Séminaires & Congrès
11, 2005, p. 171–178

EXPLICIT UPPER BOUNDS FOR THE RESIDUES AT $s = 1$ OF THE DEDEKIND ZETA FUNCTIONS OF SOME TOTALLY REAL NUMBER FIELDS

by

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Abstract. — We give an explicit upper bound for the residue at $s = 1$ of the Dedekind zeta function of a totally real number field K for which $\zeta_K(s)/\zeta(s)$ is entire. Notice that this is conjecturally always the case, and that it holds true if $K/\mathbf{Q}$ is normal or if K is cubic.

Résumé **(Bornes supérieures explicites pour les résidus en $s = 1$ des fonctions zêta de Dedekind de corps de nombres totalement réels)**
Nous donnons une borne supérieure explicite pour le résidu en $s = 1$ de la fonction zêta de Dedekind d'un corps de nombres K totalement réel pour lequel $\zeta_K(s)/\zeta(s)$ est entière. On remarque que c'est conjecturalement toujours le cas, et que c'est vrai si $K/\mathbf{Q}$ est normale ou si K est cubique.

1. Introduction

Let d_K and $\zeta_K(s)$ denote the absolute value of the discriminant and the Dedekind zeta function of a number field K of degree $m > 1$. It is important to have explicit upper bounds for the residue at $s = 1$ of $\zeta_K(s)$. As for the best general such bounds, we have (see [**Lou01**, Theorem 1]):

$$\mathrm{Res}_{s=1}(\zeta_K(s)) \leqslant \left(\frac{e \log d_K}{2(m-1)}\right)^{m-1}.$$

However, for some totally real number fields an improvement on this bound is known (see [**BL**] and [**Oka**] for applications):

Theorem 1 **(See [Lou01, Theorem 2]).** — *Let K range over a family of totally real number fields of a given degree $m \geqslant 3$ for which $\zeta_K(s)/\zeta(s)$ is entire. There exists C_m*

2000 ***Mathematics Subject Classification.*** — 11R42.
Key words and phrases. — Dedekind zeta function.

(computable) such that $d_K \geqslant C_m$ implies

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leqslant \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} \leqslant \frac{1}{\sqrt{2\pi(m-1)}}\left(\frac{e\log d_K}{2(m-1)}\right)^{m-1}.$$

Moreover, for any non-normal totally real cubic field K we have the slightly better bound

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leqslant \frac{1}{8}(\log d_K - \kappa)^2$$

where $\kappa := 2\log(4\pi) - 2 - 2\gamma = 1.90761\dots$.

Remark 2. — If $K/\mathbf{Q}$ is normal or if K is cubic, then $\zeta_K(s)/\zeta(s)$ is entire.

We will simplify our previous proof of Theorem 1 (by improving those of [**Lou98**, Theorem 5] and [**Lou01**, Theorem 2]) and we will give explicit constants C_m for which Theorem 1 holds true:

Theorem 3. — *There exists $C > 0$ (effective) such that for any totally real number field K of degree $m \geqslant 3$ and root discriminant $\rho_K := d_K^{1/m} \geqslant C^m$ we have*

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leqslant \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!},$$

provided that $\zeta_K(s)/\zeta(s)$ is entire. Moreover, $C = 3309$ will do for m large enough.

This result is not the one we would have wished to prove. It would indeed have been much more satisfactory to prove that there exists $C > 0$ (effective) such that this bound is valid for such totally real number fields K of root discriminants $\rho_K \geqslant C$ large enough. It would have been even more satisfactory to prove that this constant C is small enough to obtain that our bound is valid for all totally real number fields K for which $\zeta_K(s)/\zeta(s)$ is entire (e.g., see [**Was**, Page 224] for explicit lower bounds on root discriminants of totally real number fields K). Let us finally point out that, in the case that $K/\mathbf{Q}$ is abelian, we have an even better bound (see [**Lou01**, Corollary 8] and use [**Ram**, Corollary 1]):

$$\operatorname{Res}_{s=1}(\zeta_K(s)) \leqslant \left(\frac{\log d_K}{2(m-1)}\right)^{m-1}.$$

2. Proof of Theorem 1

Proposition 4. — *Let K be a totally real number field of degree $m \geqslant 1$, set $d = \sqrt{d_K}$, and assume that $\zeta_K(s)/\zeta(s)$ is entire. Then, $\operatorname{Res}_{s=1}(\zeta_K(s)) \leqslant \rho_{m-1}(d)$ where*

$$\rho_{m-1}(d) := \operatorname{Res}_{s=1}\Big\{s \mapsto \big(\pi^{-s/2}\Gamma(s/2)\zeta(s)\big)^{m-1}\Big(\frac{1}{s}+\frac{1}{s-1}\Big)(d^{s-1}+d^{-s})\Big\}. \tag{1}$$

Proof. — To begin with, we set some notation: if K is a totally real number field of degree $m \geqslant 1$, we set $A_K = \sqrt{d_K/\pi^m}$ and $F_K(s) = A_K^s\Gamma^m(s/2)\zeta_K(s)$. Hence, $F_K(s)$ is meromorphic, with only two poles, at $s=1$ and $s=0$, both simple, and it satisfies the functional equation $F_K(1-s) = F_K(s)$.

We then set $F_{K/\mathbf{Q}}(s) = F_K(s)/F_{\mathbf{Q}}(s)$, which under our assumption is entire, and satisfies the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$, and $A_{K/\mathbf{Q}} := A_K/A_{\mathbf{Q}} = \sqrt{d_K/\pi^{m-1}}$. Notice that $F_{K/\mathbf{Q}}(1) = \sqrt{d_K}\operatorname{Res}_{s=1}(\zeta_K(s))$. Let

$$S_{K/\mathbf{Q}}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{K/\mathbf{Q}}(s)x^{-s}ds \quad (c>1 \text{ and } x>0) \tag{2}$$

denote the Mellin transform of $F_{K/\mathbf{Q}}(s)$. Since $F_{K/\mathbf{Q}}(s)$ is entire, it follows that $S_{K/\mathbf{Q}}(x)$ satisfies the functional equation

$$S_{K/\mathbf{Q}}(x) = \frac{1}{x}S_{K/\mathbf{Q}}\Big(\frac{1}{x}\Big) \tag{3}$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (2) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, then use the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$), and

$$F_{K/\mathbf{Q}}(s) = \int_0^\infty S_{K/\mathbf{Q}}(x)x^s\frac{dx}{x} = \int_1^\infty S_{K/\mathbf{Q}}(x)(x^s+x^{1-s})\frac{dx}{x} \tag{4}$$

is the inverse Mellin transform of $S_{K/\mathbf{Q}}(x)$.

Now, set

$$\begin{aligned} F_{m-1}(s) &= F_{\mathbf{Q}}^{m-1}(s) = \big(\pi^{-s/2}\Gamma(s/2)\zeta(s)\big)^{m-1},\\ A_{m-1} &= A_{\mathbf{Q}}^{m-1} = \pi^{-(m-1)/2}\end{aligned} \tag{5}$$

and let

$$S_{m-1}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{m-1}(s)x^{-s}ds \quad (c>1 \text{ and } x>0) \tag{6}$$

denote the Mellin transform of $F_{m-1}(s)$. Here, $F_{m-1}(s)$ has two poles, at $s=1$ and $s=0$, the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ yields

$$\operatorname{Res}_{s=0}(F_{m-1}(s)x^{-s}) = -\operatorname{Res}_{s=1}(F_{m-1}(s)x^{s-1})$$

and

$$S_{m-1}(x) = \operatorname{Res}_{s=1}\{F_{m-1}(s)(x^{-s}-x^{s-1})\} + \frac{1}{x}S_{m-1}\Big(\frac{1}{x}\Big) \tag{7}$$

(shift the vertical line of integration $\Re(s) = c > 1$ in (6) leftwards to the vertical line of integration $\Re(s) = 1-c < 0$, notice that you pick up residues at $s=1$ and $s=0$, then use the functional equation $F_{m-1}(1-s) = F_{m-1}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). Finally, we set

$$H_{m-1}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma^{m-1}(s/2)x^{-s}ds \quad (c>1 \text{ and } x>0).$$

Notice that $0 < H_{m-1}(x)$ for $x > 0$ (see [**Lou00**, Proof of Theorem 2][(1)]). Now, write

$$\zeta_K(s)/\zeta(s) = \sum_{n\geqslant 1} a_{K/\mathbf{Q}}(n)n^{-s}$$

and

$$\zeta^{m-1}(s) = \sum_{n\geqslant 1} a_{m-1}(n)n^{-s}.$$

Then, $|a_{K/\mathbf{Q}}(n)| \leqslant a_{m-1}(n)$ for all $n \geqslant 1$ (see [**Lou01**, Lemma 26]). Since

$$S_{K/\mathbf{Q}}(x) = \sum_{n\geqslant 1} a_{K/\mathbf{Q}}(n)H_{m-1}(nx/A_{K/\mathbf{Q}})$$

and

$$0 \leqslant S_{m-1}(x) = \sum_{n\geqslant 1} a_{m-1}(n)H_{m-1}(nx/A_{m-1}),$$

we obtain

$$S_{K/\mathbf{Q}}(x) \leqslant S_{m-1}(x/d) \quad \text{with } d := A_{K/\mathbf{Q}}/A_{m-1} = \sqrt{d_K}. \tag{8}$$

We are now ready to proceed with the proof of Proposition 4. We have

$$\begin{aligned}
d\,\mathrm{Res}_{s=1}(\zeta_K(s)) = F_{K/\mathbf{Q}}(1) &= \int_1^\infty S_{K/\mathbf{Q}}(x)\Big(1+\frac{1}{x}\Big)dx \qquad \text{(by (4))}\\
&\leqslant \int_1^\infty S_{m-1}(x/d)\Big(1+\frac{1}{x}\Big)dx \qquad \text{(by (8))}\\
&= \int_{1/d}^\infty S_{m-1}(x)\Big(d+\frac{1}{x}\Big)dx\\
&= \int_1^\infty S_{m-1}(x)\Big(d+\frac{1}{x}\Big)dx + \int_1^d \frac{1}{x}S_{m-1}\Big(\frac{1}{x}\Big)\Big(\frac{d}{x}+1\Big)dx\\
&\leqslant (d+1)\int_1^\infty S_{m-1}(x)\Big(1+\frac{1}{x}\Big)dx\\
&\qquad -\int_1^d \mathrm{Res}_{s=1}\{F_{m-1}(s)(x^{-s}-x^{s-1})\}\Big(\frac{d}{x}+1\Big)dx\\
&\text{(by (7), and for } S_{m-1}(x)\geqslant 0 \text{ for } x>0)
\end{aligned}$$

(1)Notice the misprints in [**Lou00**, page 273, line 1] and [**Lou01**, Theorem 20] where one should read

$$(M_1 \star M_2)(x) = \int_0^\infty M_1(x/t)M_2(t)\frac{dt}{t}.$$

$$= (d+1)\int_1^\infty S_{m-1}(x)\Big(1+\frac{1}{x}\Big)dx$$
$$- \operatorname{Res}_{s=1}\Big\{F_{m-1}(s)\int_1^d (x^{-s}-x^{s-1})\Big(\frac{d}{x}+1\Big)dx\Big\}$$

(compute these residues as contour integrals along a circle of center 1 and of small radius, and use Fubini's theorem)

$$= (d+1)\left(\int_1^\infty S_{m-1}(x)\Big(1+\frac{1}{x}\Big)dx - \operatorname{Res}_{s=1}\Big\{F_{m-1}(s)\Big(\frac{1}{s}+\frac{1}{s-1}\Big)\Big\}\right)$$
$$+ \operatorname{Res}_{s=1}\Big\{F_{m-1}(s)(d^s+d^{1-s})\Big(\frac{1}{s}+\frac{1}{s-1}\Big)\Big\}.$$

The desired result now follows from Lemma 5 below. □

Lemma 5. — *Set*

$$G_{m-1}(s) := F_{m-1}(s)\Big(\frac{1}{s}+\frac{1}{s-1}\Big).$$

Then,

$$I_{m-1} := \int_1^\infty S_{m-1}(x)\Big(1+\frac{1}{x}\Big)dx = \operatorname{Res}_{s=1}(G_{m-1}(s)).$$

Proof. — By (6) and Fubini's theorem, we have

$$I_{m-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{m-1}(s)\Big(\int_1^\infty (x^{-s}+x^{-s-1})dx\Big)ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_{m-1}(s)ds$$

The functional equation $G_{m-1}(1-s) = -G_{m-1}(s)$ yields

$$I_{m-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G_{m-1}(s)ds$$
$$= \operatorname{Res}_{s=1}(G_{m-1}(s)) + \operatorname{Res}_{s=0}(G_{m-1}(s)) + \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} G_{m-1}(s)ds$$
$$= 2\operatorname{Res}_{s=1}(G_{m-1}(s)) - I_{m-1},$$

from which the desired result follows. □

Let us now complete the proof of Theorem 1. Since

(9) $$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - a + O(s-1),$$

with $a = (\log(4\pi)-\gamma)/2 = 0.97690\ldots$, using (1) we obtain

$$\rho_{m-1}(d) = \frac{1}{(m-1)!}\log^{m-1} d - \frac{c_{m-1}}{(m-2)!}\log^{m-2} d + O(\log^{m-3} d)$$

with $c_{m-1} := (m-1)a - 1 > 0$ for $m \geqslant 3$, and the desired first result follows. In the special case $m=3$, in writing

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - a + b(s-1) + O((s-1)^2),$$

with $b = 1.00024\dots$, and in setting $\kappa' = \kappa/2 := 2a-1 = \log(4\pi)-1-\gamma = 0.95380\dots$ and $\kappa'' = 3+2a^2-4b = 0.90769\dots$, we have

$$\rho_2(d) = \frac{1}{2}\big((\log d - \kappa')^2 - \kappa''\big) + \frac{1}{2d}\big((\log d + \kappa')^2 - \kappa''\big) \leqslant \frac{1}{2}(\log d - \kappa')^2$$

for $(d+1)\kappa'' \geqslant (\log d + \kappa')^2$, hence for $d = \sqrt{d_K} \geqslant \sqrt{148}$ (notice that 148 is the least discriminant of a non-normal totally real cubic field).

3. Proof of Theorem 3

Set $\kappa_K := \mathrm{Res}_{s=1}(\zeta_K(s))$, $d = \sqrt{d_K}$, $g(t) = \sum_{n\geqslant 1} e^{-\pi n^2 t}$ $(t > 0)$ and $\Lambda(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. We have

$$\Lambda(s) = 1 + s(s-1)\int_1^\infty g(t)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t} \tag{10}$$

(see [**Lan**, Page 250])[(2)] and

$$H_{m-1}(s) = \frac{2s-1}{s^m}\Lambda^{m-1}(s).$$

According to Proposition 4, we have

$$\begin{aligned}\kappa_K \leqslant \rho_{m-1}(d) &= \mathrm{Res}_{s=1}\Big\{s \mapsto \frac{1}{(s-1)^m}(d^{s-1} + d^{-s})\frac{2s-1}{s^m}\Lambda^{m-1}(s)\Big\}\\ &= \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\Big|_{s=1}(d^{s-1} + d^{-s})H_{m-1}(s)\\ &= \frac{1}{(m-1)!}\sum_{k=0}^{m-1}\binom{m-1}{k}(\log^{m-1-k} d)\big(1 + \frac{(-1)^{m-1-k}}{d}\big)\frac{\mathrm{d}^k H_{m-1}(s)}{\mathrm{d}s^k}\Big|_{s=1}.\end{aligned}$$

Now, $H_{m-1}(1) = \Lambda(1) = 1$ and

$$H'_{m-1}(1) = 1 - (m-1)(1-\Lambda'(1)) := -c_{m-1} < 0$$

for $m \geqslant 3$ (for $\Lambda'(1) = (2+\gamma-\log(4\pi))/2 = 0.02309\dots$). Using Lemma 6 below, we obtain

$$\begin{aligned}((m-1)!)\cdot\kappa_K \leqslant{} & \big(1 + \frac{(-1)^{m-1}}{d}\big)\log^{m-1} d\\ & - (m-1)\big(1 + \frac{(-1)^{m-2}}{d}\big)c_{m-1}\log^{m-2} d\\ & + ((m-1)!)\cdot\big(1 + \frac{1}{d}\big)\frac{2r+1}{(1-r)r^{m-1}}\Big(\frac{\Lambda(1+r)}{1-r}\Big)^{m-1}\sum_{k=0}^{m-3}\frac{(r\log d)^k}{k!}.\end{aligned}$$

(2) It follows that $\Lambda(s)$ is positive and convex for $s > 0$ (see [**SZ**] for a different proof, and [**Lou00**, Lemma 9] for a stronger result), for (10) yields $\Lambda^{(k)}(s) \geqslant 0$ for $s \geqslant 1/2$ and $k \geqslant 0$, and the functional equation $\Lambda(1-s) = \Lambda(s)$ then yields $(-1)^k\Lambda^{(k)}(s) \geqslant 0$ for $s \leqslant 1/2$.

Now, assume that $d \geqslant \exp(2(m-3)/r)$. Then,

$$\frac{(r\log d)^k}{k!} \leqslant \frac{(r\log d)^{m-3}}{(m-3)!}2^{-(m-3-k)} \text{ for } 0 \leqslant k \leqslant m-3$$

and

$$\begin{aligned}\frac{(m-1)!}{\log^{m-1} d}\kappa_K - 1 \leqslant \frac{1}{d} - (m-1)\Big(1-\frac{1}{d}\Big)c_{m-1}\log^{-1} d \\ + 2(m-1)(m-2)\Big(1+\frac{1}{d}\Big)\frac{2r+1}{(1-r)r^2}\Big(\frac{\Lambda(1+r)}{1-r}\Big)^{m-1}\log^{-2} d,\end{aligned}$$

and this right hand side is clearly negative for $m \geqslant 3$ and $d \geqslant d_m$ large enough. Now, we take

$$r = \frac{2}{(m-1)(1+\Lambda'(1))}$$

(hence, $0 < r < 1$ for $m \geqslant 3$) and we still assume that

$$d \geqslant \exp(2(m-3)/r) = \exp((1+\Lambda'(1))(m-3)(m-1)).$$

We have

$$\frac{2r+1}{(1-r)r^2}\Big(\frac{\Lambda(1+r)}{1-r}\Big)^{m-1} = \frac{1}{4}(1+\Lambda'(1))^2e^2m^2 + O(m)$$

and for any

$$C'\frac{(1+\Lambda'(1))^2e^2}{2(1-\Lambda'(1))} = 4.052168162\ldots$$

we obtain $((m-1)!)\cdot\kappa_K \leqslant \log^{m-1} d$ for $d \geqslant \exp(C'm^2)$ and m large enough, which proves the desired result for any $C = \exp(2C') > 3308.78497\ldots$.

Lemma 6. — *For $k \geqslant 0$ and $0 < r < 1$, it holds that*

$$\Big|\frac{\mathrm{d}^k H_{m-1}(s)}{\mathrm{d}s^k}\Big|_{s=1}\Big| \leqslant \frac{2r+1}{1-r}\Big(\frac{\Lambda(1+r)}{1-r}\Big)^{m-1}\frac{k!}{r^k}.$$

Proof. — Since $H_{m-1}(s)$ is analytic in the half plane $\Re(s) > 0$, for any $r \in (0,1)$ we have

$$\Big|\frac{\mathrm{d}^k H_{m-1}(s)}{\mathrm{d}s^k}\Big|_{s=1}\Big| = \Big|\frac{k!}{2\pi i}\int_{|z-1|=r}\frac{H_{m-1}(z)}{(z-1)^{k+1}}dz\Big| \leqslant \frac{k!}{r^k}\sup_{|z-1|=r}|H_{m-1}(z)|.$$

Since (for $t > 0$) $\sigma \mapsto t^{\sigma/2} + t^{(1-\sigma)/2}$ is convex in $(0,\infty)$, we have

$$\begin{aligned}|t^{z/2} + t^{(1-z)/2}| \leqslant t^{\sigma/2} + t^{(1-\sigma)/2} &\leqslant \max(t^{(1-r)/2} + t^{r/2}, t^{(1+r)/2} + t^{-r/2}) \\ &= t^{(1+r)/2} + t^{-r/2}\end{aligned}$$

for $\sigma = \Re(z)$ and $0 < |1-z| = r < 1$ and $t \geqslant 1$, and using (10) we obtain

$$|\Lambda(z)| \leqslant 1 + (1+r)r\int_1^\infty g(t)(t^{(1+r)/2} + t^{-r/2})\frac{dt}{t} = \Lambda(1+r)$$

and

$$|H_{m-1}(z)| \leqslant \frac{2r+1}{(1-r)^m}\Lambda^{m-1}(1+r)$$

for $0 < |z-1| = r < 1$. □

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