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EMBEDDING ARBITRARY GRAPHS INTO STRONGLY REGULAR AND DISTANCE REGULAR GRAPHS

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Abstract. We show that every simple graph on x vertices is an induced subgraph of some strongly regular graph on fewer than $4x^2$ vertices; which, up to a constant factor, coincides with the existing lower bound. We also show that every simple graph on x vertices is an induced subgraph of some distance regular graph of diameter 3 on fewer than $8x^3$ vertices, and every simple bipartite graph on x vertices is an induced subgraph of some distance regular bipartite graph of diameter 3 on fewer than $8x^2$ vertices.

Let $s(x)$ be the smallest positive integer such that each simple graph on x vertices is an induced subgraph of some strongly regular graph on at most $s(x)$ vertices. The function $s(x)$ was studied recently in [5] and [6]. It was proved in [6] that $s(x) \geq c \cdot x^2$ for some constant $c > 0$. More precisely, it was proved there that the graph $K_{n+1} \cup K_{n,n}$, the disjoint union of a complete graph and a complete bipartite graph, cannot be embedded into a strongly regular graph with fewer than n^2 vertices. On the other hand, a construction from [5] proves that $s(x) = O(x^4)$. In this note we close this gap by showing that $s(x) = O(x^2)$.

In [6], an open problem is posed: is it true that every graph can be embedded into a distance regular graph of diameter $d \geq 3$? We give here an affirmative answer to this question.

It is also mentioned there that every bipartite graph is an induced subgraph of a bipartite distance regular graph, but no bound for the size is given. We construct such embeddings with a quadratic upper bound.

Fon-Der-Flaass D.G., Embedding arbitrary graphs into strongly regular and distance regular graphs.

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We shall use Constructions 1 and 4 of [3], and a new construction of bipartite distance regular graphs found by G. Exoo [2] (the graphs of [3, Construction 1] were previously found by Wallis [7]). To make the note self-contained, we describe here, without proofs, all three constructions.

All necessary definitions and facts about strongly regular graphs can be found in [1]. By *the order* of a graph we mean the number of its vertices; by *the order* of an affine plane, the number of points on a line (in particular, an affine plane of order n has n^2 points).

Definition 1. *An* **affine design** *is a 2-design with the following two properties:*

(i) *every two blocks are either disjoint or intersect in a constant number r of points;*

(ii) *each block together with all blocks disjoint from it form a parallel class: a set of n mutually disjoint blocks partitioning all points of the design.*

Only three classes of affine designs are known:

(1) all lines of an affine plane of order n ($r = 1$);

(2) all affine hyperplanes of a d-dimensional vector space over the field $GF(n)$ ($r = n^{d-2}$);

(3) Hadamard 3-designs ($n = 2$). A Hadamard 3-design can be defined via a Hadamard matrix of size $4r$ with the constant first row. Each of the remaining rows defines a partition of the set of columns into two $2r$-sets corresponding to entries 1 and -1; the design being the collection of all these $2r$-sets.

Affine planes of order n are known to exist when n is a power of a prime. One example of a Hadamard 3-design of size 2^d is the collection of all affine hyperplanes of the d-dimensional vector space over the field $GF(2)$. It is conjectured that Hadamard 3-designs of size $4r$ exist for every $r \geq 1$ (for more details, see [4]).

Affine designs are used as building blocks in the following three constructions.

Construction 1. [3, Construction 1], [7] *Let $S_1, \ldots, S_{p+1}$ be arbitrary affine designs having the same parameters; here p is the number of parallel classes in each S_i. Let $S_i = (V_i, L_i)$. Let $I = \{1, \ldots, p+1\}$.*

For every i, denote arbitrarily the parallel classes of S_i by symbols L_{ij}, $j \in I \backslash \{i\}$. For $v \in V_i$, let $l_{ij}(v)$ denote the line in the parallel class L_{ij} which contains v.

For every pair i, j, $i \neq j$, choose an arbitrary bijection $\sigma_{ij} : L_{ij} \to L_{ji}$; we only require that $\sigma_{ji} = \sigma_{ij}^{-1}$.

Construct a graph $\Gamma_1 = \Gamma_1((S_i), (\sigma_{ij}))$ on the vertex set $X = \cup_{i \in I} V_i$. The sets V_i will be independent sets. Two vertices $v \in V_i$ and $w \in V_j$, $i \neq j$, are adjacent in Γ_1 if and only if $w \in \sigma_{ij}(l_{ij}(v))$ (or, equivalently, $\sigma_{ij}(l_{ij}(v)) = l_{ji}(w)$).

Construction 2. [3, Construction 4] *Let $\Gamma_1 = \Gamma_1((S_i), (\sigma_{ij}))$ be a graph of Construction 1 where $S_1, \ldots, S_{n+2}$ are affine planes of order n. Remove all vertices of $V_{n+1} \cup V_{n+2}$; for all $i \leq n$, add n-cliques on every line of the parallel classes $L_{i,n+1}$.*

The description of this construction here is a streamlined version of that given in [3]; it is not difficult to see that both descriptions produce the same graphs.

Construction 3. [2] *Let $A_1, \ldots, A_{p+1}, B_1, \ldots, B_{p+1}$ be arbitrary affine designs having the same parameters; here p is the number of parallel classes in each A_i, B_i. Let $A_i = (V_i, L_i)$, $B_i = (W_i, M_i)$. Let $I = \{1, \ldots, p+1\}$.*

For every i, denote arbitrarily the parallel classes of A_i, B_i correspondingly by symbols L_{ij}, M_{ij}, $j \in I \setminus \{i\}$. For $v \in V_i$, let $l_{ij}(v)$ denote the line in the parallel class L_{ij} which contains v.

For every pair i, j, $i \neq j$, choose an arbitrary bijection $\sigma_{ij} : L_{ij} \to M_{ji}$.

Construct a graph $\Gamma_3 = \Gamma_3((A_i, B_i), (\sigma_{ij}))$ on the vertex set $X = \cup_{i \in I}(V_i \cup W_i)$. The sets V_i, W_i will be independent sets. Two vertices $v \in V_i$ and $w \in W_j$, $i \neq j$, are adjacent in Γ_3 if and only if $w \in \sigma_{ij}(l_{ij}(v))$.

Construction 1 produces strongly regular graphs; Construction 2 produces distance regular graphs of diameter 3; Construction 3 produces bipartite distance regular graphs of diameter 3 which are incidence graphs of certain symmetric 2-designs.

Theorem 1. *a) If there exists a Hadamard design on x points, then every simple graph of order $\leq x$ is an induced subgraph of a strongly regular graph of order x^2.*

b) If there exists an affine plane of order x, then every simple graph of order $\leq x$ is an induced subgraph of a distance regular graph of diameter 3 and order x^3.

c) If there exists a Hadamard design on x points, then every simple bipartite graph of order $\leq x$ is an induced subgraph of a distance regular bipartite graph of diameter 3 and order $2x^2$.

In particular, every simple graph of order x is an induced subgraph of a strongly regular graph of order $< 4x^2$, and of a distance regular graph of diameter 3 and order $< 8x^3$; and every simple graph of order x is an induced subgraph of a bipartite distance regular graph of diameter 3 and order $< 8x^2$.

Proof. Let G be an arbitrary simple graph with vertices $V(G) = \{g_1, \ldots, g_m\}$, $m \leq x$. Following Construction 1, and using the notations therein, take $p+1 \geq x$ affine designs $S_1, \ldots, S_{p+1}$ having the same parameters; p is the number of parallel classes in each S_i. The names L_{ij} of parallel classes can be assigned arbitrarily.

For $i = 1, \ldots, m$, take an arbitrary point $v_i \in V_i$. Now it is easy to choose the bijections σ_{ij} in such a way that the graph induced on $\{v_1, \ldots, v_m\}$ will be isomorphic to G. Namely, for $1 \leq i < j \leq m$, let $\sigma_{ij}(l_{ij}(v_i)) = l_{ji}(v_j)$ if and only if the vertices g_i, g_j are adjacent in G.

A Hadamard design on x points has $x-1$ parallel classes. Thus, if we use in the above construction Hadamard designs, we obtain a strongly regular graph with x^2 vertices having an induced subgraph isomorphic to G.

If we use $x+2$ affine planes of order x, we get a strongly regular graph with $x^2(x+2)$ vertices having an induced subgraph isomorphic to G. When we apply Construction 2 to it (removing V_{x+1} and V_{x+2} which do not contain vertices v_i of the induced copy of G), we get a distance regular graph of diameter 3 with x^3 vertices. An induced copy of G remains there, because new edges within the sets V_i do not affect edges between vertices v_i.

If G is bipartite with parts $\{g_1, \ldots, g_l\}$, $\{h_{l+1}, \ldots, h_m\}$, we similarly use Construction 3 from Hadamard designs; choosing in V_i vertices corresponding to g_i, and in W_i vertices corresponding to h_i. Thus we obtain a distance regular bipartite graph with $2x^2$ vertices having an induced subgraph isomorphic to G.

The last assertion of the theorem follows from the fact that for every x there is a power of 2 in the interval $\{x, \ldots, 2x-1\}$. $\square$

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