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ON THE COMPLEXITY OF GRAPH MANIFOLDS

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Abstract. We provide a new formula for an upper bound of the complexity of non-Seifert graph-manifolds obtained by gluing together two Seifert manifolds fibered over the disc with two exceptional fibers. This bound turns out to be sharp for many manifolds.

The Matveev's complexity $c(M)$ of a compact 3-manifold M is equal to k if M possesses an almost simple spine with k vertices and has no almost simple spines with a smaller number of vertices [1]. In general, the problem of calculating $c(M)$ is very difficult. The exact value of the complexity is only known for a finite number of closed orientable irreducible manifolds [2], for the complements of the figure eight knot and its twin, as well as for all their finite coverings [3], and also for manifolds having special spines with one 2-cell [4]. To estimate $c(M)$ it suffices to construct an almost simple spine P of M. The number of vertices of P is an upper bound for the complexity. On one hand, an almost simple spine can be easily constructed from many presentations of the manifold [5]. On the other hand, as a rule, $c(M)$ is significantly less than such an upper bound.

Let X be some infinite set of manifolds. We say that an integral nonnegative function $\tilde{c} : X \to \mathbb{Z}$ is a k-sharp complexity bound for X if $c(M) \le \tilde{c}(M)$ for all $M \in X$, and $c(M) = \tilde{c}(M)$ for all $M \in X$ with $c(M) \le k$. First example of a k-sharp complexity bound was obtained by S. Matveev for lens spaces. Using a computer he composed a table of all closed orientable irreducible manifolds of complexity ≤ 6. Analyzing the table he proved that a function $\tilde{c}(L_{p,q}) = S(p, q) - 3$ is 6-sharp [1], where $S(p, q)$ is the sum of all partial quotients in the expansion of p/q as a regular continued fraction. Later M. Ovchinnikov and B. Martelli, C. Petronio extended the table to complexity 7 and 9, respectively, and verified that the function $S(p, q) - 3$ is 9-sharp [6]. Also Martelli and Petronio found a 9-sharp

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complexity bound for all Seifert manifolds [7]. This year S. Matveev and V. Tarkaev composed a table of all closed orientable irreducible manifolds of complexity ≤ 12. Now we can state that those complexity bounds are 12-sharp.

Denote by G the set of all non-Seifert graph-manifolds obtained by gluing together two Seifert manifolds fibered over the disc with two exceptional fibers along some homeomorphism of their boundary tori. Note that each manifold $M \in G$ can be presented in the form

$$(D^2, (p_1, q_1), (p_2, q_2 - p_2)) \bigcup_A (D^2, (p_3, q_3), (p_4, q_4 - p_4)),$$

where $p_i > q_i > 0$, $1 \leq i \leq 4$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an integer matrix with determinant (-1).

Theorem 1. *The function*

$$\tilde{c}(M) = \max\{S(|a| + |b|, |c| + |d|) - 2, 0\} - 2 + \sum_{i=1}^{4} S(p_i, q_i)$$

is a 12*-sharp complexity bound for* G*.*

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