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A NOTE ON CODES AND KETS

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Abstract. To every binary linear $[n,k]$ code C we associate a quantum state $|\Psi_C\rangle \in H^{\otimes n}$, where H is the two-dimensional complex Hilbert space associated to the spin $\frac{1}{2}$ particle. For the state $|\Psi_C\rangle$ we completely characterize all the expectation values of the products of spins measured, for each one out of the n particles, either in the x- or in the y-direction. This establishes an interesting relationship with the dual code $C^\perp$.

1. INTRODUCTION

The Pauli matrices (see [4], Chapter 4) are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In what follows we will consider special systems of spin $\frac{1}{2}$ fermions. For a single such fermion, the Hilbert space H associated to the spin has an orthonormal basis consisting of eigenvectors of σ_z:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In order to establish a relationship with linear codes, let us agree, for convenience, to use the following binary notation:

$$|0\rangle \text{ for } |+\rangle \quad \text{and} \quad |1\rangle \text{ for } |-\rangle.$$

Then the following matrix relations hold true:

$$\sigma_z|0\rangle = |0\rangle \text{ and } \sigma_z|1\rangle = -|1\rangle, \tag{1}$$

$$\sigma_x|0\rangle = |1\rangle \text{ and } \sigma_x|1\rangle = |0\rangle, \tag{2}$$

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$$\sigma_y|0\rangle = i|1\rangle \text{ and } \sigma_y|1\rangle = -i|0\rangle. \tag{3}$$

Let C be a linear binary $[n, k]$ code. That is C is a k-dimensional subspace of the n-dimensional vector space $\{0,1\}^n$ over the field with 2 elements. To C we associate the following quantum state (element of $H^{\otimes n}$) describing a quantum system of n spin $\frac{1}{2}$ particles:

$$|\Psi_C\rangle := \frac{1}{2^{k/2}} \sum_{(x_1,\ldots,x_n)\in C} |x_1\rangle \otimes |x_2\rangle \otimes \ldots \otimes |x_n\rangle. \tag{4}$$

Note that in the special case of the $[n, 1]$ code

$$C = \{(0,0,\ldots,0),(1,1,\ldots,1)\},$$

$|\Psi_C\rangle$ becomes the well known Green-Horne-Zeilinger (GHZ) quantum entangled state [1], [2]

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|00\ldots0\rangle + |11\ldots1\rangle\right).$$

In what follows we will imagine that for a system of n fermions in the state $|\Psi_C\rangle$ we measure either the x-component or the y-component of the spin for each of the n particles.

To acquire a uniform binary notation we will agree to rename the operator σ_x as σ_0 and σ_y as σ_1. Also, for a bit $x \in \{0,1\}$ we will denote by $\overline{x}$ the "flipped"bit $1+x \pmod 2$. A particular measurement of x- or y- spins for the particles in (4) can be represented as a product operator of the form

$$\sigma_A = \sigma_{a_1} \otimes \ldots \otimes \sigma_{a_n},$$

where $A := (a_1, \ldots, a_n) \in \{0,1\}^n$ indicates the measurement choice we make for every particle. For example, if $n = 3$ and $A = (1,0,1)$ we measure the x-component of the spin for the second particle and the y-components of the spin for the first and the third particles. Thus, every such binary vector A codifies a selection of x- and y- spins to be measured.

It is not difficult to see that a unified binary representation of the relations (2) and (3) is given by the following:

$$\sigma_a|b\rangle = e^{i\left(\frac{\pi a}{2}+\pi ab\right)}|\overline{b}\rangle, \tag{5}$$

where a, b are bits.

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2. CODE DUALITY AND EXPECTATION VALUES FOR PRODUCTS OF SPINS

We will now determine the expectation value, in the quantum state $|\Psi_C\rangle$, of the product $\Pi(C, A)$ of spins codified as shown above by a certain binary vector A. For the mathematical formalism of quantum mechanics see [4], Chapter 2. In other words, we will determine

$$\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle$$

By using (5), we will get:

$$\sigma_A|\Psi_C\rangle = \frac{1}{2^{k/2}} \sum_{X=(x_1,\ldots,x_n)\in C} |\sigma_{a_1}x_1\rangle \otimes \ldots \otimes |\sigma_{a_n}x_n\rangle =$$

$$= \frac{1}{2^{k/2}} \sum_{X \in C} e^{\frac{i\pi}{2}(a_1+\ldots+a_n)+i\pi(a_1x_1+\ldots+a_nx_n)} |\overline{x_1}\rangle \otimes \ldots \otimes |\overline{x_n}\rangle.$$

That is,

$$\sigma_A|\Psi_C\rangle = \frac{e^{\frac{i\pi}{2}wt(A)}}{2^{k/2}} \sum_{X \in C} e^{i\pi(A|X)}|\overline{X}\rangle, \tag{6}$$

where $wt(A)$ is the Hamming weight of the binary vector A,

$$(A|X) = a_1x_1 + \ldots + a_nx_n \pmod 2$$

is the binary dot product, and

$$|\overline{X}\rangle = |\overline{x_1}\rangle \otimes \ldots \otimes |\overline{x_n}\rangle.$$

The following technical lemma will be essential in the derivation of our main result:

Lemma 1. *If A is a binary vector of length n and if C is a binary $[n, k]$ code, then*

$$\Pi(C, A) = \frac{e^{\frac{i\pi}{2}wt(A)}}{2^k} \sum_{X,Y \in C} e^{i\pi(A|X)}\langle Y|\overline{X}\rangle. \tag{7}$$

Proof. Indeed, (7) is a straightforward consequence of (4) together with (6) and the fact that $\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle$.

Two observations are in order at this point. First, note that the kets $|X\rangle$ form an orthonormal basis in the 2^n-dimensional Hilbert space $H^{\otimes n}$. Second, note that $\overline{X}$ equals $\mathbf{1} + X$ where $\mathbf{1} = (1, 1, \ldots, 1) \in \{0, 1\}^n$. This, together with the above lemma shows that if $\mathbf{1} \notin C$, then

$$\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle = 0.$$

Indeed, if $\mathbf{1} \notin C$, since C is a linear code, $\overline{C} = \{\mathbf{1} + X | X \in C\}$ is a coset of C disjoint from C so that all the products $\langle Y|\overline{X}\rangle$ in (7) will be zero.

Consider now the case $\mathbf{1} \in C$. That is, *all codewords in the dual code $C^\perp$ have even weight* (for information on linear codes and duality, see [3], Chapter 3). In this case, $\Pi(C, A)$ expressed in (7) will become

$$\Pi(C, A) = \frac{e^{\frac{i\pi}{2}wt(A)}}{2^k} \sum_{X \in C} e^{i\pi(A|\overline{X})} = \frac{e^{\frac{i\pi}{2}wt(A)}}{2^k} \sum_{X \in C} e^{i\pi(A|\mathbf{1}+X)},$$

or, keeping in mind that $(A|\mathbf{1}) = wt(A)$,

$$\Pi(C, A) = \frac{e^{\frac{i\pi}{2}wt(A)}}{2^k} e^{i\pi wt(A)} \sum_{X \in C} e^{i\pi(A|X)}. \tag{8}$$

Note that the sum appearing in (8) is zero if $A \notin C^\perp$ and 2^k if $A \in C^\perp$. Also, note that if $\mathbf{1} \in C$, then any binary vector $A \in C^\perp$ satisfies $e^{i\pi wt(A)} = 1$ since A has even weight. These observations, together with (8) show that if $\mathbf{1} \in C$ and if $A \notin C^\perp$ then

$$\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle = 0,$$

and that if $\mathbf{1} \in C$ and if $A \in C^\perp$ then

$$\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle = e^{\frac{i\pi}{2}wt(A)}.$$

Note that if $\mathbf{1} \in C$ and if $A \in C^{\perp}$, then A has even weight and therefore $e^{\frac{i\pi}{2} wt(A)} = \pm 1$, that is, $e^{\frac{i\pi}{2} wt(A)}$ is 1 if $wt(A) \equiv 0 \pmod 4$ and -1 if $wt(A) \equiv 2 \pmod 4$. This entanglement effect appears precisely when $|\Psi_C\rangle$ has nonzero components along both "GHZ kets"$|00...0\rangle$ and $|11...1\rangle$.

The next theorem brings together all the results we got in this section, completely characterizing the values of $\Pi(C, A) = \langle\Psi_C|\sigma_A|\Psi_C\rangle$.

Theorem 2. *If A is a binary vector of length n and if C is a binary $[n, k]$ code, then the following hold true:*

$\Pi(C, A) = 0$ if $\mathbf{1} \notin C$,
$\Pi(C, A) = 0$ if $\mathbf{1} \in C$ *and* $A \notin C^{\perp}$,
$\Pi(C, A) = 1$ if $\mathbf{1} \in C$, $A \in C^{\perp}$ and $wt(A) \equiv 0 \pmod 4$,
$\Pi(C, A) = -1$ if $\mathbf{1} \in C$, $A \in C^{\perp}$ and $wt(A) \equiv 2 \pmod 4$.

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