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VIRTUAL AND WELDED LINKS AND THEIR INVARIANTS

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Virtual and welded links. Virtual link theory is introduced by Kauffman [8] as a generalization of classical link theory. He defined a virtual link diagram. On virtual link diagram there are some types of local moves (generalized Reidemeister moves): classical Reidemeister moves, virtual Reidemeister moves and mixed Reidemeister move. Two virtual link diagrams are *equivalent* if one diagram can be transformed into another by a finite sequence of generalized Reidemeister moves. *A virtual link* is an equivalence class of virtual link diagrams under generalized Reidemeister moves.

A theorem of Goussarov, Polyk and Viro [6] states that if two classical link diagrams are equivalent under generalized Reidemeister moves, then they are equivalent under the classical Reidemeister moves. In this sense virtual link theory is a nontrivial extension of the classical theory.

On the set of virtual diagrams there are so called *forbidden moves* F1 and F2 which are similar to a generalized Reidemeister move but are not part of the equivalence relation for virtual links. We can include one or both of them to obtain a quotient theory of the theory of virtual links. If we add the move F1, then we obtain the theory of *Welded links*. The theory with both forbidden moves added is called the theory of *Fused links*.

Virtual and welded braids. Recently some generalizations of classical braids were defined and studied: virtual braids [8, 10], welded braids [5]. A theorem of Markov reduces the problem of classification of links to some algebraic problems of the theory of braid groups. These problems include the word problem and the conjugacy problem. There are generalizations of Markov's theorem for virtual links, and welded links [7].

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The braid group B_n, $n \geq 2$, on n strings can be defined as a group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, with the defining relations

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i, \quad |i-j| \geq 2.$$

The virtual braid group VB_n is generated by σ_i, ρ_i, $i = 1, 2, \ldots, n-1$. Elements σ_i generate the braid group B_n and elements ρ_i generate the symmetric group S_n, and the following mixed relations hold

$$\sigma_i \, \rho_j = \rho_j \, \sigma_i, \quad |i-j| \geq 2,$$

$$\rho_i \, \rho_{i+1} \, \sigma_i = \sigma_{i+1} \, \rho_i \, \rho_{i+1}, \quad i = 1, 2, \ldots, n-2.$$

In [6] it was proved that the relations

$$\rho_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \rho_{i+1}, \qquad \rho_{i+1} \, \sigma_i \, \sigma_{i+1} = \sigma_i \, \sigma_{i+1} \, \rho_i$$

are not fulfilled in VB_n.

The welded braid group WB_n is generated by σ_i, α_i, $i = 1, 2, \ldots, n-1$. Elements σ_i generate the braid group B_n and elements α_i generate the symmetric group S_n, and the following mixed relations hold

$$\alpha_i \, \sigma_j = \sigma_j \, \alpha_i, \quad |i-j| \geq 2,$$

$$\sigma_i \, \alpha_{i+1} \, \alpha_i = \alpha_{i+1} \, \alpha_i \, \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2,$$

$$\sigma_{i+1} \, \sigma_i \, \alpha_{i+1} = \alpha_i \, \sigma_{i+1} \, \sigma_i, \quad i = 1, 2, \ldots, n-2.$$

Comparing the defining relations of VB_n with the defining relations of WB_n, we see that WB_n can be obtained from VB_n by adding some new relation. Therefore, there exists a homomorphism

$$\varphi_{VW} : VB_n \longrightarrow WB_n,$$

taking σ_i to σ_i and ρ_i to α_i for all i. Hence, WB_n is the homomorphic image of VB_n. In [5] it was proved that WB_n is isomorphic to the group of conjugating automorphisms C_n. The structure of C_n and its linear representations were studied in [2, 3, 4].

In [1] it was studied the structure of VB_n. It is similar to the braid group B_n and welded braid group WB_n. The group VB_n contains a normal subgroup VP_n which is called the *virtual pure braid group*. The quotient group VB_n/VP_n is isomorphic to the symmetric group S_n. The group VB_n is a semi–direct product of VP_n and S_n. It was proved that VP_n is representable as the following semi–direct product

$$VP_n = V_{n-1}^* \rtimes VP_{n-1} = V_{n-1}^* \rtimes (V_{n-2}^* \rtimes (\ldots \rtimes (V_2^* \rtimes V_1^*)) \ldots),$$

where V_i^* is a free subgroup of VP_n (in general infinitely generated for $i > 1$).

The representations by automorphisms. By Artin's theorem, B_n is embedded into the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n. Since $WB_n \simeq C_n$ then WB_n is also embedded into $\mathrm{Aut}(F_n)$. It is not known if VB_n is embedded into $\mathrm{Aut}(F_m)$ for some m.

Theorem 1. *There is a representation ψ of VB_n in $\mathrm{Aut}(F_{n+1})$, $F_{n+1} = \langle x_1, x_2, \dots, x_n, y\rangle$ which is defined by the following actions on the generators of VB_n:*

$$\psi(\sigma_i): \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l, \quad l \neq i, i+1; \\ y \longmapsto y, \end{cases} \qquad \psi(\rho_i): \begin{cases} x_i \longmapsto y\, x_{i+1}\, y^{-1}, \\ x_{i+1} \longmapsto y^{-1}\, x_i\, y, \\ x_l \longmapsto x_l, \quad l \neq i, i+1, \\ y \longmapsto y, \end{cases}$$

for all $i = 1, 2, \dots, n-1$.

>From this representation we constructed V. O. Manturov's invariant $\mathcal{F}$ of VB_n (see [9]).

Group of virtual link. For a virtual link L Kauffman defines a group $G_K(L)$. But if T is the virtual trefoil knot with two classical crossings and one virtual crossing and U is unknot then $G_K(T) \simeq G_K(U) \simeq \mathbb{Z}$, although it is well known that T is not equivalent to U.

We introduce another group $G(L)$ of a virtual link L. Let $L = \widehat{\beta}$ be a closed virtual braid, where $\beta \in VB_n$. Define

$$G(L) = \langle x_1, x_2, \dots, x_n, y \mid\mid x_i = \psi(\beta)(x_i), \quad i = 1, \dots, n\rangle.$$

If L is a classical link then

$$G(L) = \mathbb{Z} * \pi_1(S^3 \setminus L), \qquad \mathbb{Z} = \langle y\rangle.$$

In particular,

$$G(U) = \langle x, y\rangle \simeq F_2.$$

Theorem 2. *The group $G(L)$ is an invariant of the virtual link L.*

Since, for the virtual trefoil T we have $T = \widehat{\sigma_1^2 \rho_1}$ then

$$G(T) = \langle x, y \mid\mid x\,(y\,x\,y^{-2}\,x\,y) = (y\,x\,y^{-2}\,x\,y)\,x\rangle \not\simeq F_2.$$

Hence, T is a non-trivial knot.

Linear representations. We define a representation

$$\widehat{\rho}_B : VB_n \longrightarrow \mathrm{GL}_n(\mathbf{Z}[t^{\pm 1}, q^{\pm 1}]),$$

by the following actions on the generators of VB_n:

$$\widehat{\rho}_B(\sigma_i) = \left(\begin{array}{c|cc|c} \mathbf{E}_{i-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 1-t & t & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{n-i-1} \end{array}\right), \quad i = 1, \dots, n-1,$$

$$\widehat{\rho}_B(\rho_i) = \left(\begin{array}{c|cc|c} \mathbf{E}_{i-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & q & \mathbf{0} \\ \mathbf{0} & q^{-1} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{n-i-1} \end{array}\right), \quad i = 1, \dots, n-1,$$

that is an extension of the classical Burau representation

$$\rho_B : B_n \longrightarrow \mathrm{GL}_n(\mathbf{Z}[t^{\pm 1}]).$$

If we let $q = 1$, then we get a linear representation of VB_n and WB_n which was constructed in [10]. It is evident that $\ker(\widehat{\rho}_B) \neq 1$ for $n \geq 5$.

Question. Is it true that $\ker(\widehat{\rho}_B) = 1$ for $n = 3$?

A linear representation of C_n was constructed in [3, 4]. This representation extends (with some conditions on parameters) the known Lawrence–Krammer representation.

There is a linear representation of P_n which is called the Gassner representation. The problem of the faithfulness of this representation for $n > 3$ is still open. Using the Magnus representation we construct a linear representation

$$\mathrm{IA}(F_n) \longrightarrow \mathrm{GL}_n(\mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]),$$

which is an extension of the Gassner representation of P_n. Here $\mathrm{IA}(F_n)$ is the group of IA-automorphism, which is a subgroup of $\mathrm{Aut}(F_n)$.

The group WB_n contains as subgroup the pure welded braid group PW_n, and

$$P_n \leq PW_n \leq \mathrm{IA}(F_n).$$

We have

Theorem 3. *There is a linear representation $\varphi : PW_n \longrightarrow \mathrm{GL}_n(\mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}])$, which is an extension of the Gassner representation of the pure braid group P_n. This representation has a non-trivial kernel for every $n \geq 2$.*

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