formula $\phi(\Gamma)$ which we introduce, given a graph Γ, and the properties developed below suggest the following general question.

Question 1.1. Let Γ_1 and Γ_2 be two finite connected graphs and suppose that the formula $\phi(\Gamma_1)$ is logically equivalent to the formula $\phi(\Gamma_2)$ for all groups from some class $\mathcal{K}$. What can be said about Γ_1 and Γ_2?

We also investigate properties of centralisers of groups. Among other things we show that the class of groups which has a centraliser lattice of finite height m is

universally axiomatisable and describe the behaviour of this class under various group operations. In subsequent papers we plan to apply the results of this work to find the centraliser dimension of free partially commutative groups and to investigate the problem of universal equivalence for this class of groups. Our interest in these problems is inspired by the importance of such results in algebraic geometry over groups (see [4], [26]): and in particular over free partially commutative groups.

We begin by considering classes of groups axiomatised by certain sentences in the first order language corresponding to graphs. In this way, we arrive at certain classes of groups, some new, among which are the well-known classes of CT- and CSA-groups.

In Section 3 we turn to the study of what Myasnikov and Shumyatsky [24] call the centraliser dimension of a group. This coincides with the notion of height of the centraliser lattice of a group, introduced by R. Schmidt [27]. The lattice of centralisers of various groups, have been investigated by numerous authors: see for example [17], [33], [30], [21], [28], [32], [19], [18], [7], [8], [2], [6], [31] and [24]. In particular, a detailed account of results in the field can be found in V. A. Antonov's book [3]. Here we show that the groups which have centraliser lattice of finite height are universally axiomatisable.

Next we investigate the behavior of the centraliser dimension under several group operations: namely free products, direct products and free products with amalgamation by their centres. We also study of groups with centraliser dimension 2. The groups with trivial centre which are of centraliser dimension 2 are shown to coincide with the class of CT-groups. Examples show that when the centre of the group is non-trivial the picture is far more complex.

2. Universal classes and some notions from model theory

2.1. **Preliminaries.** We recall here some basic notions of model theory that we require. For more details we refer the reader to [10]. The standard language of group theory, which we denote by L, consists of a symbol for multiplication '$\cdot$', a symbol for inversion $^{-1}$, and a constant symbol for the identity. We take $X = \{x_1, x_2, \ldots\}$ as the set of variables of our language and define $X^{-1} = \left\{x^{-1} | x \in X\right\}$ and $X^{\pm 1} = X \cup X^{-1}$. A *term* is an element of the free semigroup on $X^{\pm 1}$. An *atomic formula* is an expression of the form $w = 1$, where w is a term. A *formula* in L is either an atomic formula or one of $\theta \vee \phi$, $\theta \wedge \phi$, $\neg \phi$, $\forall x \phi$ or $\exists x \phi$, where θ and ϕ are formulas (and $\vee$, $\wedge$, $\neg$, $\forall$ and $\exists$ have their usual meanings). If θ is a formula and S is a subset of X then we write $\theta(S)$ to indicate that the variables which occur in θ are all elements of S. It follows from standard first order logic that any formula is logically equivalent to a formula of the type

$$(2.1) \qquad Q_1 y_1 Q_2 y_1 \cdots Q_m y_m \psi(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

where $Q_i \in \{\forall, \exists\}$ and $\psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a formula. We shall therefore assume formulas have this form. Those of the x_i's occuring are called *free* variables. If (2.1) has no free variables it is called a *sentence* in L.

Let G be a group. We assume that, for a sentence ϕ in L, the meaning of "ϕ holds in G" is understood. For example the sentence $\forall x \forall y([x, y] = 1)$ holds in G if and only if G is Abelian. If $\phi(x_1, \ldots, x_m)$ is an arbitrary formula in L then we denote by $\phi(g_1, \ldots, g_m)$ the element of G obtained by substiting g_i for x_i in ϕ, whenever x_i is a free variable of ϕ. Let g be the sequence $g = g_1, \ldots, g_m \in G^m$. We

write $G \models \phi(g)$ if $\phi(g)$ holds in G. For example, let $\phi(x_1, x_2, x_3)$ be the formula $\forall x_1 \exists x_2 (x_1 \cdot x_2 = x_3)$ and let $g = g_1, g_2, g_3 \in G^3$. Then $\phi(g)$ is $\forall x_1 \exists x_2 (x_1 \cdot x_2 = g_3)$ so $G \models \phi(g)$. The *truth domain* of ϕ over G is

$$\phi(G) = \{g \in G^m | G \models \phi(g)\}.$$

If $\phi(G) = G^m$ then we write $G \models \phi$ and say that ϕ is *satisfied* by G, ϕ is *valid* in G, ϕ *holds* in G or that G is a ϕ-*group*. Of course, since a sentence has no free variables, when ϕ is a sentence, this reduces to the notion of "holds in G" that we assumed above. Let $\mathcal{K}$ be a class of groups. Then we say that θ and ϕ are *logically equivalent* in $\mathcal{K}$ if $G \models \theta$ if and only if $G \models \phi$, for all groups G from $\mathcal{K}$. We say that $\mathcal{K}$ is *axiomatisable* by a set of sentences S if $\mathcal{K}$ consists of all groups G such that $G \models s$, for all $s \in S$.

If G is a group then the set $\mathsf{Th}_\forall(G)$ of all *universal sentences* (i.e. $Q_1 = \ldots = Q_m = \forall$ in Formula (2.1)) which are valid in G is called the *universal theory* of G. By the definition, two groups G and H are *universally equivalent* if $\mathsf{Th}_\forall(G) = \mathsf{Th}_\forall(H)$, in which case we write $G \equiv_\forall H$. The *universal closure* $\mathtt{ucl}(G)$ of a group G consists of all groups H such that $\mathsf{Th}_\forall(G) \subseteq \mathsf{Th}_\forall(H)$. A class of groups $\mathcal{K}$ is *universally axiomatisable* if it can be axiomatised by a set of universal sentences. The *existential theory* $\mathsf{Th}_\exists(G)$ of G is defined analagously, as are *existential equivalence* and *existential closure*. Notice that conditions $G \equiv_\forall H$ and $G \equiv_\exists H$ are equivalent.

Let G be a group and M be a set of elements of G. Then the set M together with induced partial group operation on it is called a *partial model* of G. On the set of partial models of G the notion of isomorphism of partial models arises naturally. The following Proposition follows from well-known facts of model theory.

Proposition 2.1. *Let G and H be groups. Then $G \equiv_\forall H$ if and only if every finite partial model of G is isomorphic to a finite partial model of H, and vice-versa.*

Corollary 2.2. *Free non-Abelian groups (of arbitrary rank) are universally equivalent.*

Definition 2.3. *Let G and H be groups. We say that G is* discriminated *by H if, for every finite subset $\{g_1, \ldots, g_m\}$ of non-trivial elements of G, there exists a homomorphism $\varphi : G \to H$ such that $\varphi(g_i) \neq 1$, for $i = 1, \ldots, m$. The set of all groups discriminated by H is denoted $\mathtt{Dis}(H)$.*

If every finitely generated subgroup of G is discriminated by H then we say that G is *locally discriminated* by H. The set of all groups locally discriminated by H is denoted $\mathtt{LDis}(H)$. The next Proposition follows immediately from Proposition 2.1 and the fact that if G is locally discriminated by H then, for every finite subset $\{g_1, \ldots, g_m\}$ of non-trivial elements of G, we may choose $\varphi : G \to H$ such that $\varphi(g_i) \neq \varphi(g_j)$, when $i \neq j$.

Proposition 2.4. *Suppose that G is locally discriminated by H and that H is locally discriminated by G. Then $G \equiv_\forall H$.*

Corollary 2.5. *Arbitrary non-trivial torsion-free Abelian groups are universally equivalent.*

Доказательство. An infinite cyclic group is discriminated by any torsion-free Abelian group. Hence, by Proposition 2.4, it suffices to prove that every Abelian group of finite rank is discriminated by an infinite cyclic group, and this is easily verified. $\square$

2.2. **Logical Formulas and Universal Classes.** We next describe some important logical group formulas in the language L, involving commutation of group elements. Our conventions are that $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$.

Let $\gamma(x_1, \ldots, x_k, y)$ denote the formula

$$\bigwedge_{i=1}^{k} [x_i, y] = 1.$$

A $(k+1)$-tuple $(g_1, \ldots, g_k, g) \in G^{k+1}$ is in the truth domain $\gamma(G)$ of γ if and only if g is contained in the centraliser $C_G(g_1, \ldots, g_k)$ of $\{g_1, \ldots, g_k\}$ in G (see Section 3.1). Thus it is natural to use $y \in C(x_1, \ldots, x_k)$ to denote $\gamma(x_1, \ldots, x_k, y)$. Similarly we use $y \notin C(x_1, \ldots, x_k)$ for the negation of γ. Similarly, by $x \in Z$ we denote the formula $\forall y \, [y, x] = 1$, since the truth domain of this formula over a group G is the centre $Z(G)$ of G.

The *commutativity axiom* is the sentence $\forall x, y \, [x, y] = 1$: valid in the group G if and only if G is Abelian. The *commutative transitivity or CT axiom* is the sentence $\text{CT}(x, y, z)$ given by

$$\forall x, y, z \; (x \neq 1 \wedge x \in C(y, z) \to [y, z] = 1).$$

The CT axiom is logically equivalent (in the class of all groups) to the sentence

$$\forall x, y, z (x = 1 \vee y = 1 \vee z = 1 \vee x = y \vee x = z \vee y = z$$
$$\vee [x, y] \neq 1 \vee [x, z] \neq 1 \vee [y, z] = 1).$$

Thus G is CT-group if and only if the centraliser of every nontrivial element of G is an Abelian subgroup; which must therefore be a maximal Abelian subgroup.

The CSA *axiom* is the sentence $\text{CSA}(x, y, t)$ given by

$$\forall x, y, t(x \neq 1 \wedge x \in C(y) \wedge x^t \in C(y) \to t \in C(y)).$$

A subgroup M of a group G is *conjugacy-separable* or *malnormal* if $M \cap M^g = 1$, for all $g \in G \backslash M$. Thus a group is a CSA-group if and only if all centralisers of single elements are conjugacy-separable.

Clearly a CSA-group is a CT-group. However the converse does not hold. The free product of two cyclic groups of order two is an example of a group which is a CT-group but not a CSA-group. In fact, if the factors are generated by a and b then the centraliser of ab does not contain b but is fixed under conjugation by b and hence is not conjugacy-separable. Therefore, the class of CT-groups is wider than the class of CSA-groups.

A CSA-group also satisfies the following which we call the *unilateral-separability* or US-axiom. The US-axiom is the sentence $\text{US}(x, y)$ given by

$$\forall x, y \; (x^y \in C(x) \to y \in C(x)).$$

Again the class of US-groups is wider than the class of CSA-groups. For example if F is a free group of rank 2 and C is infinite cyclic then it is easy to see that $F \times C$ is a US-group. However $F \times C$ is not a CT-group so is not a CSA-group.

Now suppose that G is both a CT-group and a US-group. Let x, y and z be elements of G with $x \neq 1$ and both $x \in C(y)$ and $x^z \in C(y)$. Then, as G is a CT-group, $x^z \in C(x)$ and, as G is a US-group, $z \in C(x)$. Using the CT-axiom again $z \in C(y)$, so G is a CSA-group. Hence the CSA-axiom is logically equivalent, in the class of all groups, to the sentence

$$\forall x, y, z \; (\text{CT}(x, y, z) \wedge \text{US}(x, y)).$$

It is also not hard to show that a group is CSA if and only if all maximal Abelian subgroups are conjugacy-separable. For more details of CSA groups see [25].

2.3. **Paths and Cycles.** We describe a number of existential sentences which encapsulate the relations of commutativity of a finite set of elements. These sentences for commutativity relations are indexed by certain graphs and so we shall begin by defining formulas $\theta(\Gamma)$ and $\phi(\Gamma)$ corresponding to an arbitrary graph Γ. Let Γ be a graph with vertices $V(\Gamma)$ and edges $E(\Gamma)$. For notational simplicity we assume that $V(\Gamma)$ is a subset of X and that $V(\Gamma) = \{x_1, \ldots, x_n\}$. The sentence $\phi(\Gamma)$ is defined to be $\exists x_1 \ldots \exists x_n \theta(\Gamma)$, where $\theta(\Gamma)$ is the conjunction of the following formulas.

    (1) $x \neq 1$, for all $x \in V(\Gamma)$;
    (2) $x \neq y$, if x and y are disjoint vertices of Γ;
    (3) $[x, y] = 1$, whenever there is an edge in Γ connecting x and y and
    (4) $[x, y] \neq 1$ if there is no such edge.

Thus $G \models \phi(\Gamma)$ if and only if $G \models \theta(\Gamma)(g_1, \ldots, g_n)$, for some $g_1, \ldots, g_n \in G^n$. In this case we call the sequence $g_1, \ldots, g_n$ an *implementation* of Γ in G and say that G *admits* the graph Γ. Let $\Phi(\Gamma)$ be the class of all groups in which the sentence $\phi(\Gamma)$ is satisfied and let $\Phi(\neg\Gamma)$ be the complement of the class $\Phi(\Gamma)$. Clearly, since $\neg\phi(\Gamma)$ is a universal formula, the class $\Phi(\neg\Gamma)$ is a universal class.

The *path graph* $\texttt{Path}_l$ of length l is a tree with $l+1$ vertices precisely two of which have degree one. Our first family of sentences for commutativity relations is indexed by the path graphs of positive length.

The *length-one-path axiom* is the sentence $\phi(\texttt{Path}_1)$, that is

$$\exists x_1, x_2 \ (x_1 \neq 1 \wedge x_2 \neq 1 \wedge x_1 \neq x_2 \wedge [x_1, x_2] = 1).$$

The negation of this sentence $\neg\phi(\texttt{Path}_1)$ is

$$\forall x_1, x_2 \ (x_1 = 1 \vee x_2 = 1 \vee x_1 = x_2 \vee [x_1, x_2] \neq 1).$$

Clearly this sentence is satisfied by groups of order at most 2. If G is a group of order more than 2 then either G has an element g of order 3 or more, or all non-trivial elements of G have order 2. In the former case $\neg\phi(\texttt{Path}_1)$ does not hold in G since we may take $x_1 = g$ and $x_2 = g^2$. In the latter case, since the order of G is more than 2, it has non-trivial elements a and b with $a \neq b$ and $[a, b] = 1$, so $\neg\phi(\texttt{Path}_1)$ does not hold. Therefore $\Phi(\neg\texttt{Path}_1)$ consists of the trivial group and the cyclic group of order 2. It follows that $\neg\phi(\texttt{Path}_1)$ is logically equivalent, in the class of all groups, to the universal sentence

$$\forall x_1, x_2 \ (x_1 = 1 \vee x_2 = 1 \vee x_1 = x_2).$$

The *length-two-path axiom* is the sentence $\phi(\texttt{Path}_2)$ given by

$$\exists x_1, x_2, x_3 \ (\bigwedge_{i=1}^{3} x_i \neq 1 \bigwedge_{\substack{i,j=1 \\ i \neq j}}^{3} (x_i \neq x_j) \wedge$$

$$\wedge [x_1, x_2] = 1 \wedge [x_2, x_3] = 1 \wedge [x_1, x_3] \neq 1).$$

(Assuming that x_2 is the vertex of $\texttt{Path}_2$ of degree 2.) Negation of $\phi(\texttt{Path}_2)$ is a universal sentence, $\neg\phi(\texttt{Path}_2)$, logically equivalent, in the class of all groups, to the CT axiom. Therefore $\Phi(\neg\texttt{Path}_2)$ is is the class of CT-groups.

Similarly, the *length-l-path axiom*, for $l \geq 3$, is defined to be $\phi(\texttt{Path}_l)$. Now $\Phi(\texttt{Path}_l)$, the class of groups which satisfy $\phi(\texttt{Path}_l)$, clearly satisfies $\Phi(\texttt{Path}_l) \geq$

$\Phi(\texttt{Path}_{l+1})$, for all $l \geq 1$. It is easy to see that if F is a free group of rank 2 and C is infinite cyclic then $F \times C$ satisfies $\phi(\texttt{Path}_2)$ but does not satisfy $\phi(\texttt{Path}_3)$. Moreover Blatherwick [5] has shown that in fact $\Phi(\texttt{Path}_l) > \Phi(\texttt{Path}_{l+1})$, for $l \geq 1$. Thus we have the following chain of inclusions.

$$\Phi(\texttt{Path}_1) > \Phi(\texttt{Path}_2) > \Phi(\texttt{Path}_3) > \Phi(\texttt{Path}_4) > \Phi(\texttt{Path}_5) > \cdots .$$

We next consider a family of existential sentences indexed by cycle graphs. The *cycle graph*, or *l-cycle*, $\texttt{Cyc}_l$ is the connected graph with l vertices which is regular of degree 2. The *three-cycle axiom* is the sentence $\phi(\texttt{Cyc}_3)$

$$\exists x_1, x_2, x_3 \ (\bigwedge_{i=1}^{3} x_i \neq 1 \bigwedge_{\substack{i,j=1 \\ i \neq j}}^{3} (x_i \neq x_j) \wedge$$

$$\wedge \ [x_1, x_2] = 1 \wedge [x_2, x_3] = 1 \wedge [x_1, x_3] = 1).$$

Proposition 2.6. *The class $\Phi(\mathcal{Cyc}_3)$ consists of all groups except those of order less than 4 and the dihedral group D_6.*

Доказательство. Clearly $\phi(\texttt{Cyc}_3)$ does not hold in any group of order less than 4 or in D_6. Suppose that G is a group of order at least 4 and that $G \not\cong D_6$. We shall find a sequence $\mathbf{v} = a, b, c$ of elements of G which satisfy $\theta(\texttt{Cyc}_3)(x_1, x_2, x_3)$. If there are non-trivial elements a and b of G such that $a \neq b^{\pm 1}$ and $[a, b] = 1$ then we may take $\mathbf{v} = a, b, ab$. Hence we may assume that G contains no such elements. If G has an element a of order 4 or more then we can take $\mathbf{v} = a, a^2, a^3$. Assume then that all elements of G are of order at most 3. If G has no element of order 3 then G has distinct non-trivial elements a and b with $a \neq b^{\pm 1}$ and $[a, b] = 1$, a contradiction. Thus we may assume G has elements of order 3. Suppose that G has no elements of order 2. Then we may choose non-trivial elements a and b of G with $a \neq b^{\pm 1}$ and $[a, b] \neq 1$. Since a, b, ab, ab^2 and $a^2 b^2$ have order 3, it follows that $[a, a^b] = [a^2, a^b] = 1$. It is then easy to check that we may take $\mathbf{v} = a, a^2, a^b$. This leaves the case where G has elements of order 2 and 3. Let a and b be elements of G with $|a| = 2$, $|b| = 3$ and $[a, b] \neq 1$. Suppose first that $|ab| = 3$. Then $aba = b^2 ab^2$ and so $[a, a^b] = ab^2 abab^2 ab = (ab)^3 = 1$. Similarly $[a, a^{b^2}] = 1$ and a straightforward check shows that $\mathbf{v} = a, a^b, a^{b^2}$ satisfies $\theta(\texttt{Cyc}_3)$. Now suppose that $|ab| = 2$ (with $|a| = 2$, $|b| = 3$ and $[a, b] \neq 1$ as before). Then $\langle a, b \rangle \cong D_6$. As $G \not\cong D_6$ it must contain an element $c \notin \langle a, b \rangle$. Then, given our intial assumptions, we have $[a, c] \neq 1$ and $[b, c] \neq 1$. If $|c| = 3$ and $\langle a, c \rangle \not\cong D_6$ then we can find $\mathbf{v}$ satisfying $\theta(\texttt{Cyc}_3)$, as above. If $\langle a, c \rangle \cong D_6$ then $|ac| = 2$. As $ac \notin \langle a, b \rangle$ we may replace c with ac. Thus, without loss of generality, we may assume that $|c| = 2$ and that $\langle a, b \rangle \cong \langle c, b \rangle \cong D_6$. If $|ac| = 2$ then we have $a \neq c^{\pm 1}$ and $[a, c] = 1$, a contradiction. Hence $|ac| = 3$. Then $(acb)^3 = acbacbacb = ab^2 cacab^2 cb = b(ac)^3 b^2 = 1$. Similarly $(acb^2)^3 = ((ac)^2 b^2)^3 = 1$ and, as in the case where G has no element of order 2, we may now take $\mathbf{v} = ac, (ac)^2, (ac)^b$. $\square$

The *four-cycle axiom* is the sentence $\phi(\texttt{Cyc}_4)$ given by

$$\exists x_1, x_2, x_3, x_4 \ (\bigwedge_{i=1}^{4} x_i \neq 1 \bigwedge_{\substack{i,j=1 \\ i \neq j}}^{4} (x_i \neq x_j) \wedge [x_1, x_2] = 1 \wedge$$

$$[x_1, x_4] = 1 \wedge [x_2, x_3] = 1 \wedge [x_3, x_4] = 1 \wedge [x_1, x_3] \neq 1 \wedge [x_2, x_4] \neq 1).$$

If we identify x_1, x_2, x_3, x_4 with an implementation of $\mathtt{Cyc}_4$ in G then in the drawing of $\mathtt{Cyc}_4$ below letters connected by an edge commute and letters that are not connected by an edge do not.

PSfrag replacements

x_1

x_2

x_3

$x_{4\text{b}}$

a

d

c
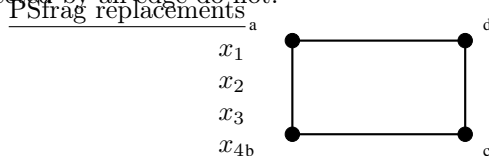
The negation of $\phi(\mathtt{Cyc}_4)$ is a universal sentence which is satisfied by G if and only if G admits no 4-cycle. It is clear that if G is a CT-group then it admits no 4-cycle. However the converse does not hold: for example D_8, the dihedral group of order 8, admits no 4-cycle and is not a CT-group. This means that $D_8 * D_8$ admits no 4-cycle, is not a CT-group and in addition has trivial centre.

Similarly the 5-*cycle axiom* is the sentence $\phi(\mathtt{Cyc}_l)$, for $l \geq 5$. Blatherwick [5] has shown that, although $\Phi(\mathtt{Cyc}_3) > \Phi(\mathtt{Cyc}_4)$, for $n \geq 5$ and $m = n \pm 1$, $\Phi(\mathtt{Cyc}_n) \not\geq \Phi(\mathtt{Cyc}_m)$.

3. Centraliser Dimension

3.1. **Definitions and Preliminaries.** The centraliser lattices of groups have been studied in numerous papers; some listed in the introduction. Here we shall consider groups which have centraliser lattice of finite height; on which there is also a considerable literature. We classify such groups according to centraliser dimension which we define in this section.

If S is a subset of a group G then the centraliser of S in G is $C_G(S) = \{g \in G : gs = sg, \text{ for all } s \in S\}$. We write $C(S)$ instead of $C_G(S)$ when the meaning is clear. The following properties of centralisers are well-known; see for example [21] or [27]. Given a family of subsets $\{S_i\}_{i \in I}$ of G indexed by a set I,

  (i) $\cap_{i \in I} C(S_i) = C(\cup_{i \in I} S_i)$;
 (ii) $\cup_{i \in I} C(S_i) \subseteq C(\cap_{i \in I} S_i)$.

Moreover (*loc. cit.*) given subsets S and T of G,

 (iii) if $S \subseteq T$ then $C(S) \supseteq C(T)$;
 (iv) $S \subseteq C(C(S))$;
  (v) $C(S) = C(C(C(S)))$;
 (vi) $C(S) \subseteq C(T)$ if and only if $C(C(S)) \supseteq C(C(T))$.

Let $\mathfrak{C}(G)$ denote the set of centralisers of a group G. The relation of inclusion then defines a partial order '$\leq$' on $\mathfrak{C}(G)$. We define the infimum of a pair of elements of $\mathfrak{C}(G)$ as obvious way:

$$C(M_1) \wedge C(M_2) = C(M_1) \cap C(M_2) = C(M_1 \cup M_2).$$

Moreover the supremum $C(M_1) \vee C(M_2)$ of elements $C(M_1)$ and $C(M_2)$ of $\mathfrak{C}(G)$ may be defined to be the intersection of all centralisers containing $C(M_1)$ and $C(M_2)$. Then $C(M_1) \vee C(M_2)$ is minimal among centralisers containing $C(M_1)$ and $C(M_2)$. These definitions make $\mathfrak{C}(G)$ into a lattice, called the *centraliser lattice* of G. This lattice is bounded as it has a greatest element, $G = C(1)$, and a least element, $Z(G)$, the centre of G. From (i) above, every subset of $\mathfrak{C}(G)$ has an infimum, so $\mathfrak{C}(G)$ is a complete lattice.

If C and C' are in $\mathfrak{C}(G)$ with C strictly contained in C' we write $C < C'$. If C_i is a centraliser, for $i = 0, \ldots, m$, with $C_0 > \cdots > C_m$ then we call $C_0, \ldots, C_m$

a *centraliser chain* of *length m*. Infinite descending, ascending and doubly-infinite centraliser chains are defined in the obvious way. A group G is said to have the *minimal condition on centralisers* min-c if every descending chain of centralisers is eventually stationary; that is if $\mathfrak{C}(G)$ satisfies the descending chain condition. The *maximal condition on centralisers* max-c is satisfied by G if the ascending chain condition holds in $\mathfrak{C}(G)$. From (vi) above a group has min-c if and only if it has max-c. Groups with the minimal condition on centralisers have been widely studied; see for instance [33], [27], [7], [8], [18]. As in many of the articles cited we consider now the restriction to groups in which there is a global bound on the length of centraliser chains.

Definition 3.1. *If there exists an integer d such that the group G has a centraliser chain of length d and no centraliser chain of length greater than d then G is said to have* centraliser dimension $\mathit{cdim}(G) = d$. *If no such integer d exists we define* $\mathit{cdim}(G) = \infty$.

If $\mathtt{cdim}(G) = d$ then every strictly descending chain of centralisers in $\mathfrak{C}(G)$ from G to $Z(G)$ contains at most d inclusions. This number is usually referred to as the *height* of the lattice; so $\mathtt{cdim}(G)$ is the height of the centraliser lattice of G.

Using Definition 3.1 we introduce the following classes of groups. For every positive integer $m \geq 0$ set

$$\mathtt{CD}_m = \{G | \mathtt{cdim}(G) \leq m\}.$$

In addition we shall sometimes wish to consider the set of all groups with finite centraliser dimension so we set

$$\mathtt{CD} = \bigcup_{i=1}^{\infty} \mathtt{CD}_i = \{G | \mathtt{cdim}(G) < \infty\}.$$

Any group from $\mathtt{CD}$ satisfies the minimal condition on centralisers. The converse is not true: Lennox and Roseblade [21, Theorem H] and Bryant [5] give examples of groups which are nilpotent of class 2 and have min-c but are not in $\mathtt{CD}$. The class of groups $\mathtt{CD}$ is nonetheless very broad as the following example shows.

Example 3.2. 1. Finitely generated Abelian-by-nilpotent groups are in $\mathtt{CD}$ as are polycyclic-by-finite groups [21].
2. A linear group of degree n has centraliser dimension at most $n^2 - 1$ [32]. Moreover, if R is a finite direct product of fields then the general linear group $GL(m, R)$ is in $\mathtt{CD}$ [24].
3. If G is a non-Abelian, hyperbolic, torsion-free group then, as shown in [25], G is a $\mathtt{CSA}$-group so, from Proposition 3.9.1, $\mathtt{cdim}(G) = 2$.
4. We are grateful to A. Yu. Ol'shanskii for the following argument showing that all hyperbolic groups (including those with torsion) have finite centraliser dimension. Suppose G is a hyperbolic group. Then there is a bound on the orders of finite subgroups of G (see for example [14, Chapter 4]). Thus it suffices to show that there is a bound on the length of strictly descending chains of infinite centralisers of G. Suppose that $C = C_G(X)$ where X is a subset of G generating a non–elementary subgroup K of G. (A group is *elementary* if it has a cyclic subgroup of finite index. Also, the elementariser $E_G(H)$ of a subgroup H of G is the set of all $x \in G$ such that x^H is finite: see [15] for details.) Then, from [15, Proposition 1], the elementariser $E_G(K)$ of K is finite, so $C = C_G(K) \subseteq E_G(K)$ is finite.

Therefore infinite centralisers in G are centralisers of elementary subgroups of G. Since there is a bound on the order of finite subgroups of G the set of lengths of chains of centralisers of finite elementary subgroups is bounded. As $E \subseteq E'$ implies $C(E) \supseteq C(E')$ it therefore suffices to bound the set of lengths of chains of infinite centralisers of infinite elementary subgroups. Suppose now that $C_G(E)$ is the centraliser of the elementary subgroup E and that a is an element of infinite order in E. Then $C_G(a)$ is an (infinite) elementary subgroup ([14, p. 156]) and $C_G(a) \supseteq C_G(E)$, so $C_G(E)$ is elementary. Let

$$(3.1) \qquad\qquad C_0 > \cdots > C_d$$

be a strictly descending chain of infinite centralisers $C_i = C_G(E_i)$, where E_i is an infinite elementary subgroup. We may assume that $E_0 < \cdots < E_d$. There is an element of infinite order in $C_0 \cap E_d$ so the group E generated by C_0 and E_d is elementary ([15, p. 375]). Thus (3.1) is a chain of centralisers of the elementary subgroup E. From [15, Lemma 19] it follows that there is an integer M such that every infinite elementary subgroup of G has an infinite cyclic subgroup of index at most M. From this and Proposition 3.8 below it follows that G is in CD, as claimed.

5. We are grateful to S. V. Ivanov for the following argument concerning free Burnside groups. The free Burnside groups of large exponent have centraliser dimension 2, when n is odd, but do not have min-c when n is even. In more detail, let $G = B(m, n)$ be the m-generator free Burnside group of exponent n. If $m > 1$ and $n \geq 665$ then centralisers of non-trivial elements of G are cyclic of order n [1]. It follows that in this case $\mathtt{cdim}G = 2$. On the other hand suppose that $n \geq 2^{48}$ and that $2^9 | n$. Then we may choose a finite 2-subgroup T_1 of G. From [16, Theorem 1 (a)], $C_G(T_1)$ contains a subgroup B isomorphic to $B(2, n)$ such that $C_G(T_1) \cap B = \{1\}$. We may now take a finite 2-subgroup D of B and set $T_2 = \langle T_1, D \rangle = T_1 \times D$. Then T_2 is a finite 2-subgroup of G. Repeating the process starting with T_2 instead of T_1 and continuing this way we see that G contains an infinite ascending chain

$$T_1 < T_2 < \cdots$$

of finite 2-subgroups. From [16, Theorem 1 (c)], for all i, $C_G(C_G(T_i) = T_i$, so

$$C_G(T_1) > C_G(T_2) > \cdots$$

is an infinite descending chain of centralisers. Therefore G does not have min-c.

6. We pose the following question. Is the centraliser dimension of a biautomatic group finite? This is related to (and stronger than) several well-known questions concerning these groups. Gersten and Short [13, Proposition 4.3] show that, in a biautomatic group, centralisers of finite subsets are biautomatic. They ask (loc. cit.) if biautomatic groups have min-c and show that if so then every Abelian subgroup of a biautomatic group is finitely generated. Moreover Mosher [23] shows that a biautomatic group has an infinitely generated Abelian subgroup if and only if it has an Abelian subgroup which is either of infinite rank or is an infinite torsion group. However, whether or not such subgroups are to be found in biautomatic groups is an open question. Another related open question asks whether or not a biautomatic group can have an element of infinite order which has infinite index in its centraliser. Mosher (loc. cit.) shows that if such

a biautomatic group exists it must contain a subgroup isomorphic to $\mathbb{Z}^2$, so the group cannot be hyperbolic.
7. Blatherwick [5] has examples showing that, for each integer $m \geq 4$ (and for $m = 2$) there exists a nilpotent group of class 2 with centraliser dimension m (see also [21]).
8. We shall show in Section 3.2 that the class $\texttt{CD}$ is closed under formation of direct sums and free products (with finitely many factors) and certain amalgamated products. Moreover if a group G has a subgroup of finite index belonging to $\texttt{CD}$ then G is in $\texttt{CD}$ (Proposition 3.8).

The first four statements of the following proposition are well-known, but we give proofs for completeness. Statements 3 and 4 follow from Lemma 3.1 and Theorem 3.2 of [27], which show that a group has distributive centraliser lattice if and only if the group is Abelian, in which case the lattice is trivial; and that no group can have centraliser lattice of height one.

Proposition 3.3. 1. *If G has min-c and C is a centraliser in G then there exists a finite subset M such that $C = C(M)$* [8].
2. *If $\texttt{cdim}(G) = m$ and*

$$G = C_0 > \cdots > C_m = Z(G)$$

is a centraliser chain of maximal length in G then C_{m-1} is Abelian [27].
3. *Let G be an Abelian group, then $\texttt{cdim}(G) = 0$* [27].
4. *If G is non-Abelian then $\texttt{cdim}(G) \geq 2$: that is $\texttt{CD}_0 = \texttt{CD}_1$* [27].
5. *In the event that $\texttt{cdim}(G) = m$ is finite, there exists an m-tuple of non-central elements $a_1, \ldots, a_m$ such that*

$$G > C(a_1) > \ldots > C(a_1, \ldots, a_m) = Z(G).$$

Доказательство. If $C = C(S)$ is not the centraliser of any finite subset then we may construct an infinite centraliser chain by choosing succesive elements $s_1, s_2, \ldots$ of S and forming centralisers $C(s_1, \ldots, s_k)$, for increasing k. This proves 1. To see 2 suppose that C_{m-1} is non-Abelian. Take a pair a, b of non-commuting elements from C_{m-1} and consider the centraliser $C = C_{m-1} \cap C(a)$. Notice that $a \in C$ but, since $[a, b] \neq 1$, $b \notin C$. Hence $C_{m-1} > C$ and we have a centraliser chain

$$G = C_0 > \cdots > C_{m-1} > C > C_m = Z(G)$$

of length greater than m. As $\texttt{cdim}(G) = m$ this is a contradiction and 2 holds.

Statement 3 is clear. For 4 observe that if G is non-Abelian then $G \neq Z(G)$ and so we may choose $a \in G \backslash Z(G)$. Then

$$G > C(a) > Z(G),$$

so $\texttt{cdim}(G) \geq 2$.

To prove statement 5 notice that there is nothing to prove if $m < 2$. Assume that $\texttt{cdim}(G) = m \geq 2$. Then there exist finite subsets $M_1, \ldots, M_m$ of G such that

(3.2) $$G > C(M_1) > C(M_2) > \ldots > C(M_m) = Z(G)$$

is a (strictly descending) centraliser chain. Take $a_1 \in M_1$ such that $C(a_1) \neq G$. Then $C(a_1) = C(M_1)$ by maximality of (3.2). Assume that elements $a_1, \ldots, a_{i-1}$ have been chosen so that $C(M_j) = C(a_1, \ldots, a_j)$, for $j = 1, \ldots, i-1$. As $C(M_i) < C(M_{i-1})$ we may choose $a_i \in M_i$ such that $C(a_i) \ngeq C(M_{i-1})$. Since $C(a_i) > C(M_i)$ and (3.2) is maximal it follows that $C(M_i) = C(a_1, \ldots, a_i)$. Hence, by induction,

we may choose such a_i for $i = 1, \ldots, m$. None of the a_i belong to $Z(G)$, since (3.2) is a strictly descending chain, hence 5 holds. $\qquad\square$

From now on we shall only consider groups with finite centraliser dimension. Next we show that for every positive integer $m \geq 0$ the class of groups $\mathtt{CD}_m$ is universally axiomatisable. We shall make use of the notation of Section 2.2 for formulas in the language L. Since $\mathtt{CD}_0 = \mathtt{CD}_1$ and $\mathtt{CD}_0$ is the class of all Abelian groups, these classes are defined by the following universal sentence,

$$\mathtt{CD}_0 = \mathtt{CD}_1 : \forall x, y([x, y] = 1),$$

which, in the notation of Section 2.2, takes the form

$$\forall x, y(x \in C(y)).$$

We next write down an axiom for $m = 2$.

$$\mathtt{CD}_2 : \forall x_0, x_1, x_2, y_1, y_2, z \ (y_1 \in C(x_0) \wedge y_1 \notin C(x_0, x_1) \wedge$$
$$\wedge \ y_2 \in C(x_0, x_1) \wedge y_2 \notin C(x_0, x_1, x_2) \wedge z \in C(x_0, x_1, x_2) \rightarrow z \in Z(G)).$$

From Proposition 3.3.5 it follows that $\mathtt{CD}_2$ is the class of groups axiomatised by this universal sentence. For $m > 2$ we have the following axiom.

$$\mathtt{CD}_m : \forall x_0, \ldots, x_m, y_1, \ldots, y_m, z \ (y_1 \in C(x_0) \wedge y_1 \notin C(x_0, x_1) \wedge$$
$$\wedge y_2 \in C(x_0, x_1) \wedge y_2 \notin C(x_0, x_1, x_2) \wedge \ldots \wedge$$
$$\wedge y_m \in C(x_0, \ldots, x_{m-1}) \wedge y_m \notin C(x_0, \ldots, x_m)$$
$$\wedge z \in C(x_0, \ldots, x_m) \rightarrow z \in Z(G)).$$

Proposition 3.4. *For $m \geq 0$, the class of groups $\mathtt{CD}_m$, defined in Section 3.1, is axiomatised by the universal sentence $\mathtt{CD}_m$ above. That is, a group G satisfies axiom $\mathtt{CD}_m$ if and only if $\mathtt{cdim}(G) \leq m$.*

Доказательство. Obviously, if $\mathtt{cdim}(G) \leq m$ then $\mathtt{CD}_m$ holds in G. Conversely, suppose that $\mathtt{CD}_m$ holds in G and $\mathtt{cdim}(G) = n > m$. Then, by Proposition 3.3.5 there exist non-central elements $a_1, \ldots, a_n$ such that

$$G > C(a_1) > C(a_1, a_2) > \ldots > C(a_1, \ldots, a_n) = Z(G).$$

Therefore there are elements $y_1, \ldots, y_{n-1} \in G$ such that $y_1 \in G$, $y_1 \notin C(a_1)$ and $y_j \in C(a_1, \ldots, a_{j-1})$, $y_j \notin C(a_1, \ldots, a_j)$, for $j = 2, \ldots, n$. In this case though $\mathtt{CD}_m$ does not hold for the $(2m+1)$-tuple $1, a_1, \ldots, a_m, y_1, \ldots y_m$ of elements of G. $\quad\square$

3.2. **The Behavior of Centraliser Dimension Under Group Operations.** In the proof of Proposition 3.6 we shall make use of the following well-known theorem.

Theorem 3.5 (Theorem 4.5 of [22]). *Let G be an amalgamated product of H_1 and H_2 with amalgamation by K, i.e. $G = H_1 *_K H_2$. Assume that the elements q and r of G commute. Then one of the following holds.*

1. *Either q or r is conjugate to an element from the subgroup K.*
2. *Both q and r lie in H_i^g, for some $g \in G$ and some fixed $i = 1$ or 2.*
3. *If the previous two conditions fail then*

$$q = gcg^{-1}u^l \quad and \quad r = gc'g^{-1}u^{l'}$$

where $c, c' \in K$, $g, u \in G$, $l, l' \in \mathbb{Z}$ and the elements gcg^{-1}, $gc'g^{-1}$ and u pairwise commute.

In order to state our next proposition we need a further definition. Let $Z(G)$ denote the centre of the group G. We define

$$\mathbf{z}(G) = \left\{ \begin{array}{ll} 0, & \text{if } Z(G) = 1_G \\ 2, & \text{otherwise} \end{array} \right. .$$

Statement 2 of the following proposition is a consequence of the fact, proved in [28], that $\mathfrak{C}(G_1 \times G_2) = \mathfrak{C}(G_1) \times \mathfrak{C}(G_2)$, for groups G_1 and G_2. We give a proof for completeness.

Proposition 3.6. *Let $G_1, G_2 \in$ CD.*

1. *If $G_1 \leq G_2$ then $cdim(G_1) \leq cdim(G_2)$.*
2. *$cdim(G_1 \times G_2) = cdim(G_1) + cdim(G_2)$ (see [28]).*
3. *$cdim(G_1 * G_2) = \max \left\{ cdim(G_1) + z(G_1), cdim(G_2) + z(G_2) \right\}.$*
4. *Let G_1 and G_2 be non-Abelian groups with $Z(G_i) = Z_i$, $i = 1, 2$, such that $Z_1 \simeq Z_2 \neq 1$, and let $G = G_1 *_{Z_1=Z_2} G_2$. Then*

$$cdim(G) = \max \left\{ cdim(G_1), cdim(G_2) \right\}.$$

Доказательство. 1. This follows from the fact that every centraliser $C_{G_1}(M)$ of a set M in G_1 is a subset of the centraliser $C_{G_2}(M)$ in G_2. Moreover if $C_{G_1}(M) < C_{G_1}(N)$ then $C_{G_2}(M) < C_{G_2}(N)$.

2. Let $G = G_1 \times G_2$. If $M \subseteq G$ and we let M_1 and M_2 be the projections of M onto G_1 and G_2, respectively, then $C_G(M) = C_{G_1}(M_1) \times C_{G_2}(M_2)$. In particular, if $N_i \subseteq G_i$, for $i = 1, 2$, then $C_G(N_1 \times N_2) = C_{G_1}(N_1) \times C_{G_2}(N_2)$.

We first show that $\mathtt{cdim}(G) \geq \mathtt{cdim}(G_1) + \mathtt{cdim}(G_2)$. Let $\mathtt{cdim}(G_1) = m$ and $\mathtt{cdim}(G_2) = n$. In this case there exist centraliser chains

$$G_1 > C_1 > \cdots > C_m = Z(G_1)$$

and

$$G_2 > D_1 > \cdots > D_n = Z(G_2).$$

Then

$$G > C_1 \times G_2 > \cdots > C_m \times G_2 > C_m \times D_1 > \cdots > C_m \times D_n$$

is a strictly descending chain of centralisers in G. Therefore $\mathtt{cdim}(G) \geq \mathtt{cdim}(G_1) + \mathtt{cdim}(G_2)$.

Next we show that $\mathtt{cdim}(G) = \mathtt{cdim}(G_1) + \mathtt{cdim}(G_2)$. Suppose we have a centraliser chain

$$(3.3) \qquad\qquad\qquad C_0 > \ldots > C_k$$

in G. Then $C_i = C_{G_1}(M_i) \times C_{G_2}(N_i)$, where $M_i \subseteq G_1$ and $N_i \subseteq G_2$, for $i = 0, \ldots, k$. Since (3.3) is strictly descending we have $C_{G_1}(M_i) \geq C_{G_1}(M_{i+1})$ and $C_{G_2}(N_i) \geq C_{G_2}(N_{i+1})$, with at least one of these inclusions strict, for $i = 0, \ldots, k - 1$. Since $\mathtt{cdim}(G_1) = m$ there are at most $m + 1$ distinct centralisers among the $C_{G_1}(M_i)$. Hence there are at most m of these inclusions with $C_{G_1}(M_i) \neq C_{G_1}(M_{i+1})$. Similarly there are at most n inclusions with $C_{G_2}(N_i) \neq C_{G_2}(N_{i+1})$. Hence the number k of inclusions in (3.3) is at most $m + n$, and it follows that $\mathtt{cdim}(G) \leq m + n$.

3. Let $G = G_1 * G_2$ and let f and g be two non-trivial elements of G such that $C(f) \neq C(g)$. Then either $C(f) \cap C(g) = 1$ or both of these elements and their centralisers lie in G_i^h, for some fixed $h \in G$ (see Theorem 3.5). This implies that if

$$(3.4) \qquad\qquad\qquad G > C_1 > \cdots > C_p > 1$$

is a strictly descending chain of centralisers with $p \geq 2$ then there are fixed i and h such that $C_j \leq G_i^h$, for all j. After conjugation by h^{-1} we may then assume that $C_j \leq G_i$, for all j. (If $p = 1$ then we may replace C_1 with the centraliser of an element of G_1 or G_2, if necessary; so we may assume that the claim holds in this case as well.)

First suppose that G_i has trivial centre. Then, for $a \in G_i$, $C_G(a) \geq G_i$ only if $a = 1$ in which case $C_G(a) = G$. Hence C_1 is strictly contained in G_i. Also, $C_p \neq 1$ implies $C_p \neq Z(G_i)$. Therefore

$$G_i > C_1 > \cdots > C_p > 1$$

is a centraliser chain of length $p + 1$ in G_i and we have $p + 1 \leq \mathtt{cdim}(G_i) = \mathtt{cdim}(G_i) + \mathtt{z}(G_i)$.

On the other hand, if $Z(G_i) \neq 1$ then $G_i = C_G(g)$, for all $1 \neq g \in Z(G_i)$. In this case it is possible that $C_1 = G_i^h$ and $C_p = Z^h(G_i)$. Thus, if $Z(G_i) \neq 1$ then $p + 1 \leq \mathtt{cdim}(G_i) + 2 = \mathtt{cdim}(G_i) + \mathtt{z}(G_i)$. It follows that $\mathtt{cdim}(G) \leq \max\{\mathtt{cdim}(G_1) + \mathtt{z}(G_1), \mathtt{cdim}(G_2) + \mathtt{z}(G_2)\}$.

Conversely suppose that

$$G_i > C_1 > \cdots > C_p = Z(G_i)$$

is a centraliser chain in G_i. If $Z(G_i) = 1$ then, replacing G_i with G in this chain we obtain a centraliser chain for G of length p. If $Z(G_i) \neq 1$ then adding G to the left and 1 to the right of this chain we obtain a centraliser chain for G of length $p + 2$. Hence $\max\{\mathtt{cdim}(G_1) + \mathtt{z}(G_1), \mathtt{cdim}(G_2) + \mathtt{z}(G_2)\} \leq \mathtt{cdim}(G)$.

4. We have $Z(G) = Z(G_1) = Z(G_2)$ and Theorem 3.5 takes the following form. If x, y are elements from G such that $xy = yx$ then

(i) x or $y \in Z(G)$; or
(ii) there is $g \in G$ such that $x \in G_i^g \smallsetminus Z(G)$ and $y \in G_i^g \backslash Z(G)$; or
(iii) (i) and (ii) do not hold, and there exists an element z, such that $x = c_1 z^k$ and $y = c_1 z^l$, with $c_1, c_2 \in Z(G)$, $k, l \in \mathbb{Z}$.

Now let f, g be two elements from $G \smallsetminus Z(G)$ such that $C_G(f) \neq C_G(g)$. Then either $C_G(f) \cap C_G(g) = Z(G)$ or f and g both lie in the same subgroup of the type G_i^h. Since neither of G_1 or G_2 is a centraliser in G the result follows as in 3. $\square$

As shown in [27], if $G = HK$ where $H \cap K \neq \emptyset$ then it is not necessarily the case that $\mathbb{C}(G) \cong \mathbb{C}(G) \times \mathbb{C}(G)$ even if H and G centralise one another. Also the relationship between $\mathbb{C}(G)$ and $\mathbb{C}(G/Z(G))$ is complicated: [8] contains an example of a group G such that G has min-c but $G/Z(G)$ does not. Moreover Example 3.11.3 below shows that centraliser dimension may increase on factoring by the centre. On the positive side it is shown in [27] that if $G = HZ$, where $Z \leq Z(G)$, then $\mathbb{C}(G) \cong \mathbb{C}(H)$; so $\mathtt{cdim}(G) = \mathtt{cdim}(H)$. We also have the following proposition.

Proposition 3.7. *Let G be a group such that every nilpotent subgroup of G is Abelian. Then $\mathit{cdim}(G) = \mathit{cdim}(G/Z(G))$.*

Доказательство. Let $\bar{G}$ denote $G/Z(G)$ and $\bar{g}$ the image of g in $\bar{G}$. If $g \in G$ then $C(\bar{g}) = \{\bar{f} \mid [g, f] \in Z(G)\}$. Let $g, f \in G$ such that $\bar{f} \in C(\bar{g})$. Let $H = \langle f, g \rangle$. Then $[[f, g], h] = 1$, for all $h \in H$, so H is nilpotent. Hence H is Abelian, so $f \in C(g)$. Therefore $C(\bar{g}) = C(g)/Z(G)$ and it follows, via a straightforward induction, that $C(\bar{X}) = C(X)/Z(G)$, for all finite sets $X \subseteq G$. Therefore $\mathtt{cdim}(\bar{G}) = \mathtt{cdim}(G)$. $\square$

The class of groups satisfying min-c is closed under the formation of finite extensions [18]. The same is true of the class CD, however in this case slightly more can be said.

Proposition 3.8. *Let H be a subgroup of finite index k in a group G. If $\mathtt{cdim}(H) = d < \infty$ then $\mathtt{cdim}(G) \leq ((d+2)k+2)k$.*

Доказательство. Let

$$G = C_0 > C_1 > \cdots > C_n = Z(G)$$

be a centraliser chain in G of length n and let elements $t_1, \ldots, t_k$ of G form a transversal for H in G, with $t_1 = 1$. Then

$$C_j = \bigcup_{i=1}^{k} (C_j \cap Ht_i), \text{ for } j = 0, \ldots, n.$$

Note that if $C_j \cap Ht_i = \emptyset$ then $C_l \cap Ht_i = \emptyset$, for all $l \geq j$.

For $i = 1, \ldots, k$ define $d(i) = n$, if $C_n \cap Ht_i \neq \emptyset$, and otherwise $d(i) = j$, where j is the unique integer such that $C_j \cap Ht_i \neq \emptyset$ and $C_{j+1} \cap Ht_i = \emptyset$. Since $t_1 = 1$ and $H \cap Z(G) \neq \emptyset$ we have $d(1) = n$. We may therefore reorder the t_i's so that

$$n = d(1) \geq d(2) \geq \cdots \geq d(k) \geq 0.$$

Now fix j with $0 \leq j < n$. If $j = d(s)$, for some s, then

$$C_j = \bigcup_{i=1}^{s} (C_j \cap Ht_i) \text{ and } C_{j+1} = \bigcup_{i=1}^{r} (C_{j+1} \cap Ht_i),$$

where $r < s$ is maximal such that $d(r) \geq j + 1$. On the other hand, if $j \neq d(s)$, for all s, then there exists s such that $d(s) > j > d(s+1)$. Then, since $C_j \cap Ht_{s+1} = \emptyset$ by definition of $d(s+1)$,

$$C_j = \bigcup_{i=1}^{s} (C_j \cap Ht_i) \text{ and } C_{j+1} = \bigcup_{i=1}^{s} (C_{j+1} \cap Ht_i).$$

Moreover, by definition of $d(s)$, $C_{j+1} \cap Ht_s \neq \emptyset$. As $C_j > C_{j+1}$ it follows that, for some i with $1 \leq i \leq s$,

$$(3.5) \qquad\qquad C_j \cap Ht_i > C_{j+1} \cap Ht_i.$$

For $j = 0, \ldots, n$ define $e(j) = 0$, if $j = d(i)$, for some i, and otherwise set $e(j) = i$, where i is chosen to satisfy (3.5). Then $|e^{-1}(0)| \leq k$ and $\sum_{i=0}^{k} |e^{-1}(i)| = n$. If l is an integer such that $n \geq (l+1)k + 1$ this implies that there is some s, with $1 \leq s \leq k$, such that $|e^{-1}(s)| \geq l$. Assume then that $n \geq (l+1)k + 1$, for some positive integer l and fix such an s. Then there are integers $j_1 \leq \cdots \leq j_l$ such that $e(j_r) = s$, for $r = 1, \ldots, l$. From (3.5) it follows that

$$(3.6) \qquad\qquad C_{j_1} \cap Ht_s > \cdots > C_{j_l} \cap Ht_s.$$

Now $C_{j_l} \cap Ht_s \neq \emptyset$, so there exists an element $y \in C_{j_l} \cap Ht_s$. As $y \in C_{j_r}$ and $Ht_s = Hy$ we have

$$C_{j_r} \cap Ht_s = C_{j_r} \cap Hy = (C_{j_r} \cap H)y,$$

for $r = 1, \ldots, l$. Hence (3.6) implies that

$$(3.7) \qquad\qquad C_{j_1} \cap H > \cdots > C_{j_l} \cap H.$$

For $r = 1, \ldots, l$ we have $C_{j_r} = C_G(X_r)$, where $X_r \subseteq G$. We may assume that $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_l$ and set $Y_r = X_{r-1} \backslash X_r$, for $r = 2, \ldots l$. Next we shall argue that we may assume that Y_r has only one element, for all r. To see this suppose that r is minimal such that $|Y_r| > 1$. There exists $h \in C_G(X_{r-1}) \cap H$ such that $h \notin C_G(X_r) \cap H$. As $X_r = X_{r-1} \cup Y_r$ this means that there is $y \in Y_r$ such that $h \notin C_G(y)$. Hence $C_G(X_{r-1}) \cap H > C_G(X_{r-1}) \cap C_G(y) \cap H \geq C_G(X_r) \cap H$. Thus we may replace Y_r with $\{y\}$. Continuing this way each Y_r may be replaced by a singleton. We may now assume that there are elements $x_1, \ldots, x_l$ of G such that $X_r = \{x_1, \ldots, x_r\}$, for $r = 1, \ldots, l$.

Define $c(r) = i$ to be the unique integer such that $x_r \in Ht_i$. Let m be an integer such that $l \geq mk + 1$. Then for some i, with $1 \leq i \leq k$, we have $|c^{-1}(i)| \geq m$. Fix s such that $|c^{-1}(s)| \geq m$ and let $c^{-1}(s) \supseteq \{r_1, \ldots, r_m\}$, where $r_1 < \cdots < r_m$. Set $x_{r_i} = y_i$ and $Y_i = \{y_1, \ldots y_i\}$, for $i = 1, \ldots, m$. Then it follows that

$$(3.8) \qquad\qquad C_G(Y_1) > \cdots > C_G(Y_m)$$

is a strictly descending chain of centralisers in G. Since $c(r_i) = s$ we have $y_i = a_i t_s$, where $a_i \in H$, for all i. A straightforward calculation shows that, for any elements $a, b, c \in G$, the identity $C_G(ab) \cap C_G(cb) = C_G(ab) \cap C_G(ac^{-1})$ holds. By induction it follows that

$$C_G(Y_i) = C_G(a_1 t_s) \cap C_G(a_1 a_2^{-1}, \ldots, a_1 a_i^{-1}),$$

for $i = 2, \ldots, m$. Setting $A_i = \{a_1 a_2^{-1}, \ldots, a_1 a_i^{-1}\} \subseteq H$, it follows from (3.8) that

$$G > C_G(A_2) > \cdots > C_G(A_m)$$

is a strictly descending chain of centralisers in G.

Now define $A_1 = \{1\}$ and $D_i = C_G(A_i)$, for $i = 1, \ldots, m$. Then $D_i \cap H = C_H(A_i)$, so

$$H = H \cap D_1 \geq \cdots \geq H \cap D_m$$

is a strictly descending chain of centralisers of length $m - 1$ in H. This occurs if $n \geq (l+1)k+1$ and $l \geq mk+1$; that is $n \geq (mk+2)k+1$. Thus if $n \geq ((d+2)k+2)k+1$ we obtain a contradiction, and the result follows. $\qquad\square$

3.3. **Groups of Centraliser Dimension** 2. In this section we concentrate attention on the class $\mathtt{CD}_2$ of groups that have centraliser dimension at most 2. There are many examples of such groups: free groups, torsion-free hyperbolic groups and free Burnside groups, of large odd exponent, have centraliser dimension 2. R. Schmidt [27] has completely classified finite groups of centraliser dimension 2 (which are $\mathfrak{M}$-groups in the terminology of [27]). Locally finite groups in $\mathtt{cdim}(G) = 2$ have also been fairly intensively studied (see Chapter 2 of [3], and [9]). Here we show that there is a connection between groups with centraliser dimension 2 and $\mathtt{CT}$-groups and give some examples.

Proposition 3.9. *Let G be a non-Abelian group.*

1. *If $\mathtt{cdim}(G) = 2$ and $Z(G) = 1$ then G is a $\mathtt{CT}$-group. Conversely, if G is a $\mathtt{CT}$-group then $\mathtt{cdim}(G) = 2$ and $Z(G) = 1$.*
2. *Suppose that every nilpotent subgroup of G is Abelian. Then $\mathtt{cdim}(G) = 2$ if and only if the factor-group $G/Z(G)$ is a $\mathtt{CT}$-group.*

Доказательство. To see 1 suppose there exists a non-Abelian $\mathtt{CT}$-group G such that $\mathtt{cdim}(G) \geq 3$. Then, from Proposition 3.3.5, there exists a chain of centralisers

$$G > C(a_1) > C(a_1, a_2) > Z(G).$$

Since the second inclusion above is strict it follows that $C(a_1) \neq C(a_2)$, and since G is a CT-group this implies $[a_1, a_2] \neq 1$. As the third inclusion is strict there is an non-trivial element $b \in C(a_1) \cap C(a_2)$. The assumption that $b \neq 1$ together with the CT axiom now imply that $[a_1, a_2] = 1$, a contradiction.

To prove the converse suppose that $\mathtt{cdim}(G) = 2$ and that G is non-Abelian group with trivial centre. Since G is non-Abelian the centraliser of a non-trivial element is a proper non-trivial subgroup. As $\mathtt{cdim}(G) = 2$ such centralisers are all Abelian subgroups, by Proposition 3.3.2. Now if b_1 and b_2 belong to the centraliser of a non-trivial element $a \in G$ then b_1 and b_2 commute, so the CT axiom is satisfied.

In the setting of 2 note that, from Proposition 3.7, $\mathtt{cdim}(G) = \mathtt{cdim}(G/Z(G))$ and also that $G/Z(G)$ is non-Abelian, for otherwise G is nilpotent and thus, by hypothesis Abelian. Suppose first that $\mathtt{cdim}(G) = 2$. Since G has no non-Abelian nilpotent subgroups the argument of the proof of Proposition 3.7 shows that the centre of $G/Z(G)$ is trivial. Therefore 1 implies that $G/Z(G)$ is a CT-group. On the other hand, if $G/Z(G)$ is a CT-group then it follows, from 1, that $\mathtt{cdim}(G) = 2$. $\square$

Remark 3.10. In the event that $\mathtt{cdim}(G) = 2$ and $Z(G) \neq 1$ the situation is much more complex (see Example 3.11.3 below).

Example 3.11. 1. Let G be a non-Abelian CT-group and A be an Abelian group then, by Propositions 3.9 and 3.6, $\mathtt{cdim}(G \times A) = 2$.

2. For every positive integer $c \geq 3$ there exists a nilpotent group G of class c such that $\mathtt{cdim}(G) = 2$. Here is an example of such group. Let $A = \mathbb{Z}^c$ be a lattice of the rank $c, c \geq 2$. Take an automorphism of ϕ of A given by the following matrix of order c in the natural base $e_1, \ldots, e_c$:

$$[\phi] = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Let α be the homomorphism from the infinite cyclic group $\langle t \rangle$ to $\mathrm{Aut}(A)$ given by $\alpha(t) = \phi$ and let G be a semi-direct product $G = \langle t \rangle \ltimes_\alpha A$. Then $Z(G) = \langle e_c \rangle$ and the upper central series of G is

$$1 \leq \langle e_c \rangle \leq \langle e_{c-1}, e_c \rangle \leq \cdots \leq \langle e_2, \cdots, e_c \rangle \leq G,$$

so G is a nilpotent group of class c. We claim that centralisers in G are either G, A, $Z(G)$ or of the form $\langle t^i a, e_c \rangle$, for some $i \in \mathbb{Z}$ and $a \in A$. To see this first note if $g \in A$ and $g \notin \langle e_c \rangle$ then the centraliser of g is A. Moreover if $g \in \langle t \rangle$ then the centraliser of g is $\langle t, e_c \rangle$. Hence it remains to calculate the centraliser of $t^r a$, where $0 \neq r \in \mathbb{Z}$ and $a \in A, a \notin \langle e_c \rangle$. Note that G is torsion-free so extraction of roots in G is unique. (For $h \in G$ and $n \in \mathbb{N}$, the equation $x^n = h$ has at most one solution. See for example [20].) Hence, for $x, c \in G$, if $c^{-1} x^s c = x^s$ then $(c^{-1} x c)^s = x^s$ so $c^{-1} x c = x$ and we conclude that $C_G(x^r) = C_G(x)$. In addition, as G is finitely generated and nilpotent it follows that torsion-free Abelian subgroups of rank 1 are infinite cyclic. Now, if $x, y \in G$ such that x is not a proper power and $x^n = y^m$, for some $m, n \in \mathbb{Z}$, then x and y belong to a torsion-free Abelian subgroup of rank 1: namely the isolator of x^n, see [20]. Since this subgroup must be cyclic it follows that $y \in \langle x \rangle$. Hence for all $y \in G$ there exists unique $x \in G$ with the property that $x^n = y$ and whenever $y = z^m$ then $z \in \langle x \rangle$: we call x a

root of y and say x is a root element of G. Since $c \geq 2$ and $G/\langle e_c \rangle$ is isomorphic to the semi-direct product of the lattice $\mathbb{Z}^{c-1}$ and the infinite cyclic group in the same way as G, the same properties hold in $G/\langle e_c \rangle$. If $g \in G$ then we may choose $h \in G$ such that $h\langle e_c \rangle$ is the root of $g\langle e_c \rangle$ in $G/\langle e_c \rangle$. Then $g = h^n z$, for some $z \in \langle e_c \rangle$, so $C_G(g) = C_G(h^n) = C_G(h)$. Thus we may assume that $t^r a$ is such that $t^r \langle e_c \rangle$ is a root element of G. Clearly $C_G(t^r a) \supseteq \langle t^r a, e_c \rangle$. Suppose that, for some $s \in \mathbb{Z}$ and $b \in A$, $t^s b \in C_G(t^r a)$. We have $t^r a t^s b = t^{r+s}(a\phi^s)b$ and $t^s b t^r a = t^{r+s}(b\phi^r)a$ so it must be that $(a\phi^s)b = (b\phi^r)a$. Writing $a = \sum_{i=1}^c \alpha_i e_i$ and $b = \sum_{i=1}^c \beta_i e_i$ (using additive notation for A) we have

$$(a\phi^s)b = (\alpha_1 + \beta_1)e_1$$
$$+ \left(\alpha_2 + \beta_2 + \binom{r}{1}\alpha_1 \right) e_2 + \cdots$$
$$\cdots + \left(a_c + \beta_c + \binom{r}{1}\alpha_{c-1} + \cdots + \binom{r}{c-1}\alpha_1 \right) e_c,$$

(where we take $\binom{s}{k} = 0$ if $k > s$) and a similar expression for $(b\phi^r)a$. Comparing coefficients of e_i's in these two expressions we see that for fixed r, s and a the elements $\beta_1, \ldots, \beta_{c-1}$ are uniquely determined. Hence, for each $s \in \mathbb{Z}$ there is at most one coset $t^s b\langle e_c \rangle$ which is contained in $C_G(t^r a)$. Now let $q = \mathrm{lcm}(r, s)$, so there are integers u and v such that $q = ur = vs$. Then $(t^r a)^u = t^q c \in C_G(t^r a)$ and $(t^s b)^v = t^q d \in C_G(t^r a)$. From the above $(t^r a)^u \langle e_c \rangle = (t^s b)^v \langle e_c \rangle$ and, since $t^r a\langle e_c \rangle$ is a root element in $G/\langle e_c \rangle$, this means that $t^s b \in \langle t^r a, e_c \rangle$. Therefore $C_G(t^r a) = \langle t^r a, e_c \rangle$. The intersection of two such subgroups is $\langle e_c \rangle$ unless both subgroups are the same. It now follows that $\mathtt{cdim}(G) = 2$, as claimed.

3. Let G be the group constructed in the previous example, with $c \geq 3$, and let $H = G *_{Z(G)} G$. Then, by Proposition 3.6, $\mathtt{cdim}(G) = 2$. Since $Z(H) = Z(G)$ we have $H/Z(H) \cong G/Z(G) * G/Z(G)$. Now $Z(G/Z(G)) \cong \mathbb{Z}$ and so it follows from Proposition 3.6.3 that $\mathtt{cdim}(H/Z(H)) = 4$. This shows that, on taking the quotient of a group by its centre the centraliser dimension may increase. This example is directly comparable to an example of R. Bryant [8] in which the original group G has min-c but the factor group $G/Z(G)$ does not.

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