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ON THE HURWITZ EXISTENCE PROBLEM

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Given a branched covering $\widetilde{\Sigma} \to \Sigma$ between closed connected surfaces, one can easily establish some relations between the Euler characteristic and orientability of $\widetilde{\Sigma}$ and Σ, the degree d of the covering, the number n of branching points, the number $\widetilde{n}$ of preimages of the branching points, and the local degrees d_{ij} at these points. A classical problem dating back to Hurwitz asks whether these necessary conditions are sufficient as well.

More precisely, let us call *branch datum* a 5-tuple $\left(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\right)$, where $\widetilde{\Sigma}$ and Σ are closed connected surfaces, $n \geqslant 0$ and $d \geqslant 2$ are integers, and $(d_{ij})_{j=1,\ldots,m_i}$ is a partition of d, for $i = 1, \ldots, n$. A *compatible* branch datum is one that satisfies the following conditions:

(1) $\chi(\widetilde{\Sigma}) - \widetilde{n} = d \cdot (\chi(\Sigma) - n)$, where $\widetilde{n} = m_1 + \ldots + m_n$;
(2) $n \cdot d - \widetilde{n}$ is even;
(3) If Σ is orientable then $\widetilde{\Sigma}$ is also orientable;
(4) If Σ is non-orientable and d is odd then $\widetilde{\Sigma}$ is also non-orientable;
(5) If Σ is non-orientable but $\widetilde{\Sigma}$ is orientable then each partition $(d_{ij})_{j=1,\ldots,m_i}$ of d refines the partition $(d/2, d/2)$.

These conditions are easily seen to be necessary for the existence of a covering realizing the branch datum. The Hurwitz existence problem asks which compatible data are realizable. Those that are not are called *exceptional*.

Thanks to the work of many authors, the problem is now completely solved (in the affirmative) when $\chi(\Sigma) \geqslant 0$. The cases where the base surface is the sphere or the projective plane remain elusive, but the latter reduces to the former, and many

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partial results are known. In this work we describe several new series of exceptional and, *vice versa*, realizable data, when the base surface is the sphere $\mathbb{S}$.

The classical approach to the problem was based on the analysis of the possible monodromies of the covering, *i.e.* of the possible representations of the fundamental group of Σ minus n points into $\mathfrak{S}_d$. We employed three alternative different techniques, one based on Grothendieck's dessins d'enfants, one involving the factorization of a covering through two non-trivial ones, and one establishing the equivalence of the Hurwitz problem with a geometric condition (the existence of certain families of graphs on $\widetilde{\Sigma}$). The former of the next two non-existence theorems is based on the dessins technique, the latter on the factorization technique:

Theorem 1.

- *If $d = kh$ and $p \geqslant 2$ the following branch datum is exceptional:*
  $$\big(\mathbb{S}, \mathbb{S}, 3, d, (k, \ldots, k), (kh_1, \ldots, kh_p), (h + p - 1, 1, \ldots, 1)\big);$$
- *If $d \geqslant 8$ is even there exists an exceptional branch datum with $n = 3$, $\Sigma = \mathbb{S}$ and $\widetilde{\Sigma} = \mathbb{T}$ (the torus),* e.g.
  $$\big(\mathbb{T}, \mathbb{S}, 3, d, (2, \ldots, 2), (5, 3, 2, \ldots, 2), (d/2, d/2)\big);$$
- *If $d \geqslant 6$ is even there exists an exceptional branch datum with $n = 4$ and $\Sigma = \widetilde{\Sigma} = \mathbb{S}$,* e.g.
  $$\big(\mathbb{S}, \mathbb{S}, 4, d, (2, \ldots, 2), (2, \ldots, 2), (2, 1, \ldots, 1), (d - 2, 1, 1)\big).$$

Theorem 2. *Suppose d and all d_{ij} for $i = 1, 2$ are even. If the branch datum $\big(\mathbb{S}, \mathbb{S}, 3, d, (d_{ij})\big)$ is realizable then (d_{3j}) refines the partition $(d/2, d/2)$.*

On the existence side, the factorization technique leads to the first one of the next three theorems, while the geometric technique leads to the last two.

Theorem 3. *A compatible branch datum $\big(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\big)$ such that all d_{ij} are divisible by some odd $p \geqslant 3$ is realizable.*

Theorem 4. *If d is odd then every compatible branch datum of the form $\big(\mathbb{S}, \mathbb{S}, 3, d, (d - 2, 2), (d_{2j}), (d_{3j})\big)$ is realizable.*

Theorem 5. *With the single exception of $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4, 2), (3, 3), (3, 3)\big)$, every compatible branch datum of the form $\big(\mathbb{T}, \mathbb{S}, 3, d, (d - 2, 2), (d_{2j}), (d_{3j})\big)$ is realizable.*

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