S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 2, стр. 156–166 (2005) УДК 517.968.23
MSC 42B20

THE CONTINUITY OF MULTILINEAR SINGULAR INTEGRAL OPERATORS WITH VARIABLE CALDERÓN–ZYGMUND KERNEL ON HARDY AND HERZ SPACES

LIU LANZHE

Abstract. We prove the continuity of some multilinear operators generated by singular integral operators with variable Calderón-Zygmund kernel and Lipschitz functions on some Hardy and Herz-type spaces.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$. By the classical result of Coifman, Rochberg and Weiss [8], wthe commutator $[b,T]$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \le 1$. But, the boundedness hold if b belongs to the Lipschitz spaces $Lip_\beta(R^n)$ [15]. This show the difference of $b \in BMO(R^n)$ and $b \in Lip_\beta(R^n)$. In [13] and [19] it is proved that if b is a Lipschitz function then the commutators are $L^p(p > 1)$-bounded. In [1] Calderón and Zygmund introduced some singular integral operators with variable kernel and discuss their boundedness. In [10] the authors obtained the boundedness for the commutators generated by the singular integral operators with variable kernel and BMO functions. In [18] the authors proved the boundedness of the multilinear oscillatory singular integral operators generated by the operators and BMO functions. The purpose of this paper is to study the continuity properties of the multilinear operators generated by the singular integral operators with variable kernel and Lipschitz functions on some Hardy and Herz-type spaces.

Liu L.Z., The continuity of multilinear singular integral operators with variable Calderón–Zygmund kernel on Hardy and Herz Spaces.

Received April 18, 2005, published September 2, 2005.

First, let us introduce some notations (see[11-12],[16-17], [19]). Throughout this paper, Q will denote a cube of R^n with side parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization (see [9],[16]). For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that (see[15])

$$||f||_{Lip_\beta} = \sup_{x,h\in R^n,\ h>0} |f(x+h) - f(x)|/|h|^\beta < \infty.$$

Definition 1. *Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, define $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.*

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_0||^p_{L^q}\right]^{1/p}.$$

Definition 2. *Let $\alpha \in R$, $0 < p, q < \infty$.*

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. *Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if*

1) $Suppa \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$),

2) $||a||_{L^q} \le |B(0, r)|^{-\alpha/n}$,

3) $\int a(x)x^\gamma dx = 0$ for $|\gamma| \le [\alpha - n(1 - 1/q)]$.

Лемма 1 ([17]**.** *] Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α,q)-atoms(or central (α,q)-atoms of restrict type)a_j supported on $B_j = B(0,2^j)$ and constants λ_j with $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(\text{ or } ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

2. Main results

In this paper we study a class of multilinear operators related to the singular integral operators with variable kernel, whose definitions are as follows.

Definition 4. *Let $k(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \longrightarrow R$. k is said to be a Calderón-Zygmund kernel if*

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;

(b) Ω is homogeneous of degree zero;

(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N\bigcup\{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.

Definition 5. *Let $k(x,y) = \Omega(x,y)/|y|^n : R^n \times (R^n \setminus \{0\}) \longrightarrow R$. k is said to be a variable Calderón-Zygmund kernel if*

(d) $k(x,\cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;

(e) $\max_{|\gamma|\leq 2n} \left|\left| \frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x,y)\right|\right|_{L^\infty(R^n\times\Sigma)} = M < \infty$.

Let m_j be the positive integers($j = 1,\cdots,l$), $m_1 + \cdots + m_l = m$ and A_j be the functions on $R^n(j = 1,\cdots,l)$. Denote that

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\gamma|\leq m_j} \frac{1}{\gamma!}D^\gamma A_j(y)(x-y)^\gamma$$

and

$$Q_{m_j+1}(A_j;x,y) = R_{m_j}(A_j;x,y) - \sum_{|\gamma|=m_j} \frac{1}{\gamma!}D^\gamma A_j(x)(x-y)^\gamma.$$

The multilinear singular integral operator with variable Calderón-Zygmund kernel is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\Omega(x,x-y)}{|x-y|^{n+m}} \prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)f(y)dy,$$

where $\Omega(x,y)/|y|^n$ is a variable Calderón-Zygmund kernel. We also define that

$$T(f)(x) = \int_{R^n} \frac{\Omega(x,x-y)}{|x-y|^n} f(y)dy,$$

which is the singular integral operator with variable Calderón-Zygmund kernel (see [1]). We also consider the variant of T^A, which is defined by

$$\tilde{T}^A(f)(x) = \int_{R^n} \frac{\Omega(x,x-y)}{|x-y|^{n+m}} \prod_{j=1}^{l} Q_{m_j+1}(A_j;x,y)f(y)dy.$$

Note that when $m = 0$, T^A is just higher order commutator of the operators T and A (see [10],[13],[19]), while when $m > 0$, it is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$(see [2],[4-7],[9]). In [2] the $L^p(p > 1)$-boundedness of multilinear singular integral operators generated by some singular integrals operators and Lipschitz functions are obtained. The main purpose of this paper is to prove the continuity properties of the multilinear singular integral operators with variable Calderón-Zygmund kernel on Hardy and Herz-type spaces.

We shall prove the following theorems in Section 3.

Theorem 1. *Let $D^\gamma A_j \in Lip_\beta(R^n)$ for all γ with $|\gamma| = m_j$ and $j = 1, \cdots, l$.*

(a) If $0 < \beta \le 1$, $n/(n + \beta) < p \le 1$ and $1/p - 1/q = l\beta/n$, then T^A maps $H^p(R^n)$ continuously into $L^q(R^n)$;

(b) If $0 < \beta \le 1/l$, then $\tilde{T}^A$ maps $H^{n/(n+l\beta)}(R^n)$ continuously into $L^1(R^n)$.

Theorem 2. *Let $D^\gamma A_j \in Lip_\beta(R^n)$ for all γ with $|\gamma| = m_j$ and $j = 1, \cdots, l$.*

(i) If $0 < \beta \le 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = l\beta/n$ and $n(1 - 1/q_1) \le \alpha < n(1 - 1/q_1) + l\beta$, then T^A maps $H\dot{K}^{\alpha,p}_{q_1}(R^n)$ continuously into $\dot{K}^{\alpha,p}_{q_2}(R^n)$;

(ii) If $0 < \beta \le 1/l$, $0 < p \le 1$, $1 < q_1, q_2 < \infty$ and $1/q_1 - 1/q_2 = l\beta/n$, then $\tilde{T}^A$ maps $H\dot{K}^{n(1-1/q_1)+l\beta,p}_{q_1}(R^n)$ continuously into $\dot{K}^{n(1-1/q_1)+l\beta,p}_{q_2}(R^n)$.

Remark. Theorem 2 also hold for the nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of main results

We begin with a preliminary lemma.

Лемма 2 ([6]**.** *] Let A be a function on R^n such that $D^\gamma A \in L^q_{loc}(R^n)$ for $|\gamma| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\gamma|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\gamma A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Proof of Theorem 1.

(a) It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a, there is

$$||T^A(a)||_{L^q} \le C.$$

Without loss of generality, we may assume $l = 2$. Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, d)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int a(x)x^\eta dx = 0$ for $|\eta| \le [n(1/p - 1)]$. We write

$$\int_{R^n} |T^A(a)(x)|^q dx = \left(\int_{|x-x_0|\le 2d} + \int_{|x-x_0|>2d} \right) |T^A(a)(x)|^q dx = I_1 + I_2.$$

For I_1, an analog of the proof in [14], it follows that

$$|T^A(f)(x)| \le C \prod_{j=1}^{2} \left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j} A_j||_{Lip_\beta} \right) I_{2\beta}(|f|)(x),$$

where I_μ is the fractional integral operator of order μ, thus T^A is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any r, s with $1 < r < n/2\beta$ and $1/r - 1/s = 2\beta/n$ by [20]. Taking $q_1 > q$ and $1 < p_1 < n/2\beta$ such that $1/p_1 - 1/q_1 = 2\beta/n$, by Hölder's inequality and the (L^{p_1}, L^{q_1}) -boundedness of T^A, we have

$$I_1 \le C||T^A(a)||_{L^{q_1}}^q |2Q|^{1-q/q_1} \le C||a||_{L^{p_1}}^q |Q|^{1-q/q_1} \le C.$$

To estimate I_2, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. By [3], we know that

$$\begin{aligned} T^A(f)(x) &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x)\int_{R^n}\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}\prod_{j=1}^{2} R_{m_j+1}(A_j;x,y)f(y)dy \\ &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x)U_{hk}^A(f)(x), \end{aligned}$$

where $g_k \le Ck^{n-2}$, $||a_{hk}||_{L^\infty} \le Ck^{-2n}$, $|Y_{hk}(x-y)| \le Ck^{n/2-1}$ and

$$\left|\frac{Y_{hk}(x-y)}{|x-y|^n} - \frac{Y_{hk}(x-x_0)}{|x-x_0|^n}\right| \le Ck^{n/2}|x_0-y|/|x-x_0|^{n+1}$$

for $|x-x_0| > 2|x_0-y| > 0$. Let $\tilde{A}_j(x) = A_j(x) - \sum_{|\gamma|=m_j}\frac{1}{\gamma!}(D^\gamma A_j)_Q x^\gamma$. Then $R_{m_j}(A_j;x,y) = R_{m_j}(\tilde{A}_j;x,y)$ and $D^\gamma \tilde{A}_j = D^\gamma A_j - (D^\gamma A_j)_Q$ for $|\gamma| = m_j$. We write, by the vanishing moment of a,

$$\begin{aligned} &U_{hk}^A(a)(x) = \\ &\int_{R^n}\left[\frac{Y_{hk}(x-y)}{|x-y|^{m+n}} - \frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}\right] R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)a(y)dy+ \\ &+\int_{R^n}\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}[R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x,x_0)]R_{m_2}(\tilde{A}_2;x,y)a(y)dy+ \\ &+\int_{R^n}\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}[R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x,x_0)]R_{m_1}(\tilde{A}_1;x,x_0)a(y)dy- \\ &-\sum_{|\gamma_2|=m_2}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_2}}{|x-y|^{m+n}} - \frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_2}}{|x-x_0|^{m+n}}\right]\times \\ &\times R_{m_1}(\tilde{A}_1;x,y)D^{\gamma_2}\tilde{A}_2(y)a(y)dy- \\ &-\sum_{|\gamma_2|=m_2}\int_{R^n}\frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_2}}{|x-x_0|^{m+n}}[R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x,x_0)]\times \\ &\times D^{\gamma_2}\tilde{A}_2(y)a(y)dy- \\ &-\sum_{|\gamma_1|=m_1}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_1}}{|x-y|^{m+n}} - \frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_1}}{|x-x_0|^{m+n}}\right]\times \\ &\times R_{m_2}(\tilde{A}_2;x,y)D^{\gamma_1}\tilde{A}_1(y)a(y)dy- \\ &-\sum_{|\gamma_1|=m_1}\int_{R^n}\frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_1}}{|x-x_0|^{m+n}}[R_{m_1}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x,x_0)]\times \\ &\times D^{\gamma_1}\tilde{A}_1(y)a(y)dy+ \\ &+\sum_{|\gamma_1|=m_1,|\gamma_2|=m_2}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_1+\gamma_2}}{|x-y|^{m+n}} - \frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_1+\gamma_2}}{|x-x_0|^{m+n}}\right]\times \\ &\times D^{\gamma_1}\tilde{A}_1(y)D^{\gamma_2}\tilde{A}_2(y)a(y)dy. \end{aligned}$$

By Lemma 2 and the following inequality

$$|b(x) - b_Q| \leq \frac{1}{|Q|}\int_Q ||b||_{Lip_\beta}|x-y|^\beta dy \leq ||b||_{Lip_\beta}(|x-x_0|+d)^\beta,$$

we get

$$|R_{m_j}(\tilde{A}_j;x,y)| \leq \sum_{|\gamma|=m_j} ||D^\gamma A_j||_{Lip_\beta}(|x-y|+d)^{m_j+\beta};$$

On the other hand, by the formula (see [6]):

$$R_{m_j}(\tilde{A}_j;x,y) - R_{m_j}(\tilde{A}_j;x,x_0) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m_j-|\eta|}(D^\eta \tilde{A}_j;x_0,y)(x-x_0)^\eta,$$

note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus 2Q$, we obtain

$$|U^A_{hk}(a)(x)(x)| \leq Ck^{n/2}\prod_{i=j}^{2}\left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{Lip_\beta}\right)\times$$

$$\times\int_Q\left[\frac{|y-x_0|}{|x-x_0|^{n+1-2\beta}} + \frac{|y-x_0|^\beta}{|x-x_0|^{n-\beta}} + \frac{|y-x_0|^{2\beta}}{|x-x_0|^n}\right]|a(y)|dy \leq$$

$$\leq Ck^{n/2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j} ||D^{\gamma_j}A_j||_{Lip_\beta}\right)\times$$

$$\times\left[\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-2\beta}} + \frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\beta}} + \frac{|Q|^{2\beta/n+1-1/p}}{|x-x_0|^n}\right];$$

Thus

$$\begin{aligned}
|T^A(f)(x)| &\leq C\sum_{k=1}^\infty\sum_{h=1}^{g_k}|a_{hk}(x)|k^{n/2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\\
&\quad\times\left[\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-2\beta}} + \frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\beta}} + \frac{|Q|^{2\beta/n+1-1/p}}{|x-x_0|^n}\right]\\
&\leq C\sum_{k=1}^\infty k^{-2n+n/2+n-2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\\
&\quad\times\left[\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-2\beta}} + \frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\beta}} + \frac{|Q|^{2\beta/n+1-1/p}}{|x-x_0|^n}\right]\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\\
&\quad\times\left[\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-2\beta}} + \frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\beta}} + \frac{|Q|^{2\beta/n+1-1/p}}{|x-x_0|^n}\right],
\end{aligned}$$

and recall that $n/(n+\beta) < p \leq 1$, $1/p - 1/q = l\beta/n$, we obtain

$$I_2 \leq \sum_{i=1}^\infty \int_{2^{i+1}Q\setminus 2^iQ}|T^A(a)(x)|^q dx \leq$$

$$\leq C\left[\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\right]^q\sum_{i=1}^{\infty}\left[2^{iqn(1/p-(n+1)/n)}+2^{iqn(1/p-(n+\beta)/n)}\right]\leq$$

$$\leq C\left[\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\right]^q\leq C,$$

which together with the estimate for I_1 yields the desired result.

(b) Without loss of generality, we may assume $l=2$. It is only to prove that there exists a constant $C>0$ such that for every $H^{n/(n+2\beta)}$-atom a supported on $Q=Q(x_0,d)$, there is

$$|\tilde{T}^A(a)||_{L^1}\leq C.$$

We write

$$\int_{R^n}|\tilde{T}^A(a)(x)|dx=\left[\int_{|x-x_0|\leq 2d}+\int_{|x-x_0|>2d}\right]|\tilde{T}^A(a)(x)|dx:=J_1+J_2.$$

For J_1, by the following equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)-\sum_{|\gamma|=m}\frac{1}{\gamma!}(x-y)^\gamma(D^\gamma A(x)-D^\gamma A(y)),$$

we get

$$|\tilde{T}^A(a)(x)|\leq C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)I_{2\beta}(|a|)(x),$$

thus, $\tilde{T}^A$ is bounded from $L^r(R^n)$ to $L^s(R^n)$ for any r, s with $1<r<n/2\beta$ and $1/r-1/s=2\beta/n$ by [20]. We get, for $1<p<n/2\beta$ and $1/q=1/p-2\beta/n$, we

$$J_1\leq C||\tilde{T}^A(a)||_{L^q}|2Q|^{1-1/q}\leq C||a||_{L^p}|Q|^{1-1/q}\leq C.$$

Let us obtain the estimate for J_2. Put

$$\tilde{A}_j(x)=A_j(x)-\sum_{|\gamma|=m_j}\frac{1}{\gamma!}(D^\gamma A_j)_{2Q}x^\gamma.$$

Then $Q_{m_j}(A_j;x,y)=Q_{m_j}(\tilde{A}_j;x,y)$ and

$$Q_{m_j+1}(A_j;x,y)=R_{m_j}(A_j;x,y)-\sum_{|\gamma|=m_j}\frac{1}{\gamma!}D^\gamma A_j(x)(x-y)^\gamma.$$

By [3], we know that

$$\begin{aligned}\tilde{T}^A(f)(x) &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k}a_{hk}(x)\int_{R^n}\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}\prod_{j=1}^{2}Q_{m_j+1}(A_j;x,y)f(y)dy\\ &=\sum_{k=1}^{\infty}\sum_{h=1}^{g_k}a_{hk}(x)V_{hk}^A(f)(x),\end{aligned}$$

we write, by the vanishing moment of a and for $x\in(2Q)^c$,

$$V_{hk}^A(a)(x)=$$

$$\int_{R^n}\left[\frac{Y_{hk}(x-y)}{|x-y|^{m+n}}-\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}\right]R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)a(y)dy+$$

$$+\int_{R^n}\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}[R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x,x_0)]R_{m_2}(\tilde{A}_2;x,y)a(y)dy+$$
$$+\int_{R^n}\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}[R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x,x_0)]R_{m_1}(\tilde{A}_1;x,x_0)a(y)dy-$$
$$-C\sum_{|\gamma_2|=m_2}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_2}}{|x-y|^{m+n}}-\frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_2}}{|x-x_0|^{m+n}}\right]\times$$
$$\times R_{m_1}(\tilde{A}_1;x,y)D^{\gamma_2}\tilde{A}_2(x)a(y)dy-$$
$$-C\sum_{|\gamma_2|=m_2}\int_{R^n}\frac{Y_{hk}(x-y)(x-x_0)^{\gamma_2}}{|x-x_0|^{m+n}}[R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x,x_0)]\times$$
$$\times D^{\gamma_2}\tilde{A}_2(x)a(y)dy-$$
$$-C\sum_{|\gamma_1|=m_1}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_1}}{|x-y|^{m+n}}-\frac{Y_{hk}(x-x_0)(x-x_0)^{\gamma_1}}{|x-x_0|^{m+n}}\right]\times$$
$$\times R_{m_2}(\tilde{A}_2;x,y)D^{\gamma_1}\tilde{A}_1(x)a(y)dy-$$
$$-C\sum_{|\gamma_1|=m_1}\int_{R^n}\frac{Y_{hk}(x-y)(x-x_0)^{\gamma_1}}{|x-x_0|^{m+n}}[R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x,x_0)]\times$$
$$\times D^{\gamma_1}\tilde{A}_1(x)a(y)dy+$$
$$+C\sum_{|\gamma_1|=m_1,|\gamma_2|=m_2}\int_{R^n}\left[\frac{Y_{hk}(x-y)(x-y)^{\gamma_1+\gamma_2}}{|x-y|^{m+n}}-\frac{Y_{hk}(x-y)(x-x_0)^{\gamma_1+\gamma_2}}{|x-x_0|^{m+n}}\right]\times$$
$$\times D^{\gamma_1}\tilde{A}_1(x)D^{\gamma_2}\tilde{A}_2(x)a(y)dy.$$

Then as in the proof of (a) we obtain

$$\begin{aligned}|V_{hk}^A(a)(x)| &\le Ck^{n/2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\int_Q\frac{|y-x_0|}{|x-x_0|^{n+1-2\beta}}|a(y)|dy\\ &\le Ck^{n/2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\frac{|Q|^{(1-2\beta)/n}}{|x-x_0|^{n+1-2\beta}},\end{aligned}$$

thus

$$\begin{aligned}|\tilde{T}^A(a)(x)| &\le C\sum_{k=1}^{\infty}k^{-2n+n/2+n-2}\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\frac{|Q|^{(1-2\beta)/n}}{|x-x_0|^{n+1-2\beta}}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\frac{|Q|^{(1-2\beta)/n}}{|x-x_0|^{n+1-2\beta}},\end{aligned}$$

and

$$J_2\le C\prod_{j=1}^{2}\left(\sum_{|\gamma_j|=m_j}||D^{\gamma_j}A_j||_{Lip_\beta}\right)\sum_{i=1}^{\infty}2^{i(2\beta-1)}\le C,$$

which together with the estimate for J_1 yields the desired result. This completes the proof of Theorem 1.

Proof of Theorem 2.

(i) Without loss of generality, we may assume $l = 2$. Let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned} \|T^A(f)\|_{\dot{K}_{q_2}^{\alpha,p}}^p &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|T^A(a_j)\chi_k\|_{L^{q_2}} \right)^p \\ &\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|T^A(a_j)\chi_k\|_{L^{q_2}} \right)^p \\ &= L_1 + L_2. \end{aligned}$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of T^A, we have

$$L_2 \le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}} \right)^p \le$$

$$\left\{ \begin{array}{ll} C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right), & 0 < p \le 1 \\ C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2} \right)^{p/p'}, & p > 1 \end{array} \right. \le$$

$$\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C\|f\|_{H\dot{K}_{q_1}^{\alpha,p}}^p.$$

For L_1 as in the proof of Theorem 1 (a) for $x \in C_k$ and $j \le k-3$ we obtain

$$\begin{aligned} |T^A(a_j)(x)| &\le C\left(\frac{|B_j|^{1/n}}{|x|^{n+1-2\beta}} + \frac{|B_j|^{\beta/n}}{|x|^{n-\beta}} + \frac{|B_j|^{2\beta/n}}{|x|^n} \right) \int_{R^n} |a_j(y)|dy \\ &\le C\left(\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{|x|^{n+1-2\beta}} + \frac{2^{j(\beta+n(1-1/q_1)-\alpha)}}{|x|^{n-\beta-n}} \right), \end{aligned}$$

thus

$$\|T^A(a_j)\chi_k\|_{L^{q_2}} \le C2^{-k\alpha} \left(2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \right);$$

To be simply, denote $W(j,k) = 2^{(j-k)(1+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\beta+n(1-1/q_1)-\alpha)}$ and recall that $\alpha < n(1-1/q_1) + \beta$, then

$$L_1 \le C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| W(j,k) \right)^p \le$$

$$\le \left\{ \begin{array}{ll} C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} W(j,k)^p, & 0 < p \le 1 \\ C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left[\sum_{k=j+3}^{\infty} W(j,k)^{p/2} \right] \left[\sum_{k=j+3}^{\infty} W(j,k)^{p'/2} \right]^{p/p'}, & p > 1 \end{array} \right. \le$$

$$\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C\|f\|_{H\dot{K}_{q_1}^{\alpha,p}}^p.$$

These yield the desired result.

(ii) Without loss of generality, we assume $l = 2$. Let $f \in H\dot{K}_{q_1}^{n(1-1/q_1)+2\beta,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. Write

$$||\tilde{T}^A(f)||^p_{\dot{K}_{q_2}^{n(1-1/q_1)+2\beta,p}} \leq \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| ||\tilde{T}^A(a_j)\chi_k||_{L^{q_2}} \right)^p +$$

$$+ \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||\tilde{T}^A(a_j)\chi_k||_{L^{q_2}} \right)^p = M_1 + M_2.$$

For M_2, by the (L^{q_1}, L^{q_2}) boundedness of $\tilde{T}^A$, we get

$$\begin{aligned} M_2 &\leq C \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}} \right)^p \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)p(n(1-1/q_1)+2\beta)} \right) \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C||f||^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+2\beta,p}}. \end{aligned}$$

For M_1, as in the proof of Theorem 1 (b), we obtain

$$\begin{aligned} |\tilde{T}^A(a)(x)| &\leq C \frac{|B_j|^{1/n}}{|x|^{n+1-2\beta}} \int_{R^n} |a_j(y)| dy \\ &\leq C \frac{2^{j(1-2\beta)}}{|x|^{n+1-2\beta}}. \end{aligned}$$

for $x \in C_k$ and $j \leq k-3$. Thus

$$\begin{aligned} M_1 &\leq C \sum_{k=-\infty}^{\infty} 2^{kp(n(1-1/q_1)+2\beta)} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p \frac{2^{j(1-2\beta)}}{2^{k(n+1-2\beta)}} \right)^p 2^{knp/q_2} \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{p(1-2\beta)(j-k)} \\ &\leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C||f||^p_{H\dot{K}_{q_1}^{n(1-1/q_1)+2\beta,p}}. \end{aligned}$$

These yield the desired result and finish the proof of Theorem 2.

References

[1] A. P. Calderón and A. Zygmund, *On singular integrals with variable kernels*, Appl. Anal., **7**(1978), 221–238.
[2] W. G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, **16**(2000), 613–626.
[3] F. Chiarenza, M. Frasca and P. Longo, *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat., **40**(1991), 149–168.
[4] J. Cohen, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J., **30**(1981), 693–702.

[5] J. Cohen and J. Gosselin, *On multilinear singular integral operators on R^n*, Studia Math., **72**(1982), 199–223.
[6] J. Cohen and J. Gosselin, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30**(1986), 445–465.
[7] R. Coifman and Y. Meyer, *Wavelets, Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Math., 48, Cambridge University Press, Cambridge , 1997.
[8] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103**(1976), 611–635.
[9] Y. Ding and S. Z. Lu, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica, **17**(2001), 517–526.
[10] G. Di Fazio and M. A. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Func. Anal., **112**(1993), 241–256.
[11] J. Garcia-Cuerva and M. L. Herrero, *A theory of Hardy spaces associated to the Herz spaces*, Proc. London Math. Soc., **69**(1994), 605–628.
[12] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math., 16, Amsterdam, 1985.
[13] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math., **16**(1978), 263–270.
[14] S. Z. Lu, H. X. Wu and P. Zhang, *Multilinear singular integrals with rough kernel*, Acta Math. Sinica, **19**(2003), 51–62.
[15] S. Z. Lu, Q. Wu and D. C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. in China (ser. A), **45**(2002), 984–997.
[16] S. Z. Lu and D. C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (ser. A), **38**(1995), 147–158.
[17] S. Z. Lu and D. C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. in China (ser. A), **38**(1995), 662–673.
[18] S. Z. Lu, D. C. Yang and Z. S. Zhou, *Oscillatory singular integral operators with Calderón-Zygmund kernels*, Southeast Asian Bull. of Math., **23**(1999), 457–470.
[19] M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman*, Rochberg and Weiss, Indiana Univ. Math. J., **44**(1995), 1–17.
[20] E. M. Stein, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

Liu Lanzhe
College of Mathematics,
Changsha University of Science and Technology,
Changsha 410077, P.R. of China
E-mail address: lanzheliu@263.net