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# ON A NONLINEAR EIGENVALUE PROBLEM IN SOBOLEV SPACES WITH VARIABLE EXPONENT

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Abstract. We consider a class of nonlinear Dirichlet problems involving the $p(x)$–Laplace operator. Our framework is based on the theory of Sobolev spaces with variable exponent and we establish the existence of a weak solution in such a space. The proof relies on the Mountain Pass Theorem.

## 1. Introduction

The Mountain Pass Theorem is due to Ambrosetti and Rabinowitz [1] and is one of the most powerful tools in Nonlinear Analysis for proving the existence of critical points of energy functionals. One of the simplest versions of the Mountain Pass Theorem asserts that if a continuously differential functional has two local minima, then (under some natural assumptions) such a function has a third critical point. This fact is elementary for functions of one real variable. However, even for functions on the plane the proof of such a theorem requires deep topological ideas. The Mountain Pass Theorem has numerous generalizations and has been applied in the treatment of various classes of boundary value problems. We refer to the recent monograph by Jabri [11] for an excellent survey of some of the most interesting applications of this abstract result. We do not intend to insist on the wide spectrum of applications of the Mountain Pass Theorem. We remark only that this theorem has been applied in the last few years in very concrete situations. For instance, in Lewin [14] it is considered a neutral molecule that possesses two distinct stable positions for its nuclei, and it is looked for a mountain pass point between the two minima in the non-relativistic Schrödinger framework.

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As showed in [1], one of the simplest applications of the Mountain Pass Theorem implies the existence of solutions for the Dirichlet problem

$$
\begin{cases}
-\Delta u = u^{p-1} & \text{in } \Omega\\
u > 0 & \text{in } \Omega\\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2 < p < 2N/(N-2)$ if $N \geq 3$ and $p \in (2, \infty)$ if $N = 1$ or $N = 2$.

This equation is called the *Kazdan–Warner equation* and the existence results are related not only to the values of $p$, but also to the geometry of $\Omega$. For instance, problem (1) has no solution if $p \geq 2N/(N-2)$ and if $\Omega$ is a *starshaped domain* with respect to a certain point (the proof uses the *Pohozaev identity*, which is obtained after multiplication in (1) with $x \cdot \nabla u$ and integration by parts). If $\Omega$ is **not** starshaped, Kazdan and Warner proved in [12] that problem (1) has a solution for **any** $p > 2$, where $\Omega$ is an **annulus** in $\mathbb{R}^N$.

Under the same assumptions on $p$, similar arguments show that the boundary value problem

$$
\begin{cases}
-\Delta u - \lambda u = u^{p-1} & \text{in } \Omega\\
u > 0 & \text{in } \Omega\\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

has a solution for any $\lambda < \lambda_1$, where $\lambda_1$ denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. The proof of this result relies on the fact that the operator $(-\Delta - \lambda I)$ is coercive if $\lambda < \lambda_1$. Moreover, a multiplication by $\varphi_1$ and integration on $\Omega$ implies that there is no solution if $\lambda \geq \lambda_1$, where $\varphi_1$ stands for the first eigenfunction of the Laplace operator. We refer to [19] for interesting localization results of solutions to problems of the above type, as well as for a lower bound of all nontrivial solutions.

The main purpose of this paper is to study a related problem, but for a more general differential operator, the so-called $p(x)$–Laplace operator. This degenerate differential operator is defined by $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ (where $p(x)$ is a certain function whose properties will be stated in what follows) and that generalizes the celebrated $p$–Laplace operator $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a constant. The $p(x)$–Laplace operator possesses more complicated nonlinearity than the $p$–Laplacian, for example, it is inhomogeneous. We only recall that $\Delta_p$ describes a variety of phenomena in the nature. For instance, the equation governing the motion of a fluid involves the $p$–Laplace operator. More exactly, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla u$ of the fluid are related in the manner that $\vec{\tau}(x) = r(x)|\nabla u|^{p-2}\nabla u$, where $p = 2$ (resp., $p < 2$ or $p > 2$) if the fluid is Newtonian (resp., pseudoplastic or dilatant). Other applications of the $p$–Laplacian also appear in the study of flow through porous media ($p = 3/2$), Nonlinear Elasticity ($p \geq 2$), or Glaciology ($1 < p \leq 4/3$).

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## 2. Auxiliary results

In this section we recall the main properties of Lebesgue and Sobolev spaces with variable exponent. We point out that these functional spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [17], who proved various results (including Hölder's inequality) in a discrete framework. Orlicz also considered the variable exponent function space $L^{p(x)}$ on the real line, and proved the Hölder inequality in this setting, too. Next, Orlicz abandoned the study of variable exponent spaces, to concentrate on the theory of the function spaces that now bear his name. The first systematic study of spaces with variable exponent (called *modular spaces*) is due to Nakano [16]. In the appendix of this book, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers [16, p. 284]. Despite their broad interest, these spaces have not reached the same main-stream position as Orlicz spaces. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Polish mathematicians. We refer to the book by Musielak [15] for a nice presentation of modular function spaces. This book, although not dealing specifically with the spaces that interest us, is still specific enough to contain several interesting results regarding variable exponent spaces. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov. These investigations originated in a paper by Tsenov [23]. The question raised by Tsenov and solved by Sharapudinov [22] is the minimization of $\int_a^b |u(x)-v(x)|^{p(x)}dx$, where $u$ is a fixed function and $v$ varies over a finite dimensional subspace of $L^{p(x)}([a,b])$. Sharapudinov also introduces the Luxemburg norm for the Lebesgue space and shows that this space is reflexive if the exponent satisfies $1<p^-\le p^+<\infty$. In the 80's Zhikov started a new line of investigation, that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions.

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$.

Set

$$C_+(\overline{\Omega}) = \{h;\ h\in C(\overline{\Omega}),\ h(x)>1 \text{ for all } x\in\overline{\Omega}\}.$$

For any $h\in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x)\in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue and Sobolev spaces

$$L^{p(x)}(\Omega) =$$

$$\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx<\infty\}$$

and

$$W^{1,p(x)}(\Omega) = \left\{u\in L^{p(x)}(\Omega);\ |\nabla u|\in L^{p(x)}(\Omega)\right\}.$$

On these spaces we define, respectively, the following norms

$$|u|_{p(x)} = \inf\left\{\mu>0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx\le 1\right\} \qquad \text{(called Luxemburg norm)}$$

and

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Variable exponent Lebesgue and Sobolev spaces resemble classical Lebesgue and Sobolev spaces in many respects: they are Banach spaces [13, Theorem 2.5], the Hölder inequality holds [13, Theorem 2.1], they are reflexive if and only if $1<p^-\le p^+<\infty$ [13, Corollary 2.7] and continuous functions are dense if $p^+<\infty$ [13, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally

[13, Theorem 2.8]: if $0 < |\Omega| < \infty$ and $p_1, p_2 \in C_+(\overline{\Omega})$ are variable exponent so that $p_1(x) \le p_2(x)$ in $\Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{2}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $(u_n)$, $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations holds true

$$\begin{aligned} |u|_{p(x)} > 1 \quad &\Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \\ |u|_{p(x)} < 1 \quad &\Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \\ |u_n - u|_{p(x)} \to 0 \quad &\Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \end{aligned} \tag{3}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [4].

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. On this space we can use the equivalent norm $\|u\| = |\nabla u|_{p(x)}$. The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. The dual of this space is denoted by $W_0^{-1,p'(x)}(\Omega)$. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact, while $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $L^{p^*(x)}(\Omega)$, where $p^\star(x)$ denotes the critical Sobolev exponent, that is, $p^\star(x) = Np(x)/(N - p(x))$, provided that $p(x) < N$ for all $x \in \overline{\Omega}$. We refer to [3, 5, 6, 7, 8, 9, 13, 21] for further properties and applications of variable exponent Lebesgue–Sobolev spaces.

## 3. The main result

Assume throughout this paper that $\Omega$ is a smooth bounded open set in $\mathbb{R}^N$ $(N \ge 2)$, $\lambda$ is a real parameter and $p \in C_+(\overline{\Omega})$.

Consider the boundary value problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda u^{p(x)-1} + u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \ge 0,\ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{4}$$

where $p \in C_+(\overline{\Omega})$ such that $p^+ < N$, and $q$ is a real number.

**Definition 1.** *Let $\lambda$ be a real number. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a solution of Problem (4) if $u \ge 0$, $u \not\equiv 0$ in $\Omega$ and*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda \int_\Omega u^{p(x)-1} v dx + \int_\Omega u^{q-1} v dx, \qquad \forall v \in W_0^{1,p(x)}(\Omega)\,.$$

A crucial role in the statement of our result will be played by the nonlinear eigenvalue problem

$$(5)\qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

It follows easily that if $(u,\lambda)$ is a solution of (5) and $u \not\equiv 0$ then

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx}$$

and hence $\lambda > 0$. Let $\Lambda$ denote the set of eigenvalues of (5), that is,

$$\Lambda = \Lambda_{p(x)} = \{\lambda \in \mathbb{R};\ \lambda \text{ is an eigenvalue of Problem (5)}\}\,.$$

In [10] it is showed that if the function $p(x)$ is a constant $p > 1$ (we refer to [2] for the linear case $p(x) \equiv 2$), then Problem (5) has a sequence of eigenvalues, $\sup \Lambda = +\infty$ and $\inf \Lambda = \lambda_1 = \lambda_{1,p} > 0$, where $\lambda_{1,p}$ is the first eigenvalue of $(-\Delta_p)$ in $W_0^{1,p}(\Omega)$ and

$$\lambda_1 = \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega\setminus\{0\})} \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx}\,.$$

In [8] it is showed that for general functions $p(x)$ the set $\Lambda$ is infinite and $\sup \Lambda = +\infty$. Moreover, it may arise that $\inf \Lambda = 0$. Set

$$\lambda^* = \lambda^*_{p(x)} = \inf \Lambda\,.$$

In [8] it is argued that if $N = 1$ then $\lambda^* > 0$ if and only if the function $p(x)$ is monotone. In arbitrary dimension, $\lambda^* = 0$ provided that there exist an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) <$ (or $>$) $p(x)$ for all $x \in \partial U$.

**Theorem 1.** *Assume that $\lambda < \lambda^*$ and $p^+ < q < Np^-/(N-p^-)$. Then Problem (4) has at least a solution.*

We cannot expect that Problem (4) has a solution for any $\lambda \geq \lambda^*$. Indeed, consider the simplest case $p(x) \equiv 2$, take $\lambda \geq \lambda_1$ and multiply the equation in (4) by $\varphi_1 > 0$. Integrating on $\Omega$ we find

$$(\lambda - \lambda_1)\int_\Omega u\varphi_1 dx + \int_\Omega u^{q-1}\varphi_1 dx = 0$$

which yields a contradiction.

The proof of the above result relies on the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] in the following variant.

**Theorem 2.** *Let $X$ be a real Banach space and let $F : X \to \mathbb{R}$ be a $C^1$–functional. Suppose that $F$ satisfies the Palais-Smale condition and the following geometric assumptions:*

$$(6)\qquad \begin{cases} \text{there exist positive constants } R \text{ and } c_0 \text{ such that} \\ F(u) \geq c_0,\ \text{for all } u \in X \text{ with } \|u\| = R\,; \end{cases}$$

$$(7)\qquad F(0) < c_0 \text{ and there exists } v \in X \text{ such that } \|v\| > R \text{ and } F(v) < c_0\,.$$

*Then the functional $F$ possesses at least a critical point.*

We recall the celebrated "compactness condition" introduced by Palais and Smale [18]: the functional $F \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition provided that any sequence $(u_n)$ in $X$ such that

$$\sup_n |F(u_n)| < \infty \qquad \text{and} \qquad \|F'(u_n)\| \to 0$$

has a convergent subsequence.

The name of the above result is a consequence of a simplified visualization for the objects from theorem. Indeed, consider the set $\{0,\, v\}$, where 0 and $v$ are two villages, and the set of all paths joining 0 and $v$. Then, assuming that $F(u)$ represents the altitude of point $u$, assumptions (6) and (7) are equivalent to say that the villages 0 and $v$ are separated by a mountains chain. So, the conclusion of the theorem tells us that there exists a path between the villages with a minimal altitude. With other words, there exists a "mountain pass".

## 4. Proof of Theorem 1

Our hypothesis $\lambda < \lambda^*$ implies that there exists $C_0 > 0$ such that

$$\int_\Omega (|\nabla v|^{p(x)} - \lambda |v|^{p(x)})dx \geq C_0 \int_\Omega |\nabla v|^{p(x)} dx \qquad \text{for all } v \in W_0^{1,p(x)}(\Omega)\,. \tag{8}$$

Set

$$g(u) = \begin{cases} u^{q-1}, & \text{if } u \geq 0\,, \\ 0, & \text{if } u < 0 \end{cases}$$

and $G(u) = \int_0^u g(t)dt$. Define the energy functional associated to Problem (4) by

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} - \lambda |u|^{p(x)}\right) dx - \int_\Omega G(u) dx \qquad \text{for all } u \in W_0^{1,p(x)}\,.$$

Observe that

$$|G(u)| \leq C\, |u|^q$$

and, by our hypotheses on $p(x)$ and $q$, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^q(\Omega)$, which implies that $J$ is well defined on $W_0^{1,p(x)}(\Omega)$.

A straightforward computation shows that $J$ is of class $C^1$ and, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$J'(u)(v) = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v - \lambda\, |u|^{p(x)-2} uv) dx - \int_\Omega g(u) v dx\,.$$

We prove in what follows that $J$ satisfies the hypotheses of the Mountain Pass Theorem.

Verification of (6). We may write, for every $u \in \mathbb{R}$,

$$|g(u)| \leq |u|^{q-1}\,.$$

Thus, for every $u \in \mathbb{R}$,

$$|G(u)| \leq \frac{1}{q}\, |u|^q\,. \tag{9}$$

Next, by (8) and (9),

$$J(u) \geq \frac{C_0}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - C \int_\Omega |u|^q dx = C_1 \int_\Omega |\nabla u|^{p(x)} dx - C_2\, \|u\|^q_{L^q} dx\,, \tag{10}$$

for every $u \in W_0^{1,p(x)}(\Omega)$, where $C_1$ and $C_2$ are positive constants. So, by relation (3) and using the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^q(\Omega)$ combined with the assumption $p^+ < q$ we find, for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = |\nabla u|_{p(x)} = R$ sufficiently small,

$$J(u) \geq C_1\, |\nabla u|_{p(x)}^{p^+} - C_3\, |\nabla u|_{p(x)}^{q} \geq c_0 > 0\,.$$

Verification of (7). Choose $u_0 \in W_0^{1,p(x)}(\Omega)$, $u_0 > 0$ in $\Omega$. Since $p^+ < q$, it follows that if $t > 0$ is large enough then

$$J(tu_0) = \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\nabla u_0|^{p(x)} - \lambda |u_0|^{p(x)}\right) dx - \frac{t^q}{q} \int_\Omega u_0^q dx < 0\,.$$

Verification of the Palais-Smale condition. Let $(u_n)$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\sup_n |J(u_n)| < +\infty \tag{11}$$

$$\|J'(u_n)\|_{W^{-1,p'(x)}} \to 0 \qquad \text{as } n \to \infty\,. \tag{12}$$

We first prove that $(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$. Remark that (12) implies that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla v - \lambda\, |u_n|^{p(x)-2}\, u_n v) dx = \int_\Omega g(u_n) v dx + o(1)\, \|v\| \ \text{ as } n \to \infty\,. \tag{13}$$

Choosing $v = u_n$ in (13) we find

$$\int_\Omega \left(|\nabla u_n|^{p(x)} - \lambda\, |u_n|^{p(x)}\right) dx = \int_\Omega g(u_n) u_n dx + o(1)\, \|u_n\|\,. \tag{14}$$

Relation (11) implies that there exists $M > 0$ such that, for any $n \geq 1$,

$$\left| \int_\Omega \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} - \lambda\, |u_n|^{p(x)}\right) dx - \int_\Omega G(u_n) dx \right| \leq M\,. \tag{15}$$

But a simple computation yields

$$\int_\Omega g(u_n) u_n dx = q \int_\Omega G(u_n) dx\,. \tag{16}$$

Combining (14), (15) and (16) and using our assumption $p^+ < q$ we find

$$\int_\Omega G(u_n) dx = O(1) + o(1)\, \|u_n\|\,. \tag{17}$$

Thus, by (14) and (17),

$$\int_\Omega |\nabla u_n|^{p(x)} dx = O(1) + o(1)\, \|u_n\|\,,$$

which means that $(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$.

It remains to prove that $(u_n)$ is relatively compact. We first remark that (13) may be rewritten as

$$\int_\Omega |\nabla u_n|^{p(x)-2}\, \nabla u_n \cdot \nabla v dx = \int_\Omega h(x, u_n) v dx + o(1)\, \|v\|\,, \tag{18}$$

for every $v \in W_0^{1,p(x)}(\Omega)$, where

$$h(x,u) = g(u) + \lambda\,|u|^{p(x)-2}\,u\,,$$

where $\lambda < \lambda^*$ is fixed. Obviously, $h$ is continuous and, since $q < Np(x)/(N-p(x))$ for all $x \in \overline{\Omega}$, there exists $C > 0$ such that

$$(19)\quad |h(x,u)| \le C\left(1 + |u|^{(Np(x)-N+p(x))/(N-p(x))}\right) \qquad \text{for all } x \in \overline{\Omega} \text{ and } u \in \mathbb{R}\,.$$

Moreover

$$(20)\qquad h(x,u) = o\left(|u|^{Np(x)/(N-p(x))}\right) \qquad \text{as } |u| \to \infty, \text{ uniformly for } x \in \overline{\Omega}\,.$$

Define $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ by $Au = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. Then $A$ is invertible and $A^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is a continuous operator. Thus, by (18), it suffices to show that $h(x,u_n)$ is relatively compact in $W^{-1,p'(x)}(\Omega)$. By continuous embeddings for Sobolev spaces with variable exponent, this will be achieved by proving that a subsequence of $h(x,u_n)$ is convergent in

$$(L^{Np(x)/(N-p(x))}(\Omega)\,)^\star = L^{Np(x)/(Np(x)-N+p(x))}(\Omega)\,.$$

Since $(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega) \subset L^{Np(x)/(N-p(x))}(\Omega)$ we can suppose that, up to a subsequence,

$$u_n \to u \in L^{Np(x)/(N-p(x))}(\Omega) \qquad \text{a.e. in } \Omega\,.$$

Moreover, by Egorov's Theorem, for each $\delta > 0$, there exists a subset $A$ of $\Omega$ with $|A| < \delta$ and such that

$$u_n \to u \qquad \text{uniformly in } \Omega \setminus A\,.$$

So, it is sufficient to show that

$$\int_A |h(u_n) - h(u)|^{Np(x)/(Np(x)-N+p(x))}\,dx \le \eta\,,$$

for any fixed $\eta > 0$. But, by (19),

$$\int_A |h(u)|^{Np(x)/(Np(x)-N+p(x))}dx \le C\int_A (1 + |u|^{Np(x)/(N-p(x))})dx\,,$$

which can be made arbitrarily small if we choose a sufficiently small $\delta > 0$.

We have, by (20),

$$\int_A |h(u_n) - h(u)|^{Np(x)/(Np(x)-N+p(x))}dx \le \varepsilon\int_A |u_n - u|^{Np(x)/(N-p(x))}dx + C_\varepsilon\,|A|\,,$$

which can be also made arbitrarily small, by continuous embeddings for Sobolev spaces with variable exponent combined with the boundedness of $(u_n)$ in $W_0^{1,p(x)}(\Omega)$. Hence, $J$ satisfies the Palais-Smale condition. Thus, by Theorem 2, the boundary value problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u + g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$. It remains to show that $u \ge 0$. Indeed, multiplying the equation by $u^-$ and integrating we find

$$\int_\Omega |\nabla u^-|^{p(x)}dx - \lambda\int_\Omega (u^-)^{p(x)}dx = 0\,.$$

Thus, since $\lambda < \lambda^*$, we deduce that $u^- = 0$ in $\Omega$ or, equivalently, $u \geq 0$ in $\Omega$. $\square$

A careful analysis of the above proof shows that the existence result stated in Theorem 1 remains valid if $u^{q-1}$ is replaced by the more general nonlinearity $f(x,u)$, where $f(x,u) : \overline{\Omega} \to \mathbb{R}$ is a continuous functions satisfying

$$|f(x,u)| \leq C(|u| + |u|^{q-1}), \qquad \forall x \in \Omega,\ \forall u \in \mathbb{R}$$

with $p^- < q < Np^+/(N-p^+)$ if $N \geq 3$ and $q \in (p^-, \infty)$ if $N = 1$ or $N = 2$,

$$\lim_{\varepsilon \searrow 0} \sup \left\{ \left| \frac{f(x,t)}{t} \right| ;\ (x,t) \in \overline{\Omega} \times (-\varepsilon, \varepsilon) \right\} = 0 \qquad \text{uniformly for } x \in \overline{\Omega}$$

and

$$0 \leq \mu F(x,u) \leq u f(x,u) \qquad \text{for } 0 < u \text{ large and some } \mu > 2,$$

where $F(x,u) = \int_0^u f(x,t)dt$.

The following result shows that Theorem 1 still remains valid if the right hand-side is affected by a small perturbation. Consider the boundary value problem

$$\text{(21)} \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda |u|^{p(x)-2} u + |u|^{q-2} u + a(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $a \in L^\infty(\Omega)$, $p \in C_+(\overline{\Omega})$ such that $p^+ < N$, and $q$ is a real number.

**Corollary 1.** *Assume that $\lambda < \lambda^*$ and $p^+ < q < Np^-/(N-p^-)$. There exists $\delta > 0$ such that if $\|a\|_{L^\infty} < \delta$ then Problem (21) has at least a solution.*

*Proof.* For any $u \in W_0^{1,p(x)}(\Omega)$ define the energy functional

$$E(u) = \int_\Omega \frac{1}{p(x)} \left( |\nabla u|^{p(x)} - \lambda |u|^{p(x)} \right) dx - \frac{1}{q} \int_\Omega |u|^q dx - \int_\Omega a(x) u dx\,.$$

We have already seen that if $a = 0$ then Problem (21) has a nontrivial and nonnegative solution. If $\|a\|_{L^\infty}$ is sufficiently small then the verification of the Palais-Smale condition, as well as of the geometric assumptions (6) and (7) can be made following the same ideas as in the proof of Theorem 1. Thus, by Theorem 2, the functional $E$ has a nontrivial critical point $u \in W_0^{1,p(x)}(\Omega)$, which is a solution of Problem (21). However, we are not able to decide if this solution is nonnegative. This result remains true if $a \geq 0$, as we can see easily after multiplication with $u^-$ and integration. $\square$

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