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THREE KEY THEOREMS ON INFINITELY NEAR SINGULARITIES

by

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Abstract. **—** The notion of infinitely near singular points is classical and well understood for plane curves. We generalize the notion to higher dimensions and to develop a general theory, in terms of *idealistic exponents* and certain graded algebras associated with them. We then gain a refined generalization of the classical notion of first characteristic exponents. On the level of technical base in the higher dimensional theory, there are some powerful tools, referred to as *Three Key Theorems*, which are namely *Differentiation Theorem*, *Numerical Exponent Theorem* and *Ambient Reduction Theorem*.

Résumé **(Trois théorèmes-clefs sur les singularités infiniment proches). —** La notion de points singuliers infiniment proches est classique et bien comprise pour les courbes planes. On généralise cette notion aux plus grandes dimensions et on développe une théorie générale, en termes de d'*exposants idéalistes* et certaines algèbres graduées associées. Ainsi on obtient une généralisation raffinée de la notion classique des premiers exposants caractéristiques. Au niveau technique de base dans la théorie de dimension plus grande, on a des outils puissants, appelés les *Trois théorèmes-clefs*. Ce sont *le Théorème de différenciation*, *le Théorème de l'exposant numérique* et *le Théorème de réduction de l'espace ambiant*.

Introduction

The notion of infinitely near singular points is classical and well understood for plane curves. In order to generalize the notion to higher dimensions and to develop a general theory, we introduced the notion of *idealistic exponents* which, in the plane curve case, correspond to the first characteristic exponents. On the level of technical base in the higher dimensional theory, there are some powerful tools, referred to as

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Three Key Theorems, which are namely *Differentiation Theorem*, *Numerical Exponent Theorem* and *Ambient Reduction Theorem*. In this paper the three key theorems will be proven for singular data on an ambient regular scheme of finite type over any perfect field of any characteristics. In the proofs, the role played by differential operators will be ubiquitous and indispensable. The notion and basic properties of differential operators will be reviewed in the first chapter, in a manner that is purely algebraic and abstract. In the last two chapters, we state and prove the *Finite Presentation Theorem* as an application of the Key Theorems. The finite presentation is the first step and is believed by the author to be an important milestone in the development of a general theory of infinitely near singular points, giving an *algebraic presentation of finite type* to the total aggregate of all the trees of infinitely near singular points, geometrically diverse and intricate. The original proof of this theorem is contained in a paper which is going to be published in the Journal of the Korean Mathematical Society, but it is repeated here for the sake of emphasizing how important are the roles played by the key theorems. Technically in this work at least, the general theory of infinitely near singular points in higher dimensions heavily depends upon the use of partial differential operators. This approach is interesting in its own right, for instance as was shown by Jean Giraud in connection with the theory of *maximal contact*, [**3, 4**]. As as final comment, now that the algebraic presentation of finite type is known, the next charming project will be the study of structure theorems of the presentation algebras which contain rich information on the given singular data.

Notation. — Our terminal interest of this paper concerns with schemes of finite type over a perfect base field $\mathbf{k}$, which may have any characteristic. However, our interest beyond this paper will be about schemes of finite type over any excellent Dedeking domain, which will be denoted by $\mathbb{k}$. For examples, $\mathbb{k}$ could be any field or the ring of integers in any algebraic number field. From time to time, however, we choose to work on a more abstract and general scheme when possible and desirable. For instance, our schemes may be finite type over any noetherian ring, denoted by B. This B could be the completion of a local ring of a scheme.

1. Differential operators

For the sake of generality, let R be any commutative B-algebra, where B is a commutative ring, and we first define a left R-algebra by the action of the elements of R on the left:

$$\Omega^{(\mu)}_{R/B} = \big(R \otimes_B R\big)/D_R^{\mu+1}$$

where μ is any non-negative integer and D_R denotes the diagonal ideal in the tensor product, which means the kernel

$$D_R = \operatorname{Ker}\big(R \otimes_B R \longrightarrow R\big)$$

of the map induced by the multiplication law of R. We also have

$$D_R = \{\delta(f) \mid f \in R\} \subset R \otimes_B R, \quad \text{where } \delta(f) \text{ denotes } 1 \otimes f - f \otimes 1$$

The differential operators of orders $\leqslant \mu$ are defined to be the elements of the dual of $\Omega^{(\mu)}_{R/B}$. Namely, they are the elements of

$$\mathrm{Diff}^{(\mu)}_{R/B} = \mathrm{Hom}_R\big(\Omega^{(\mu)}_{R/B}, R\big)$$

We often identify elements of $\mathrm{Diff}^{(\mu)}_{R/B}$ with R-homomorphism from $R \otimes_B R$ to R via the natural homomorphism $R \otimes_B R \to \Omega^{(\mu)}_{R/B}$. In this sense, we have canonical inclusions

$$\mathrm{Diff}^{(\mu)}_{R/B} \subset \mathrm{Diff}^{(\nu)}_{R/B} \quad \text{whenever } \mu \leqslant \nu$$

Accordingly we sometimes write

$$\mathrm{Diff}_{R/B} \quad \text{for} \quad \bigcup_{\forall\, \nu \geqslant 0} \mathrm{Diff}^{(\nu)}_{R/B}$$

Furthermore, an element $\partial \in \mathrm{Diff}^{(\mu)}_{R/B}$ acts on elements of R by

$$f \in R \longmapsto \partial(1 \otimes f) \in R$$

in which sense ∂ will be often viewed as an element of $\mathrm{Hom}_B(R, R)$. It is B-linear but hardly R-linear. When a B-subalgebra S of R is given, we have a natural epimorphism $R \otimes_B R \to R \otimes_S R$ which maps the diagonal ideal of the former to that of the latter. Hence we get epimorphisms $\Omega^{(\mu)}_{R/B} \to \Omega^{(\mu)}_{R/S}, \forall\, \mu$, so that we have canonical monomorphisms $\mathrm{Diff}^{(\mu)}_{R/S} \to \mathrm{Diff}^{(\mu)}_{R/B}$. In this sense, we will often view $\mathrm{Diff}^{(\mu)}_{R/S}$ as an R-submodule of $\mathrm{Diff}^{(\mu)}_{R/B}$.

Lemma 1.1. — *Let T be any multiplicative subset of R. Then, viewing $\Omega^{(\mu)}_{R/B}$ and $\mathrm{Diff}^{(\mu)}_{R/B}$ as left R-modules, we have the following compatibility with localizations by T:*

$$\Omega^{(\mu)}_{(T^{-1}R)/B} = T^{-1}\Omega^{(\mu)}_{R/B}$$

and if $\Omega^{(\mu)}_{R/B}$ is finitely generated as an R-module then

$$\mathrm{Diff}^{(\mu)}_{(T^{-1}R)/B} = T^{-1}\,\mathrm{Diff}^{(\mu)}_{R/B}$$

Proof. — For every $t \in T$, we have $f \otimes 1 + \delta(t) = 1 \otimes t$. Here the multiplication by $f \otimes 1$ on $T^{-1}\Omega^{(\mu)}_{R/B}$ is invertible while that by $\delta(t)$ is nilpotent. Hence the multiplication by $1 \otimes t$ is invertible. Namely $(1 \otimes T)^{-1}\big(T^{-1}\Omega^{(\mu)}_{R/B}\big) = T^{-1}\Omega^{(\mu)}_{R/B}$. Moreover, we have

$$\Omega^{(\mu)}_{(T^{-1}R)/B} = (1 \otimes T)^{-1}(T \otimes 1)^{-1}\Omega^{(\mu)}_{R/B} = (1 \otimes T)^{-1}\big(T^{-1}\Omega^{(\mu)}_{R/B}\big)$$

which proves the first assertion of the lemma. The second assertion is by the commutativity of *Hom* and localizations for finitely generated modules. ∎

Lemma 1.2. — *Let $P = B[z]$ be the polynomial ring of independent variables $z = (z_1, \ldots, z_N)$. Then*

$$\Omega^{(m)}_{P/B} = P[\delta(z)]/\big(\delta(z)\big)^{m+1}P[\delta(z)]$$

which is freely generated as P-module by the images of the monomials of degrees $\leqslant m$ in the independent variables $\delta(z)$ over P. Consequently,

$$\mathrm{Diff}^{(m)}_{P/B} = \sum_{\substack{\alpha\in\mathbb{Z}^N \\ |\alpha|\leqslant m}} P\partial_\alpha$$

where

$$\partial_\alpha z^\beta = \begin{cases} \binom{\beta}{\alpha} z^{\beta-\alpha} & \textit{if } \beta\in\alpha+\mathbb{Z}^N_0 \\ 0 & \textit{if otherwise} \end{cases}$$

Moreover, for $\zeta \in \mathrm{Spec}(P)$ and $A = P/\zeta$, we have

$$\Omega^{(m)}_{A/B} = \Omega^{(m)}_{P/B}\Big/\Big(\zeta\Omega^{(m)}_{P/B} + P\delta(\zeta)\Big)$$

and therefore $\mathrm{Diff}^{(m)}_{A/B}$ is a finite A-module.

Proof. — In fact, in $P \otimes_B P$ as left P-algebra, we may identify $z \otimes 1$ with z itself and therefore $P \otimes_B P$ with $P[1 \otimes z] = P[\delta(z)]$, where $\delta(z) = \big(\delta(z_1), \ldots, \delta(z_N)\big)$. Hence

$$\Omega^{(m)}_{P/B} = P[\delta(z)]/\big(\delta(z)\big)^{m+1}P[\delta(z)]$$

which has the asserted property. Hence, its dual $\mathrm{Diff}^{(m)}_{P/B}$ has also the asserted property. As for the assertion on $\Omega^{(m)}_{A/B}$, it is enough to see that

$$\big((\zeta\otimes 1) + (1\otimes\zeta)\big)P\otimes_B P = \zeta(P\otimes_B P) + P\delta(\zeta) \qquad \square$$

Now, in the case of an affine scheme $Z = \mathrm{Spec}(A)$ where A is finitely generated as B-algebra and B is noetherian, we define $\Omega^{(\mu)}_{Z/B}$ to be the *coherent $\mathcal{O}_Z$-algebra* which corresponds to the finite A-algebra $\Omega^{(\mu)}_{A/B}$. Similarly, we define $\mathrm{Diff}^{(\mu)}_Z$ to be the *coherent $\mathcal{O}_Z$-module* which correspond to the finite A-module $\mathrm{Diff}^{(\mu)}_{A/B}$. The finiteness and coherency are due to Lemma 1.2. Since the definition of these A-modules commutes with localizations of A by Lemma 1.1, the definitions of $\Omega^{(\mu)}_{Z/B}$ and $\mathrm{Diff}^{(\mu)}_{Z/B}$ are naturally globalized for any scheme Z, not necessarily affine, of finite type over B. We call $\Omega^{(\mu)}_{Z/B}$ the *$\mathcal{O}_Z$-algebra of μ-jets of Z over B* and $\mathrm{Diff}^{(\mu)}_{Z/B}$ the *$\mathcal{O}_Z$-module of differential operators of orders $\leqslant \mu$ of Z over B*. We sometimes write $\mathrm{Diff}^{(\mu)}_Z$ for $\mathrm{Diff}^{(\mu)}_{Z/B}$ if the reference to B is clear from the context.

Back to a general commutative B-algebra R and $Z = \mathrm{Spec}(R)$, we will prove two useful lemmas on $\mathrm{Diff}^{(\mu)}_{Z/B}$, the first one is about *compositions* and the second about *commutators* of differential operators of R over B. The third lemma is a consequence of the two which we need later. In the proofs of the first two lemmas, we will follow the following chain of R-homomorphisms for a pair of differential operators ∂ and ∂':

$$R\otimes_B R \xrightarrow{(1,3)} R\otimes_B R\otimes_B R \xrightarrow{(1,\partial)} R\otimes_B R \xrightarrow{\partial'} R \tag{1.1}$$

where $(1,3): f\otimes g\mapsto f\otimes 1\otimes g$ and $(1,\partial): f\otimes g\otimes h\mapsto f\otimes\partial(f\otimes g)$. Here $\partial\in\mathrm{Diff}^{(\mu)}_{R/B}$ is viewed as an R-homomorphism from $R\otimes_B R$ to R through the natural surjection $R\otimes_B R\to\Omega^{(\mu)}_{R/B}$. Likewise for ∂'. It should be noted that for every $f\in R$ the end image of $1\otimes f$ by the above (1.1) is exactly $(\partial'\circ\partial)(f)$ in the sense of composition $\partial'\circ\partial$ of the two differential operators as being viewed as endomorphisms of R. When there is no ambiguity, we sometimes write $\partial'\partial$ for $\partial'\circ\partial$.

Lemma 1.3. — *Viewing $\partial\in\mathrm{Diff}^{(\mu)}_{R/B}$ and $\partial'\in\mathrm{Diff}^{(\mu')}_{R/B}$ as endomorphisms of R, we have the composition $\partial'\circ\partial$ belong to $\mathrm{Diff}^{(\mu+\mu')}_{R/B}$. Namely we have a natural homomorphism $\mathrm{Diff}^{(\mu)}_{R/B}\times\mathrm{Diff}^{(\mu')}_{R/B}\to\mathrm{Diff}^{(\mu+\mu')}_{R/B}$.*

Proof. — What we want is that if γ denotes the composition of the chain of homomorphisms of (1.1) then $\gamma(D_R^{\mu+\mu'+1})=0$. Define $(i,j): R\otimes_B R\to R\otimes_B R\otimes_B R$ for $1\leqslant i<j\leqslant 3$ in the same way as the above $(1,3)$ and let $D_{i,j}=(i,j)(D_R)$. Then we have $D_{1,3}\subset D_{1,2}+D_{2,3}$ because

$$1\otimes 1\otimes f - f\otimes 1\otimes 1 = (1\otimes f\otimes 1 - f\otimes 1\otimes 1)+(1\otimes 1\otimes f-1\otimes f\otimes 1)$$

We then obtain

$$D_{1,3}^{\mu+\mu'+1}\subset (D_{1,2}+D_{2,3})^{\mu+\mu'+1}\subset D_{1,2}^{\mu'+1}+D_{2,3}^{\mu+1}$$

Since $\partial'(D_R^{\mu'+1})=\partial(D_R^{\mu+1})=0$, there follows $\gamma(D_R^{\mu+\mu'+1})=0$. □

Lemma 1.4. — *For ∂ and ∂' as above, we have the following inclusion of the commutator:*

$$[\partial',\partial]=\partial'\circ\partial-\partial\circ\partial'\in\mathrm{Diff}^{(\mu'+\mu-1)}_{R/B}$$

Proof. — Pick any system of $\mu'+\mu$ elements $g_j\in R$. Let γ be the composition of (1.1) as before, and let γ' be the similar composition when ∂ and ∂' are exchanged in (1.1). It is then enough to prove that

$$\gamma\Big(\prod_{j=1}^{\mu'+\mu}\delta(g_j)\Big)=\gamma'\Big(\prod_{j=1}^{\mu'+\mu}\delta(g_j)\Big) \tag{1.2}$$

Now, writing $\delta_{i,j}=(i,j)\circ\delta$, we obtain

$$\prod_{j=1}^{\mu'+\mu}\delta_{1,3}(g_j)\equiv\sum_{\substack{I_1\cup I_2=[1,\mu'+\mu]\\ I_1\cap I_2=\varnothing\\ |I_1|=\mu',|I_2|=\mu}}\Big(\prod_{k\in I_1}\delta_{1,2}(g_k)\Big)\Big(\prod_{l\in I_2}\delta_{2,3}(g_l)\Big)\qquad\text{modulo } D_{1,2}^{\mu'+1}+D_{2.3}^{\mu+1}$$

Since $(1,\partial)$ is $R\otimes_B R$-linear from the left,

$$\begin{aligned}\partial'\circ(1,\partial)\Big(\prod_{j=1}^{\mu'+\mu}\delta_{1,3}(g_j)\Big) &= \sum_{\substack{I_1\cup I_2=[1,\mu'+\mu]\\ I_1\cap I_2=\varnothing\\ |I_1|=\mu',|I_2|=\mu}} \partial'\Big[\Big(\prod_{k\in I_1}\delta(g_k)\Big)\{1\otimes\partial(\prod_{l\in I_2}\delta(g_l))\}\Big] \\ &= \sum_{\substack{I_1\cup I_2=[1,\mu'+\mu]\\ I_1\cap I_2=\varnothing\\ |I_1|=\mu',|I_2|=\mu}} \partial\Big(\prod_{l\in I_2}\delta(g_l)\Big)\partial'\Big(\prod_{k\in I_1}\delta(g_k)\Big)\end{aligned}\tag{1.3}$$

where the last equality is because $|I_1|=\mu'$ and $\partial'\in\mathrm{Diff}^{(\mu')}_{R/B}$. In fact, since $G=\prod_{k\in I_1}\delta_{1,2}(g_k)\in D_R^{\mu'}$ and $F=\partial\big(\prod_{l\in I_2}\delta(g_l)\big)\in R$, we have $G(1\otimes F)\equiv G(F\otimes 1)=FG$ modulo $D_R^{\mu'+1}$. Now, the end of (1.3) is unchanged if ∂ and ∂' are interchanged and therefore we get (1.2), which completes the proof. $\square$

Lemma 1.5. — *For any R-submodule $\mathcal{D}^{(a)}\subset\mathrm{Diff}^{(a)}_{R/B}$, where $a\geqslant 0$, we have*

$$f_1^{-c}\,\mathcal{D}^{(a)}\circ f_1^d\subset\sum_{k=0}^{a} f_1^k\,\mathrm{Diff}^{(k)}_{R/B}\qquad \textit{if } d\geqslant 0 \textit{ and } c\leqslant d-a$$

Proof. — This is proven by induction on ad, noting that if $d=0$ then it is obvious because then $a=c=0$ too, and if $a=0$ then it is so by the commutativity in R. So assume that both a and d are positive. Now, by Lemma 1.4,

$$\begin{aligned}f_1^{-c}\,\mathcal{D}^{(a)}\circ f_1^d &= f_1^{-c}\big(\mathcal{D}^{(a)}\circ f_1\big)\circ f_1^{d-1}\\ &= f_1^{-c}\big(f_1\mathcal{D}^{(a)}+\mathcal{D}^{(a-1)}\big)\circ f_1^{d-1}\\ &= f_1^{-(c-1)}\mathcal{D}^{(a)}\circ f_1^{d-1}+f_1^{-c}\mathcal{D}^{(a-1)}\circ f_1^{d-1}\end{aligned}\tag{1.4}$$

where $\mathcal{D}^{(a-1)}$ is a certain R-submodule of $\mathrm{Diff}^{(a-1)}_{R/B}$ which arises from the commutation of Lemma 1.4. The exponent condition of the lemma is satisfied by each summand of the bottom line of (1.4). The proof is now clear by induction. $\square$

2. Idealistic exponents and their equivalences

We now review our way of generalizing the notion of infinitely near singular points. We formulate the notion in terms of *permissible diagrams of local sequence of smooth blowing-ups* as will be made precise by the series of definitions stated below. In this section, quite generally, Z will be any regular noetherian scheme.

Definition 2.1. — An *idealistic exponent* $E=(J,b)$ *on* Z is nothing but a pair of a coherent ideal sheaf J on Z and a positive integer b.

When Z is an affine scheme, say $Z = \mathrm{Spec}(A)$, we will identify J with the ideal in A which generates the $\mathcal{O}_Z$-module J. We will also consider any finite system of indeterminates $t = (t_1, t_2, \ldots, t_a)$ and denote by $Z[t]$ the product of Z with $\mathrm{Spec}(\mathbb{Z}[t])$ over the ring of integers $\mathbb{Z}$. This means that if $Z = \bigcup_\alpha \mathrm{Spec}(A_\alpha)$ is any expression as a union of open affine subschemes then $Z[t] = \bigcup_\alpha \mathrm{Spec}(A_\alpha[t])$ in a natural sense. We also let $E[t]$ denote the pair $(J[t], b)$ where $J[t]$ denotes the ideal sheaf on $Z[t]$ generated by J with respect to the canonical projection $Z[t] \to Z$.

Definition 2.2. — A *local sequence of smooth blowing-ups over* Z, called *LSB over* Z for short, means a diagram of the following type:

$$\begin{array}{lllll} Z_r \xrightarrow{\pi_{r-1}} & U_{r-1} \subset Z_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} & U_1 \subset Z_1 \xrightarrow{\pi_0} & U_0 \subset Z_0 = Z \\ & \cup & \cup & \cup \\ & D_{r-1} & D_1 & D_0 \end{array}$$

where U_i is an open subscheme of Z_i, D_i is a regular closed subscheme of U_i and the arrows mean that $\pi_i : Z_{i+1} \to U_i \subset Z_i$ is the blowing-up with center D_i.

Definition 2.3. — We define the *singular locus* $\mathrm{Sing}(E)$ of E to be the following closed subset of Z:

$$\mathrm{Sing}(E) = \{\eta \in Z \mid \mathrm{ord}_\eta(J) \geqslant b\}$$

We now want to define the notion of *permissibility* of *LSB* for a given idealistic exponent $E = (J, b)$ on Z. This will be done inductively. For that, it is enough to have two notions for a single blowing-up, one being that of *permissibility* for a blowing-up and the other being that of the *transform* by a permissible blowing-up. For an open subset $U_0 \subset Z$, we simply replace E by its restriction $E|U_0 = (J|U_0, b)$. So it is enough to have the notions for the case of $Z = U_0$.

Definition 2.4. — A blowing-up $Z_1 \to Z$ with center D_0 is said to be *permissible for* E if D_0 is regular by itself and contained in $\mathrm{Sing}(E)$ (see Def. 2.3).

Definition 2.5. — By such a permissible blowing-up as above, the *transform* $E_1 = (J_1, b)$ *of* E is defined by saying that $J_1 P^b$ is equal to the ideal sheaf on Z_1 generated by J with respect to the blowing-up morphism $Z_1 \to Z$, where P denotes the ideal sheaf of the exceptional divisor, *i.e.*, the locally principal ideal sheaf on Z_1 generated by the ideal defining $D_0 \subset Z$.

Note that J_1 is an ideal sheaf uniquely determined by the above equality.

Definition 2.6. — For a pair of idealistic exponents $E_i = (J_i, b_i), i = 1, 2$, we define the *inclusion*

$$E_1 \subset E_2$$

to mean the following relation: *Pick any finite system of indeterminates* $t = (t_1, \dots, t_a)$ *and let* $E_i[t] = (J_i[t], b_i), i = 1, 2$.

(2.1) *If any LSB over* $Z[t]$ *in the sense of Def. 2.2 is permissible for* $E_1[t]$*, it is so for* $E_2[t]$.

The above notation for the inclusion relation is rather conventional. There, to be precise, we should think of (J, b) as being identified with the following indexed family of sets:

$$\text{“}(J,b)\text{”} = \bigcup_t \{LSB \text{ permissible for } (J[t], b)\}_t$$

where, for each system t, $\{\}_t$ is the set of all those LSB's which are permissible for the idealistic exponent $(J[t], b)$ on $Z[t]$ as in the Def. 2.2 and the union is taken disjointly for all t taken as set-indices. The inclusion relations, such as those in Defs. 2.6 above and 2.7 below, must be checked for every t individually.

Definition 2.7. — The equivalence

$$E_1 \sim E_2$$

means that $E_1 \subset E_2$ and $E_1 \supset E_2$ at the same time. It means “E_1” = “E_2”. The notion of equivalence will be extended to such a statement as $E_1 \cap E_2 \sim E_3$ which will mean “E_1” $\cap$ “E_2” = “E_3”.

It should be noted that “E_1” = “E_2” does not imply $E_1 = E_2$ as will be clearly seen in the later discussions. This is indeed why we avoid the word *equality* and use *equivalence* in the Def. (2.7). In other words, we conventionally omit “ ” only in dealing with those symbols as $\subset, \supset, \cap$ and $\cup$.

Definition 2.8. — For an idealistic exponent $E = (J, b)$ on Z, we define its *order at a point* $\xi \in Z$ as follows:

$$\mathrm{ord}_\xi(E) = \begin{cases} \mathrm{ord}_\xi(J)/b & \text{if } \mathrm{ord}_\xi(J) \geqslant b \\ 0 & \text{if } \mathrm{ord}_\xi(J) < b \end{cases}$$

Hence we have $\mathrm{Sing}(E) = \{\xi \in Z \mid \mathrm{ord}_\xi(E) \geqslant 1\}$. (*cf.* Def. 2.3)

What follows are most of the elementary but basic *facts* on relations among idealistic exponents.

[F1] $(J^e, eb) \sim (J, b)$ for every positive integer e.

[F2] For every common multiple m of b_1 and b_2, we have

$$(J_1, b_1) \cap (J_2, b_2) \sim (J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m)$$

In particular if $b_1 = b_2 = b\ (= m)$ and $J_1 \subset J_2$ then we have $(J_1, b) \supset (J_2, b)$. It also follows that the intersection of any finite number of idealistic exponents is equivalent to an idealistic exponent.

[F3] We always have

$$(J_1J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$$

The reversed inclusion does not hold in general. However, if $\mathrm{Sing}(J_i, b_i + 1)$ are both *empty* for $i = 1, 2$, then the left hand side becomes *equivalent* to the right hand side. Moreover, we always have

$$(J, b) \subset (J_k, b_k), 1 \leqslant k \leqslant r, \implies (J, b) \subset \Big(\prod_{1\leqslant k\leqslant r} J_k \ , \sum_{1\leqslant k\leqslant r} b_k \Big)$$

[F4] Let us compare two idealistic exponents having the same ideal but different b's, say $F_1 = (J, b_1)$ and $F_2 = (J, b_2)$ with $b_1 > b_2$. Then we have

1) $F_1 \subset F_2$.

2) *For any LSB permissible for F_1, and hence so for F_2, their final transforms differ only by a locally principal non-zero factor supported by exceptional divisors.* To be precise, their final transforms being denoted by $F_1^* = (J_1^*, b_1)$ and $F_2^* = (J_2^*, b_2)$, we have $J_1^* = MJ_2^*$ where M is a positive power product of the ideals of the strict transforms of the exceptional divisors created by the blowing-ups belonging to the LBS. Incidentally, changing the number b turns out to be a useful technique in connection with the problem of transforming singular data into normal crossing data which appears in a process of desingularization.

[F5] We have $(J_1, b) \supset (J_2, b)$ if J_1 is contained in the *integral closure* of J_2 in the sense of *integral dependence* (after Oscar Zariski) defined in the theory of *ideals*. Recall the definition: For ideals $H_i, i = 1, 2$, in a commutative ring R, H_1 is *integral* over H_2 in the sense of the *ideal theory* if and only if $\sum_{a\geqslant 0} H_1^a T^a$ is *integral* over $\sum_{a\geqslant 0} H_2^a T^a$ in the sense of the *ring theory*, where T is an indeterminate over R. In our case, since Z is regular and hence normal, if $\rho : \widetilde{Z} \to Z$ is any proper birational morphism such that $\widetilde{Z}$ is normal and $J_2 O_{\widetilde{Z}}$ is locally non-zero principal, then the direct image $\rho_*(J_2 O_{\widetilde{Z}})$ is equal to the *integral closure* of J_2. As an example of such ρ, we could take the *normalized blowing-up* of J_2, *i.e.*, the blowing-up of J_2 followed by normalization.

In what follows, we will state and prove the three key theorems, most important from the technical point of view in the theory of idealistic exponents. They are called the *Differentiation (or Diff) Theorem* (*cf.* Theorem 1, section 8 in [**3**] of the Reference), the *Numerical Exponent Theorem* (*cf.* Proposition 8, section 2 in [**3**]) and *Ambient Reduction Theorem* (*cf.* Th. 5, section 8 in [**3**]).

3. Differentiation Theorem

The theorem stated below and its proof work well for a general regular scheme, need not be of finite type over any $\Bbbk$. Instead of $\Bbbk$, we take any commutative ring B and we work with a commutative B-algebra R, or with an affine scheme $Z = \mathrm{Spec}(R)$.

Lemma 3.1. — *Let R be a commutative B-algebra and let M be an ideal in R. Then for every $\partial \in \mathrm{Diff}^{(i)}_{R/B}$ and for every positive integer $l \geqslant i$, we have $\partial(M^l) \subset M^{l-i}$.*

Proof. — Pick any $f_j \in M, 1 \leqslant j \leqslant l$, and we have

$$\begin{aligned}1 \otimes \prod_{j=1}^{l} f_j = \prod_{j=1}^{l}(1 \otimes f_j) &= \prod_{j=1}^{l}\big(f_j \otimes 1 + \delta(f_j)\big)\\ &\equiv \sum_{\substack{I \subset [1,l] \\ |I| \leqslant i}} \Big(\prod_{k \in [1,l]-I} (f_k \otimes 1)\Big)\Big(\prod_{j \in I} \delta(f_j)\Big)\\ &= \sum_{\substack{I \subset [1,l] \\ |I| \leqslant i}} \Big(\prod_{k \in [1,l]-I} f_k\Big)\Big(\prod_{j \in I} \delta(f_j)\Big) \qquad \text{modulo } D_R^{i+1}\end{aligned}$$

where D_R denotes the diagonal ideal in $R \otimes_B R$. Note that

$$\prod_{k \in [1,l]-I} f_k \in M^{l-i}, \quad \forall\, I$$

Since ∂ is an R-homomorphism from $R \otimes_B R$,

$$\partial\big(1 \otimes \textstyle\prod_{j=1}^{l} f_j\big) \in M^{l-i}$$

which proves the lemma. □

Lemma 3.2. — *Let R be the same as the one in Lemma 3.1 and let $f = (f_1, \dots, f_m)$ be a finite system of elements of R. Write $\widetilde{R} = R[f f_1^{-1}]$. This means the subalgebra generated by the ratios $f_i f_1^{-1}, 2 \leqslant i \leqslant m$, over the canonical image of R in the localization $R[f_1^{-1}]$ of R by the powers of f_1. The canonical homomorphism $R \to \widetilde{R}$ induces canonical left $\widetilde{R}$-homomorphisms*

$$\phi^{(\mu)} : \widetilde{R} \otimes_R \Omega^{(\mu)}_{R/B} \longrightarrow \Omega^{(\mu)}_{\widetilde{R}/B}$$

for integers $\mu \geqslant 0$. We then have

$$\phi^{(\mu)}\big(\widetilde{R} \otimes_R \Omega^{(\mu)}_{R/B}\big) \supset f_1^{\mu} \Omega^{(\mu)}_{\widetilde{R}/B}, \quad \forall\, \mu$$

Proof. — Let us recall that $\Omega^{(\mu)}_{R/B} = (R \otimes_B R)/D_R^{\mu+1}$ and $\Omega^{(\mu)}_{\widetilde{R}/B} = \widetilde{R} \otimes_B \widetilde{R}/D_{\widetilde{R}}^{\mu+1}$ where D stands for the diagonal ideal. It follows that

$$\Omega^{(\mu)}_{\widetilde{R}/B} = \Big(\widetilde{R} \otimes_R \Omega^{(\mu)}_{R/B}\Big)[1 \otimes f f_1^{-1}]$$

Therefore, for a proof of the lemma, the following claim clearly suffices:

For every integer $N > 0$ and for every map $\alpha : [1, N] \to [1, m]$, the class of

$$f_1^{\mu} \prod_{i=1}^{N}\Big(1 \otimes \frac{f_{\alpha(i)}}{f_1}\Big)$$

modulo $D_{\widetilde{R}}^{\mu+1}$ is contained in the image $\phi^{(\mu)}\big(\widetilde{R} \otimes_R \Omega^{(\mu)}_{R/B}\big)$.

This will be proven as follows: In general $\delta(F) = 1 \otimes F - F \otimes 1$ and we get

$$
\begin{aligned}
f_1^{\mu} \prod_{i=1}^{N} \Big(1 \otimes \frac{f_{\alpha(i)}}{f_1}\Big) &= f_1^{\mu} \prod_{i=1}^{N} \Big(\frac{f_{\alpha(i)}}{f_1} \otimes 1 + \delta(\frac{f_{\alpha(i)}}{f_1})\Big) \\
&\equiv \sum_{I\subset[1,N],|I|\leqslant\mu} \Big(\prod_{j\in[1,N]-I} \frac{f_{\alpha(i)}}{f_1}\Big)\Big(f_1^{\mu} \prod_{i\in I} \delta(\frac{f_{\alpha(i)}}{f_1})\Big) \quad \text{modulo } D_{\widetilde{R}}^{\mu+1}
\end{aligned}
$$

We have $f_1 \otimes 1 = 1 \otimes f_1 - \delta(f_1)$ and hence

$$
\begin{aligned}
f_1^{\mu} \prod_{i\in I} \delta(\frac{f_{\alpha(i)}}{f_1}) &= (f_1 \otimes 1)^{\mu} \prod_{i\in I} \delta(\frac{f_{\alpha(i)}}{f_1}) \\
&= \big(1 \otimes f_1 - \delta(f_1)\big)^{\mu} \prod_{i\in I} \delta(\frac{f_{\alpha(i)}}{f_1}) \\
&\equiv \sum_{j=0}^{\mu-|I|} (-1)^j \delta(f_1)^j (1 \otimes f_1)^{\mu-j} \prod_{i\in I} \delta(\frac{f_{\alpha(i)}}{f_1}) \\
&= \sum_{j=0}^{\mu-|I|} (-1)^j \delta(f_1)^j (1 \otimes f_1)^{\mu-|I|-j} \prod_{i\in I} (1 \otimes f_1)\delta(\frac{f_{\alpha(i)}}{f_1}) \\
&= \sum_{j=0}^{\mu-|I|} (-1)^j \delta(f_1)^j (1 \otimes f_1)^{\mu-|I|-j} \prod_{i\in I} \big(1 \otimes f_{\alpha(i)} - \frac{f_{\alpha(i)}}{f_1} \otimes f_1\big) \\
&\qquad\qquad \text{modulo } D_{\widetilde{R}}^{\mu+1}
\end{aligned}
$$

Observe that the bottom line is expressible as linear combination of $\delta(f_1)$ and $\delta(f_{\alpha(i)})$ with coefficients in $\widetilde{R} = R[ff_1^{-1}]$. Now our claim follows from the combination of these two congruence-equality equations. □

Corollary 3.3. — *By means of the canonical R-homomorphisms*

$$
\begin{aligned}
\widetilde{R} \otimes_R \mathrm{Diff}_{R/B}^{(\mu)} &= \widetilde{R} \otimes_R \mathrm{Hom}_R(\Omega_{R/B}^{(\mu)}, R) \\
&\to \mathrm{Hom}_{\widetilde{R}}(\widetilde{R} \otimes_R \Omega_{R/B}^{(\mu)}, \widetilde{R}) \\
&\to \mathrm{Hom}_{\widetilde{R}}(f_1^{\mu}\Omega_{\widetilde{R}/B}^{(\mu)}, \widetilde{R}) = f_1^{-\mu}\,\mathrm{Diff}_{\widetilde{R}/B}^{(\mu)}
\end{aligned}
$$

the lemma gives us the following natural homomorphism:

$$f_1^{\mu}\Big(\mathrm{Diff}_{R/B}^{(\mu)}\Big) \longrightarrow \mathrm{Diff}_{\widetilde{R}/B}^{(\mu)}$$

***Theorem 3.4* (Diff Theorem)**. — *Assume that Z is regular. If $\mathcal{D}$ is any left $\mathcal{O}_Z$-submodule of $\mathrm{Diff}_Z^{(i)}$ then we have the following inclusion in the sense of Def. 2.6*

$$(J, b) \subset (\mathcal{D}J, b - i)$$

or equivalently

$$(J, b) \cap (\mathcal{D}J, b - i) \sim (J, b)$$

Proof. — The problem is local and hence we take the affine case where $Z = \mathrm{Spec}(R)$ with an B-algebra R. For any finite system of indeterminates $t = (t_1, \ldots, t_r)$, every elements of $\mathrm{Diff}_{R/B}$ can be uniquely extended to an element of $\mathrm{Diff}_{R'/B}$ with $R' = R[t]$ under the condition that it acts trivially on t, so that $R' \, \mathrm{Diff}_{R/B} \subset \mathrm{Diff}_{R'/B}$. We denote the submodule by $\mathcal{D}'$. Moreover we have $(\mathcal{D}J)R' = \mathcal{D}[t](JR')$. In other words, the propositional set-up for $(J[t], b)$ with respect to $\mathcal{D}[t]$ is the same as for (J, b) with respect to $\mathcal{D}$. Hence we may only investigate the blowing-up diagrams over Z and their effects on (J, b). The part of the restriction to an open set is trivially manageable. Therefore, for the proof of the theorem, it is enough to verify the following two statements:

1) $\mathrm{Sing}(J, b) \subset \mathrm{Sing}(\mathcal{D}J, b - i)$

2) *If W is a regular closed subscheme of Z such that $W \subset \mathrm{Sing}(J, b)$, then there exists a certain $\widetilde{\mathcal{D}} \subset \mathrm{Diff}^{(i)}_{\widetilde{Z}}$ such that $(\widetilde{\mathcal{D}}\widetilde{J}, b - i)$ is the transform of $(\mathcal{D}J, b - i)$ by the blowing-up $\pi : \widetilde{Z} \to Z$ with center W, where $(\widetilde{J}, b)$ denotes the transform of (J, b) by π.*

Now 1) can be easily deduced from Lemma 3.1. Next we want to prove 2) and we can easily reduce the problem to the case in which Z is an affine $\mathrm{Spec}(R)$. The proof will be made by means of the Lemma 3.2, in which f is chosen to be any system of generators of the ideal of W in $Z = \mathrm{Spec}(R)$. Pick any member of the f and call it f_1 by reordering f if necessary. We then see that, within an affine open subset $\mathrm{Spec}(\widetilde{R})$ where we choose $\widetilde{R} = R[ff^{-1}]$, $(f_1^{-(b-i)}(\mathcal{D}J)\widetilde{R}, b - i)$ is the transform of $(\mathcal{D}J, b - i)$ and $(f_1^{-b}J\widetilde{R}, b)$ is that of (J, b). So we want to compare the two ideals $f_1^{-b}J\widetilde{R}$ and $f_1^{-(b-i)}(\mathcal{D}J)\widetilde{R}$. Let $\widetilde{J} = f_1^{-b}J\widetilde{R}$. Now, we have

$$\begin{aligned} f_1^{-(b-i)}\Big(\mathcal{D}J\Big)\widetilde{R} &= \widetilde{R}\, f_1^{-(b-i)}\Big(\mathcal{D}J\Big)\widetilde{R} \\ &= \widetilde{R}\Big(f_1^{-(b-i)}\mathcal{D} \circ f_1^b\Big)\Big(f_1^{-b}J\Big)\widetilde{R} \\ &= \Big(\widetilde{R}\big(f_1^{-(b-i)}\mathcal{D} \circ f_1^b\big)\Big)\widetilde{J} \end{aligned}$$

and by Lemma 1.5

$$\widetilde{R}\Big(f_1^{-(b-i)}\mathcal{D} \circ f_1^b\Big) \subset \widetilde{R} \sum_{k=0}^{i} f_1^k \, \mathrm{Diff}^{(k)}_{R/B} \subset \mathrm{Diff}^{(i)}_{\widetilde{R}/B}$$

Therefore, by Cor. (3.3),

$$\widetilde{\mathcal{D}} = \widetilde{R}\Big(f_1^{-(b-i)}\mathcal{D} \circ f_1^b\Big)$$

will do for the claim 2). Once 1) and 2) are proven, the rest is an easy step by step induction. □

4. NC-divisorial exponent

Let Z be any regular scheme. Let $\Gamma = \{\Gamma_d, 1 \leqslant d \leqslant l\}$, be a finite system of hypersurfaces in Z. Then Γ is said to have *(only) normal crossings* at a point $\xi \in Z$ if there exists a regular system of parameters $x = (x_1, \ldots, x_n)$ of the local ring $\mathcal{O}_{Z,\xi}$ such that either $\xi \notin \Gamma_d$ or $\Gamma_d = (x_{\alpha(d)})\mathcal{O}_{Z,\xi}$ at ξ for every $d, 1 \leqslant d \leqslant l$, where $d \neq d' \Rightarrow \alpha(d) \neq \alpha(d')$. We say that a regular subscheme C of Z has *normal crossings with* Γ at ξ if such an x exists and satisfies an additional condition that the ideal of C in $\mathcal{O}_{Z,\xi}$ is generated by a subsystem of x. If the *normal crossings* are everywhere, they are simply said to have *normal crossings*.

***Definition 4.1* (NC-divisor)**. — Let Γ have normal crossings as above. Then a linear combination $\sum_{d=1}^{l} \gamma_d \Gamma_d$ with integers $\gamma_d \geqslant 0$ is called an *NC-divisor with support* Γ *on* Z. It should be kept in mind that an NC-divisor carries a specific support Γ and the ordering of the members of Γ is also important. These data are specified and included in the notion of an NC-divisor.

***Definition 4.2* (NC-divisorial exponent)**. — An *NC-divisorial (idealistic) exponent* on Z means a triple $(\mathfrak{D}, H, b)$ such that $\mathfrak{D}$ is an NC-divisor on Z in the sense of Def. 4.1 and (H,b) is an idealistic exponent in the sense of Def. 2.1. Its affiliated definitions are as follows:

(1) The *order* and the *singular locus* are defined by

$$\mathrm{ord}_\xi(\mathfrak{D}, H, b) = \Big(\mathrm{ord}_\xi(H) + \sum_{\xi \in \Gamma_d} \gamma_d\Big)/b$$
$$\mathrm{Sing}(\mathfrak{D}, H, b) = \{\xi \in Z \mid \mathrm{ord}_\xi(\mathfrak{D}, H, b) \geqslant 1\}$$

(2) A blowing-up $\pi : Z' \to Z$ with center C is *permissible* for $(\mathfrak{D}, H, b)$ if C is a closed regular subscheme of Z which is contained in $\mathrm{Sing}(\mathfrak{D}, H, b)$ and has normal crossings with Γ everywhere.

(3) The *transform* $(\mathfrak{D}', H', b)$ of $(\mathfrak{D}, H, b)$ by the permissible blowing-up π is defined by saying:

$$(I_{\pi^{-1}(C),Z'})^\delta H' = H\mathcal{O}_{Z'} \quad \text{with } \delta = \mathrm{ord}_C(H)$$
$$\mathfrak{D}' = \sum_{d=1}^{l+1} \gamma_d \Gamma'_d \quad \text{with } \gamma_{l+1} = \delta - b + \sum_{C \subset \Gamma_d} \gamma_d$$

where Γ'_d is the strict transform of Γ_d by π, $1 \leqslant d \leqslant l$, and $\Gamma'_{l+1} = \pi^{-1}(C)$. It should be noted that $\Gamma' = \{\Gamma'_e, 1 \leqslant e \leqslant l+1\}$ has normal crossings in Z' everywhere and the new exceptional divisor Γ'_{l+1} is placed at the end of the new ordered system of normal crossings.

(4) Given a LSB in the sense of Def. 2.2, its *permissibility* for $(\mathfrak{D}, J, b)$ is defined inductively by these 2) and 3).

(5) Given an NS-divisorial exponent $(\mathfrak{D}, H, b)$, we have its *affiliated idealistic exponent* which is defined to be:

$$(J, b) \quad \text{with } J = H \prod_{d=1}^{l} \Gamma_d^{\gamma_d}$$

It should be noted that $\mathrm{ord}_\xi(\mathfrak{D}, H, b) = \mathrm{ord}_\xi(J, b)$ and that, for an LSB over Z which has normal crossings of centers with the transforms of Γ, it is permissible for $(\mathfrak{D}, H, b)$ if and only if it is so for its affiliated (J, b).

(6) For any finite system of indeterminates $t = (t_1, \ldots, t_r)$, we define $\Gamma[t] = \{\Gamma_i[t], 1 \leqslant i \leqslant l\}$, $\mathfrak{D}[t] = \sum_{i=1}^{l} \gamma_i \Gamma_i$, and $(\mathfrak{D}, H, b)[t] = (\mathfrak{D}[t], H[t], b)$.

(7) Given two NC-divisorial exponents $(\mathfrak{D}_i, H_i, b_i), i = 1, 2$, where the two $\mathfrak{D}_i$, $i = 1, 2$, have the same support Γ, the *inclusion*

$$(\mathfrak{D}_1, H_1, b_1) \subset (\mathfrak{D}_2, H_2, b_2)$$

means that every LSB permissible for the first is permissible to the second, even after we pick any finite system of indeterminates t and replace all the data by applying $[t]$.

(8) The *equivalence*

$$(\mathfrak{D}_1, H_1, b_1) \sim (\mathfrak{D}_2, H_2, b_2)$$

is defined to mean the both way inclusions in the sense of (7).

Remark 4.1. — For any given idealistic exponent (J, b) on Z, we get an NC-divisorial exponent $(\varnothing, J, b)$ where $\varnothing$ denotes the zero divisor supported by the empty system of hypersurfaces. However, it should be noted that the rule of transforms by permissible blowing-ups is essentially different for the NC-divisorial case. For instance, when we apply LSB to an NC-divisorial exponent, the exceptional divisors are all marked and stored in the order of their creation.

Remark 4.2. — Given an NC-divisorial $(\mathfrak{D}, H, b)$, if a blowing-up $\pi : Z' \to Z$ with center C is permissible for the affiliated (J, b), then there exists a closed subset $S \subset C$, nowhere dense in C, such that π restricted to $Z - S$ is permissible for the NC-divisorial $(\mathfrak{D}, H, b)$.

Proof. — Let ξ be the generic point of C. Then it is clear by definition that C has normal crossings with Γ at ξ. If x is a chosen regular system of parameters of $\mathcal{O}_{Z,\xi}$ for this purpose as was in the first paragraph of this section, then there exists a closed subset $S_1 \subset C$, nowhere dense in C, such that for every point $\eta \in C - S_1$ the x is extendable to a regular system of parameters of $\mathcal{O}_{Z,\eta}$. Let $S_2 = \bigcup_{\xi \notin \Gamma_d} \big(\Gamma_d \cap C\big)$, again nowhere dense in C. Let $S = S_1 \cup S_2$. Then C has normal crossings with Γ at every point of $C - S$. This S will do for the remark. $\square$

Remark 4.3. — Consider a LSB over Z in the way that was described in the Def. 2.2. Assume that it is permissible for the affiliated (J, b). Assume further that for every $i < j$, $0 \leqslant i, j \leqslant r - 1$, the generic point of D_j is mapped either to the generic point of D_i or into $U_i - D_i$. Then we can choose a closed subset $S_i \subset D_i$, nowhere dense in D_i, such that if we replace Z_i by $Z_i - S_i$, U_i by $U_i - S_i$ and D_i by $D_i - S_1$ for all i, then new LSB of the new $Z_{i+1} \to U_i$ with the new center D_i for all i is permissible for $(\mathfrak{D}, H, b)$. We will make use of the following special cases:

case 1) For every i, $1 \leqslant i \leqslant r-1$, the generic point of D_i is mapped to the generic point of D_{i-1}, in which case we need suitable open restrictions.

case 2) All the centers D_i are closed points of the Z_i.

case 3) For every i, the center D_i is one of the member hypersurface of the NC-divisor $\mathfrak{D}^{(i)}$ of the transform $(\mathfrak{D}^{(i)}, H^{(i)}, b)$ in Z_i of the given $(\mathfrak{D}, H, b)$.

case 4) Any combination of cases 2) and 3).

Proof. — Apply Rem. 4.2 repeatedly and inductively, each time deleting all the inverse images of S_i in Z_j, $\forall\, j \geqslant i+1$. ☐

5. Numerical Exponent Theorem

The second of the important technical theorems in dealing with idealistic exponents is about the numerical order evaluation at their singular points. Here is the statement to this effect in its full generality.

***Theorem 5.1* (Numerical Exponent Theorem)**. — *We assume that Z is excellent so that every scheme of finite type over Z has a closed non-regular locus. Let $T_i = (\mathfrak{D}, H_i, b_i), i = 1, 2$, be two NC-divisorial exponents on Z with the same NC-divisor D supported by normal crossings Γ in the sense of Def. 4.2. Let $E_i = (J_i, b_i)$ be the affiliated idealistic exponent of T_i for $i = 1, 2$ in the sense of 5) of Def. 4.2. If $T_1 \subset T_2$ in the sense of Def. 4.2, then we have*

$$\mathrm{ord}_\xi(E_1) = \mathrm{ord}_\xi(T_1) \leqslant \mathrm{ord}_\xi(T_2) = \mathrm{ord}_\xi(E_2) \tag{5.1}$$

for every point $\xi \in Z$. In particular, we have the same inequality if $E_1 \subset E_2$ in the sense of Def. 2.6. Consequently, we have

$$T_1 \sim T_2 \ (\textit{or } E_1 \sim E_2) \Longrightarrow \mathrm{ord}_\xi(T_1) = \mathrm{ord}_\xi(E_1) = \mathrm{ord}_\xi(E_2) = \mathrm{ord}_\xi(T_2)$$

for every point $\xi \in Z$.

Remark 5.1. — In the proof of (5.1) given below, we will use only LSB's which are compositions of two portions as follows:

1) *The first portion is a sequence of permissible blowing-ups whose centers are all quasi-finite over and generically surjective to the closure of ξ in Z. (See the sequence (5.2) below.)*

2) *The second portion is a sequence consisting blowing-ups whose centers are all generically isomorphic to the last exceptional divisor created by the first portion. (See the sequence (5.4) below.)*

In general, therefore, the first portion is in the case 1) and the second in the case 3) in the sense of Remark 4.3. Therefore, the NC-divisorial permissibility is all guaranteed with such *LSB* so long as open restrictions are chosen to be sufficiently small. The point is that *normal crossings* is an open condition.

Remark 5.2. — If the ambient scheme Z is of finite type over $\Bbbk$ and $H \subset \mathcal{O}_Z$ is a coherent ideal sheaf, then $\mathrm{ord}_\zeta(H)$ for a $\zeta \in Z$ is equal to $\mathrm{ord}_\xi(H)$ for almost all closed points ξ in the closure of ζ in Z. Therefore, for the inequality of the Numerical Exponent Theorem it is enough to prove it only for closed points. What is more important, when ξ is a closed point, it should be noted that our proof of the theorem below use only LSB's which are combination of two portions, the first being in the case 2) and the second being in the case 3) in the sense of Remark 4.3. Once again, the NC-divisorial permissibility is guaranteed.

Proof of Theorem 5.1. — It is clearly enough to consider $\xi \in \mathrm{Sing}(E_1)$. Let X denote the closure of ξ in Z. It is a reduced irreducible subscheme of Z and its locus of non-regular points $\mathrm{Sing}(X)$ is closed and nowhere dense in X. We can also find a closed nowhere dense subset S of X such that, within $X - S$, both E_1 and E_2 have respective constant orders and moreover Γ has normal crossings with X. Let $U = Z - (\mathrm{Sing}(X) \cup S)$ and $X_0 = X \cap U$. Let t be an indeterminate and take products $\times \mathrm{Spec}(\Bbbk[t])$, which will be denoted by $[t]$ for short as before. Let $Z'_0 = U[t]$, $L'_0 = X_0[t]$, and $X'_0 = L'_0 \cap \{t = 0\}$. Here it should be noted that the canonical projection $X'_0 \to X_0$ is an isomorphism as was pointed out in Remark 5.1. Let $E(j)'_0$ be the restriction of $E_j[t]$ to Z'_0, $j = 1, 2$. Let us take an LSB in the sense of Def. 2.2,

$$
\begin{array}{ccccccc}
Z'_r \longrightarrow & Z'_{r-1} & \longrightarrow \cdots \longrightarrow & Z'_1 & \longrightarrow & Z'_0 \\
 & \cup & & \cup & & \cup \\
 & X'_{r-1} & & X'_1 & & X'_0
\end{array}
\tag{5.2}
$$

where, for every index $i \geqslant 1$, L'_i being the strict transform of L'_{i-1} by the blowing-up $Z'_i \to Z'_{i-1}$ with center X'_{i-1}, X'_i is the isomorphic inverse image of X'_{i-1} by the isomorphic blowing-up $L'_i \to L'_{i-1}$ with the same center. Here it should be noted that the ideal of $X_{i-1} \subset L_{i-1}$ is principal and that, with the strict transform $L'_i \subset Z'_i$ of L'_{i-1}, $L'_i \to L'_{i-1}$ is the blowing-up induced by $Z'_i \to Z'_{i-1}$ with the same center. The sequence (5.2) is permissible for both $E(j)'_0$, $j = 1, 2$, and for all $r \geqslant 1$. Let $E(j)'_i$ be the transform of $E(j)'_{i-1}$ for each $i \geqslant 1$. We have

$$
\begin{gathered}
\mathrm{ord}_{\xi'_i}(E(j)'_i) = i\Big(\mathrm{ord}_\xi(E_j) - 1\Big) + \mathrm{ord}_\xi(E_j) \\
\textit{where } \xi'_i \textit{ is the generic point of } X'_i,\ \forall\, i \geqslant 0
\end{gathered}
\tag{5.3}
$$

After each (5.2), we continue the following sequence of blowing-ups

$$
\begin{array}{lccccccccc}
(5.4) \qquad & Z'_{r,s} & \longrightarrow & Z'_{r,s-1} & \longrightarrow \cdots \longrightarrow & Z'_{r,1} & \longrightarrow & Z'_{r,0} = Z'_r - L'_r \subset Z'_r \\
& & & \cup & & \cup & & \cup \\
& & & Y'_{r,s-1} & & Y'_{r,1} & & Y'_{r,0}
\end{array}
$$

where L'_r is the strict transform of L'_0 by (5.2), $Y'_{r,0}$ is the restriction to $Z'_{r,0}$ of the total transform of X'_{r-1} by the last blowing-up of (5.2), $Z'_{r,l} \to Z'_{r,l-1}$ is the blowing-up with center $Y'_{r,l-1}$, and $Y'_{r,l}$ is the total transform of $Y'_{r,l-1}$ by the blowing-up for all $l \geqslant 1$. Letting $E(j)'_{r,0}$ be the restriction of $E(j)'_r$ to $Z'_{r,0}$, we define $E(j)'_{r,l}$ be the idealistic exponent on $Z'_{r,l}$ which is the transform of $E(j)'_{r,l-1}$ by the blowing-up $Z'_{r,l} \to Z'_{r,l-1}$ for every $l \geqslant 1$. Note that $Y'_{r,l-1}$ is a regular irreducible hypersurface in $Z'_{r,l-1}$ and all the blowing-ups of (5.4) are isomorphisms and the total transforms $Y'_{r,l}$, $l \geqslant 1$, are all isomorphic to $Y'_{r,0}$. Here, an important point is that (5.4) can be and will be prolonged so long as we have the permissibility for $E(j)'_{r,l}$. Note that the permissibility for $E(1)'_{r,l}$ implies the same for $E(2)'_{r,l}$. For each $r \geqslant 0$, we take the maximally prolonged (5.4) for E_j, $j = 1$ and 2, and the maximal number s will be called $s_j(r)$. We have $s_1(r) \leqslant s_2(r)$ for every $r \geqslant 0$. Let η'_l be the generic point of $Y'_{r,l}$, $l \geqslant 0$, and by (5.3) we have

$$
\begin{aligned}
\operatorname{ord}_{\eta'_0}\big(E(j)'_{r,0}\big) = \operatorname{ord}_{\eta'_0}\big(E(j)'_r\big) &= \operatorname{ord}_{\xi'_{r-1}}\big(E(j)'_{r-1}\big) - 1 \\
&= (r-1)\big(\operatorname{ord}_\xi(E_j) - 1\big) + \operatorname{ord}_\xi(E_j) - 1 \\
&= r\big(\operatorname{ord}_\xi(E_j) - 1\big)
\end{aligned}
$$

and

$$
\operatorname{ord}_{\eta'_l}\big(E(j)'_{r,l}\big) = \operatorname{ord}_{\eta'_{l-1}}\big(E(j)'_{r,l-1}\big) - 1, \forall\, l \geqslant 1
$$

so that

$$
\begin{aligned}
\operatorname{ord}_{\eta'_{s(r)}}\big(E(j)'_{r,s(r)}\big) &= \operatorname{ord}_{\eta'_0}\big(E(j)'_{r,0}\big) - s_j(r) \\
&= r\big(\operatorname{ord}_\xi(E_j) - 1\big) - s_j(r)
\end{aligned}
$$

Now, for $j = 1, 2$, we take $r \gg 1$ and the maximality of $s_j(r)$ implies that

$$
0 \leqslant r\big(\operatorname{ord}_\xi(E_j) - 1\big) - s_j(r) < 1
$$

Dividing this by r and letting $r \to \infty$, we get

$$
\operatorname{ord}_\xi(E_j) = \lim_{r\to\infty} \frac{s_j(r)}{r}
$$

Since $s_1(r) \leqslant s_2(r), \forall\, r$, we obtain the asserted inequality $\operatorname{ord}_\xi(E_1) \leqslant \operatorname{ord}_\xi(E_2)$. The rest of the theorem follows. □

6. Birational Ubiquity of Point Blowing-ups

In this section, we will be particularly interested in blowing-ups whose centers are closed points on a regular scheme Z.

Definition 6.1. — An LSB of Def. 2.2 will be called $pLSB$ when all the centers of blowing-ups in it are closed points which correspond to each others by the blowing-up morphisms.

***Theorem 6.1* (Point Blow-up Equivalence Theorem)**. — *Let us assume that Z is a regular scheme of finite type over a noetherian ring B and that for every closed point $\xi \in Z$ the image of B into the residue field $\mathcal{O}_{Z,\xi}/\max(\mathcal{O}_{Z,\xi})$ is a field. Let $E_i = (J_i, b_i)$ be idealistic exponents on Z for $i = 1, 2$. Consider the following condition: Over $Z[t]$ with any finite system t of indeterminates,*

$$\text{(6.1)} \qquad \textit{every } pLSB \textit{ permissible for } E_1[t] \textit{ is permissible for } E_2[t].$$

This condition implies $E_1 \subset E_2$ in the sense of Def. 2.6. Hence, if we have:

$$\text{(6.2)} \qquad a\ pLSB \textit{ is permissible for } E_1[t] \Longleftrightarrow \textit{it is so for } E_2[t]$$

then we have $E_1 \sim E_2$ in the sense of Def. 2.7. Moreover, for the conditions 6.1 and 6.2, we only need all those $pLSB$ of the type described in the Lemma 6.2 below, that is a repeated blowing-up along a formal regular scheme.

A proof of the theorem will be given after the following lemma.

***Lemma 6.2* (Point-blow domination)**. — *Let $\sigma : V \to Z$ be a birational morphism, where V is also a scheme of finite type over a noetherian B which is reduced and irreducible. Let $D \subset V$ be a closed irreducible hypersurface in V and let $\xi \in D$ be a closed point. Let K_{-1} be the canonical image of B into the residue field of the local ring $\mathcal{O}_{Z,\xi}$, and assume that K_{-1} is a field. Let $z \in D$ denote the generic point. Let C_0 be the closure of $\sigma(z)$ in Z. Then we can find a germ of regular curve (*possibly formal*) Γ in D, whose closed point $\widehat{\xi}$ is mapped to ξ and whose generic point $\widehat{z}$ is to z, together with a $pLSB$ over Z which is written as:*

$$\text{(6.3)} \qquad \begin{array}{ccccccccc} Z'_s & \xrightarrow{\pi_{s-1}} & Z'_{s-1} & \xrightarrow{\pi_{s-2}} & \cdots & \xrightarrow{\pi_1} & Z'_1 & \xrightarrow{\pi_0} & Z'_0 = Z \\ \cup & & \cup & & & & \cup & & \cup \\ \eta_s \in C'_s & & \eta_{s-1} \in C'_{s-1} & & & & \eta_1 \in C'_1 & & \eta_0 \in C'_0 \end{array}$$

which has the following properties:

(1) *$\eta_k \in C'_k$ is a closed point, π_k is the blowing-up with center $\eta_k, \forall\, k \geqslant 0$, and $\pi_{k-1}(\eta_k) = \eta_{k-1}, \forall\, k > 0$,*

(2) *C'_k is the closure of the image into Z'_k of $\widehat{z} \in \Gamma$ while η_k is the image of $\widehat{\xi}$, by the morphism $\Gamma \to Z_k$ induced by the given birational correspondence between V and Z'_k, and*

(3) *the birational correspondence between Z'_s and V is a well-defined morphism to V from a neighborhood of $\eta_s \in Z'_s$, where η_s is necessarily mapped to the given $\xi \in V$.*

Here the last property is the important point of this lemma.

Proof. — By localizing problems about the point ξ, we may assume that V and Z are both affine schemes, say $Z = \mathrm{Spec}(A)$ and

$$V = \mathrm{Spec}(A[g_1/g_0, \ldots, g_m/g_0])$$

where $(g_0, g_1, \ldots, g_m)$ is a finite system of elements of A. Let G denote the ideal generated by these elements in A. To begin with, we choose any regular formal curve through ξ in D which is Zariski-dense in a neighborhood of $\xi \in D$. Namely, let $\mathfrak{p}$ be any prime ideal in the completion $\widehat{R}$ of the local ring $\mathcal{O}_{D,\xi}$ such that $\widehat{S} = \widehat{R}/\mathfrak{p}$ is a regular local ring of dimension one, *i.e.*, a discrete valuation ring of rank one, and moreover $\mathfrak{p} \cap \mathcal{O}_{D,\xi} = (0)$. The existence of such $\mathfrak{p}$ is known well, which is due to the fact that the formal power series ring of one variable has infinite transcendence degree over the polynomial ring of the same variable. Let $\widehat{\xi}$ be the closed point and $\widehat{z}$ the generic point of $\mathrm{Spec}(\widehat{S})$. We then define the wanted *pLSB* (6.3) to be as follows: η_0 is the image of ξ as well as that of $\widehat{\xi}$. Clearly $\eta_0 \in C'_0$. Define π_0 accordingly. The birational correspondence between V and Z'_1 is a well-defined morphism at $z \in V$ because the local ring $\mathcal{O}_{V,z}$ is a valuation ring, and moreover we get a well-defined canonical morphism $\rho_1 : \mathrm{Spec}(\widehat{S}) \to Z'_1$ because π_0 is proper. Then C'_1 is the closure of $\rho_1(\widehat{z})$ and $\eta_1 = \rho_1(\widehat{\xi})$. This continues with a well-defined morphism $\rho_k : \mathrm{Spec}(\widehat{S}) \to Z'_k$ and $\eta_k = \rho_k(\widehat{\xi}) \in C'_k$ which is the closure of $\rho_k(\widehat{z}) \in Z_k, \forall\, k$. All we need to prove is that for $s \gg 0$ the birational correspondence between Z_s and V is a well-defined morphism from a neighborhood of η_s, while η_s is then automatically mapped to ξ. For each $k \geqslant 0$, let $m_k = \mathrm{ord}_{\widehat{\xi}}(\max(\mathcal{O}_{Z_k,\eta_k})\widehat{S})$ which is monotone decreasing with respect to k until it reaches the minimum by the discreteness of the orders. So let us assume that the minimum is attained for $\forall\, k \geqslant l$. For every pair of elements $a, b \in \mathcal{O}_{Z_l,\eta_l}$ such that $\widehat{a} \neq (0)$, where $\widehat{\ }$ indicates the natural image into $\widehat{S}$, we find $a', b' \in \mathcal{O}_{Z_{l+1},\eta_{l+1}}$. such that $b/a = b'/a'$ and $\mathrm{ord}_{\widehat{\xi}}(\widehat{a}') = \mathrm{ord}_{\widehat{\xi}}(\widehat{a}) - m_l$ unless we have $\mathrm{ord}_{\widehat{\xi}}(\widehat{a}) = 0$ in which case $b/a \in \mathcal{O}_{Z_l,\eta_l}$. The reason for this is that $\max(\mathcal{O}_{Z_l,\eta_l})\mathcal{O}_{Z_{l+1},\eta_{l+1}}$ is a principal ideal, say generated by $w_l \in \max(\mathcal{O}_{Z_l,\eta_l})$, then we have $a' = a/w_l \in \mathcal{O}_{Z_{l+1},\eta_{l+1}}$ and $b' = b/w_l \in \mathcal{O}_{Z_{l+1},\eta_{l+1}}$. We repeat this if possible, but it cannot continue forever once again by the discreteness. So we will have $b/a \in \mathcal{O}_{Z_k,\eta_k}$ for all $k \gg l$. Applying this to the pair g_0, g_i for each $i, 1 \leqslant i \leqslant m$, we conclude $g_i/g_0 \in \mathcal{O}_{Z_s,\eta_s}, \forall\, i, \forall\, s \gg 0$. This means that, for $s \gg 0$, the birational correspondence between V and Z_s is a well-defined morphism from a neighborhood of $\eta_s \in Z_s$. The proof is done. $\square$

We are now ready to prove the theorem 6.1.

Proof. — Pick any t and any LSB in the sense of Def. 2.2 over $Z[t]$ which is permissible for $E_1[t]$. This LSB will be called $\mathfrak{S}$. Assuming the condition (6.1), we want to prove that $\mathfrak{S}$ is permissible for $E_2[t]$, too. For this end, it is enough to prove that $\mathfrak{S}[t']$ is permissible for $E_2[t, t']$ with an additional indeterminate t'. The reason is that the operation $[t']$ in general transforms data and processes in a manner of if-and-only-if in terms of permissibility. By doing this, the last center for blowing-up of $\mathfrak{S}[t']$ has positive dimensional image down into $Z[t,t']$. By renaming data for simplicity, we drop the furniture $[t,t']$ in the rest of the proof. Namely, express our LSB just as in Def. 2.2:

$$\begin{array}{ccccccc} Z_r \longrightarrow & U_{r-1} \subset Z_{r-1} & \longrightarrow \cdots \longrightarrow & U_1 \subset Z_1 & \longrightarrow & U_0 \subset Z_0 = Z \\ & \cup & & \cup & & \cup \\ & D_{r-1} & & D_1 & & D_0 \end{array} \tag{6.4}$$

and assume that

(6.5) *the image of D_{r-1} into Z has a positive dimension.*

Let $E_j^{(k)}$ denote the successive transforms of $E_j = E_j^{(0)}$ into Z_k along $\mathfrak{S}$, $j = 1, 2$, as long as it makes sense for $j = 2$. Write $E_j^{(k)} = (J_j^{(k)}, b_j), \forall\, k$. We then want to prove that we have

$$\operatorname{ord}_\zeta(E_1^{(k)}) \leqslant \operatorname{ord}_\zeta(E_2^{(k)}) \quad \text{or} \quad \operatorname{ord}_\zeta(J_1^{(k)})/b_1 \leqslant \operatorname{ord}_\zeta(J_2^{(k)})/b_2 \tag{6.6}$$

for the generic point $\zeta \in D_k$ and for every k, $0 \leqslant k \leqslant r-1$, so that the $\mathfrak{S}$ is also permissible for E_2 and the proof of the theorem is done. Thanks to the assumption that all of our schemes are of finite type over B, the claim for the generic point is equivalent to the same for all closed points within an open dense subset. Therefore, for instance, (6.6) is true for $k = 0$ by Th. 5.1. By the induction on the length r of the $\mathfrak{S}$, we assume that (6.6) is true for all $k < r-1$ and want to prove it for $k = r-1$. Let γ denote the last blowing-up $Z_r \to U_{r-1}$ and let $D = \gamma^{-1}(D_{r-1})$, the last exceptional divisor. Give a new name V for Z_r. Since D is a hypersurface in a regular scheme V, we have an open dense subset $D(0) \subset D$ such that

$$\begin{array}{c} J_j^{(r-1)}\mathcal{O}_{V,\xi} \textit{ is principal and equal to a power of} \\ \textit{the prime ideal of } D \textit{ at } \forall\, \xi \in D(0) \textit{ and } \forall\, j. \end{array} \tag{6.7}$$

Pick and fix a closed point $\xi \in D(0)$ and apply Lemma 6.2 to $\xi \in D \subset V$. We then get a $pLSB$:

$$\begin{array}{ccccccc} Z'_s & \xrightarrow{\pi_{s-1}} & Z'_{s-1} & \xrightarrow{\pi_{s-2}} \cdots \xrightarrow{\pi_1} & Z'_1 & \xrightarrow{\pi_0} & Z'_0 = Z \\ \cup & & \cup & & \cup & & \cup \\ \eta_s \in C'_s & & \eta_{s-1} \in C'_{s-1} & & \eta_1 \in C'_1 & & \eta_0 \in C'_0 \end{array} \tag{6.8}$$

which has all the properties stated in Lemma 6.2. In particular, we have an open neighborhood H_s of $\eta_s \in Z'_s$ such that the canonical birational correspondence induces a morphism $\lambda_s : H_s \to V$ such that $\lambda_s(\eta_s) = \xi$. Moreover, the C'_k is the closure of the image of the generic point z of D. In particular, by (6.5), we have $\dim(C'_0) > 0$. Since the centers η_k of the blowing-ups of (6.8) are all closed points, the morphisms π_{k-1} are all isomorphic at the generic points of the $C'_k, \forall\, k > 0$. Hence, (6.8) is permissible for E_1 and hence for E_2, too. If $F_j^{(k)} = (I_j^{(k)}, b_j)$ denote the successive transform in

Z'_k of $E = F_j^{(0)} = (J_j, b_j) = (I_j^{(0)}, b_j)$ along (6.8), then by the assumptions of our theorem and by Th. 5.1 we obtain

$$\operatorname{ord}_{\eta_s}(F_1^{(s)}) \leqslant \operatorname{ord}_{\eta_s}(F_2^{(s)}) \quad \text{or} \quad \operatorname{ord}_{\eta_s}(I_1^{(s)})/b_1 \leqslant \operatorname{ord}_{\eta_s}(I_2^{(s)})/b_2 \tag{6.9}$$

The idealistic exponents $E_j, j = 1, 2$, follow the same law of transformation along (6.4) as well as along (6.8), and therefore by the existence of the morphism λ_s we obtain:

$$(6.9) \quad \Longrightarrow \quad \operatorname{ord}_{\eta_s}(J_1^{(r-1)}\mathcal{O}_{Z_s,\eta_s})/b_1 \leqslant \operatorname{ord}_{\eta_s}(J_2^{(r-1)}\mathcal{O}_{Z_s,\eta_s})/b_2$$

This then implies (6.6) for $\zeta = \xi$ because the ideals of (6.6) are powers of the same prime element by (6.7). The proof of the theorem is done. □

7. Universally regular Extensions

Definition 7.1. — A morphism of schemes $\lambda : Z^\# \to Z$ is said to be *universally regular* if for every point $\eta^\# \in Z^\#$ the local homomorphism $\mathcal{O}_{Z,\eta} \to \mathcal{O}_{Z^\#,\eta^\#}$ with $\eta = \lambda(\eta^\#)$ is *universally regular* in the sense that is defined by the following conditions:

(1) The local homomorphism is flat, and

(2) for every finitely generated $\mathcal{O}_{Z,\eta}$-algebra R and for every $q \in \operatorname{Spec}(R)$, the localization $R_q = (R-q)^{-1}R$ is regular if and only if $(R-q)^{-1}R^\#$ is so, where $R^\# = R \otimes_{\mathcal{O}_{Z,\eta}} \mathcal{O}_{Z^\#,\eta^\#}$.

Remark 7.1. — Assuming the universal regularity of the above definition, we can also see that

(1) For every $q \in \operatorname{Spec}(R)$, R/qR is regular if and only if $R^\#/qR^\#$ is so.

(2) For every $q^\# \in \operatorname{Spec}(R^\#)$ mapped to $q \in \operatorname{Spec}(R)$, R_q is regular if and only if $R^\#_{q^\#}$ is so and moreover $\operatorname{ord}_{\max(R_q)}(h) = \operatorname{ord}_{\max(R^\#_{q^\#})}(h)$ for every $h \in R_q$.

Example 7.1. — Any smooth morphism is universally regular, and in particular any etale morphism is universally regular.

Example 7.2. — Let Z be any excellent scheme. In particular, let it be any scheme of finite type over the ring of integers $\mathbb{Z}$. Let us pick any point $\xi \in Z$ and let $Z^\# = \operatorname{Spec}(\widehat{\mathcal{O}}_{Z,\xi})$ where the *hat* symbol denotes the completion of the local ring. We then have the canonical morphism $Z^\# \to Z$ which is universally regular in our sense.

Remark 7.2. — For an NSB on Z in the sense of Def. 2.2, the *pull-back of LSB* by a universally regular $Z^\# \to Z$ means what follows: The LSB being written as in Def. 2.2, its pull-back is by applying $\times_Z Z^\#$ to all the components. Namely, the pull-back is defined as follows:

$$\begin{array}{ccccccccc} Z_r^\# & \longrightarrow & U_{r-1}^\# \subset Z_{r-1}^\# & \longrightarrow \cdots \longrightarrow & U_1^\# \subset Z_1^\# & \longrightarrow & U_0^\# \subset Z_0^\# = Z^\# \\ & & \cup & & \cup & & \cup \\ & & D_{r-1}^\# & & D_1^\# & & D_0^\# \end{array} \tag{7.1}$$

where $U_i^{\#} = U_i \times_Z Z^{\#}$, $Z_i^{\#} = Z_i \times_Z Z^{\#}$, and $D_i^{\#} = D_i \times_Z Z^{\#}$. Here it should be noted that $D_i^{\#}$ is regular because D_i is so for every i.

Remark 7.3. — Let $\lambda : Z^{\#} \to Z$ be a universally regular morphism in the sense of Def. 7.1. For a system of normal crossings $\Gamma = \{\Gamma_i\}$ on Z, its pull-back by λ, $\lambda^{-1}(\Gamma) = \{\lambda^{-1}(\Gamma_i)\}$, is also a system of normal crossings on $Z^{\#}$. The pull-back of an NC-divisor $\mathfrak{D} = \sum_i \gamma_i \Gamma_i$ on Z is an NC-divisor $\lambda^{-1}(\mathfrak{D}) = \sum_i \gamma_i \lambda^{-1}(\Gamma_i)$ on $Z^{\#}$. It follows that if $(\mathfrak{D}, H, b)$ is an NC-divisorial exponent on Z, then its pull-back

$$\lambda^{-1}(\mathfrak{D}, H, b) = (\lambda^{-1}(\mathfrak{D}), \lambda^{-1}(H), b)$$

is an NC-divisorial exponent on $Z^{\#}$, where $\lambda^{-1}(H) = H\mathcal{O}_{Z^{\#}}$. For every finite system of indeterminates $t = (t_1, \ldots, t_r)$, the extended morphism $\lambda[t] : Z^{\#}[t] \to Z[t]$ is also universally regular.

Theorem 7.1. — *Let $\lambda : Z^{\#} \to Z$ be a universally regular morphism in the above sense where both Z and $Z^{\#}$ are regular schemes. Let $E = (J, b)$ be an idealistic exponent on Z and let $E^{\#} = (J^{\#}, b)$ be the pull-back of E by λ, where $J^{\#} = J\mathcal{O}_{Z^{\#}}$. Similarly, let $T = (\mathfrak{D}, H, b)$ be an NC-divisorial exponent on Z and let $T^{\#} = (\mathfrak{D}^{\#}, H^{\#}, b)$ be its pull-back by λ. Then, for any finite system of indeterminates $t = (t_1, \ldots, t_r)$ and given any LSB over $Z[t]$ in the sense of Def. 2.2, its pull-back by λ in the sense of Rem. 7.2 is permissible for $E^{\#}[t]$ (respectively for $T^{\#}[t]$) if and only if the LSB is permissible for $E[t]$ (respectively for $T[t]$) within a sufficiently small neighborhood of the image of the morphism $\lambda[t]$.*

Proof. — The proof is easily reduced to the case in which the LSB consists of a single blowing-up. Moreover, all the assumptions and conditions are preserved by the application of $[t]$ and hence it is enough to consider the case without $[t]$. Pick any point $\eta^{\#} \in Z^{\#}$ and let $\eta = \lambda(\eta^{\#}) \in Z$. Then the local injection $\mathcal{O}_{Z,\eta} \hookrightarrow \mathcal{O}_{Z^{\#},\eta^{\#}}$ is faithfully flat, so that a regular system of parameters x of $\mathcal{O}_{Z,\eta}$ mapped to a regular sequence in $\mathcal{O}_{Z^{\#},\eta^{\#}}$. Because of the universal regularity assumption, $\mathcal{O}_{Z^{\#},\eta^{\#}} / \max(\mathcal{O}_{Z,\eta})\mathcal{O}_{Z^{\#},\eta^{\#}}$ is regular and hence x extends to a regular system of parameters (x, y) of $\mathcal{O}_{Z^{\#},\eta^{\#}}$. Using this (x, y) by which means the initial forms of elements can be polynomially expressible, it is shown that for every element $f \in \mathcal{O}_{Z,\eta}$ its order in there is equal to that in $\mathcal{O}_{Z^{\#},\eta^{\#}}$. It follows that $\lambda^{-1}(\mathrm{Sing}(E)) = \mathrm{Sing}(E^{\#})$. Moreover, for a subscheme $D \subset Z$ and for any pair of points $\eta = \lambda(\eta^{\#})$ as above, $D^{\#}$ is regular at $\eta^{\#}$ if and only if D is regular at η. Thus, so long as we restrict our attention to a sufficiently small neighborhood of $\lambda(Z^{\#}) \subset Z$, the last if-and-only-if assertion of the theorem follows immediately. $\square$

Corollary 7.2. — *Assume that Z is excellent, i.e., its affine rings are all excellent so that the completion of every local ring of Z is universally regular. Let $E_i, i = 1, 2$, be idealistic exponents on Z and $T_i, i = 1, 2$, NC-divisorial exponents on Z. For each point $\xi \in Z$, let $\widehat{\lambda}_\xi : \widehat{Z}_\xi \to Z$ be the canonical morphism with $\widehat{Z}_\xi = \mathrm{Spec}(\widehat{\mathcal{O}}_{Z,\xi})$*

where $\widehat{\ }$ *denotes the completion of a local ring. We then assert that* $E_1 \subset E_2$ *in the sense of Def. 2.6 (respectively* $T_1 \subset T_2$ *in the sense of Def. 4.2) if* $\widehat{\lambda}_\xi^{-1}(E_1) \subset \widehat{\lambda}_\xi^{-1}(E_2)$ *(respectively* $\widehat{\lambda}_\xi^{-1}(T_1) \subset \widehat{\lambda}_\xi^{-1}(T_2)$*) for all* $\xi \in Z$.

Proof. — Immediate from the theorem 7.1. □

8. Ambient Reductions and Expansions

Let Z be a regular scheme and W a closed regular subscheme of Z. We keep this general situation until we need to come down to more specific cases which are more geometrical and fit to our primary purposes. We are either given with an idealistic exponent $E = (J, b)$ on Z and search for its reduction to W or, the other way around, we are given with an idealistic exponent $F = (H, a)$ on W and look for its expansions to Z. Let us first define what *reduction* and *expansion* mean.

Definition 8.1. — Given $E = (J, b)$ on Z, $F = (H, a)$ on W is called an *ambient reduction* of E from Z to W if the following condition is satisfied:

> *Pick any finite system of indeterminates* $t = (t_1, \ldots, t_r)$ *and any* LSB *on* $Z[t]$ *in the sense of Def. 2.2, subject to the condition that all the centers contained in the strict transforms of* $W[t]$*. Then the* LSB *permissible for* $E[t]$ *if and only if its induced* LSB *on* $W[t]$ *is permissible for* $F[t]$.

Conversely, if F is given on W and E on Z has the said property then E is called an *ambient expansion of* F *from* W *to* Z.

Definition 8.2. — We say that E on Z is a *maximal ambient expansion* of F on W if for every finite system of indeterminates t we have

> *an* LSB *on* Z *is permissible for* $E[t]$ *if and only if*
> *its centers are all contained in the strict transforms of* W *and the induced* LSB *on* W *is permissible for* $F[t]$.

Here the induced LSB have the same centers as the given LSB but are all viewed as subschemes of the strict transforms of W.

Remark 8.1. — It is not true in general that F of the Def. 8.1 can be written as (H, b) with the same integer b of E and some coherent ideal sheaf H on W. For instance, examine the following case: $Z = \mathrm{Spec}(\Bbbk[x,t])$, $W = \mathrm{Spec}(\Bbbk[x,t]/(t))$, $E = (tx^b\Bbbk[x,t], b)$ and $F = (x^b\Bbbk[x,t]/(t), b-1)$ where $b > 2$. Given E, its ambient reductions are not unique as idealistic exponents but they are mutually equivalent in the sense of Def. 2.7. Given F, however, its ambient expansions E are not in general equivalent to each other. For instance, $E' = ((tx^b, t^b)\Bbbk[x,t], b)$ is also an ambient expansion of the same F as above but it is not equivalent to the $E = (tx^b\Bbbk[x,t], b)$.

It is an interesting question to ask if there exists a canonical and constructive ambient reductions or ambient expansions. This question is going to be addressed in what follows. At any rate, we first propose a candidate for a canonical ambient reduction of E from Z to W by means of differentiation as follows.

Definition 8.3. — We define the *ambient Diff-reduction of* $E = (J, b)$ *to* W as follows:

$$\mathrm{DRed}_W(E) = \bigcap_{0 \leqslant j < b} \Big(\big(\mathrm{Diff}_Z^{(j)}\, J\big) O_W, b - j \Big)$$

which is an idealistic exponent on W.

We must admit that $\mathrm{DRed}_W(E)$ is not in general any ambient reduction of E to W in the sense of Def. 8.1. This is seen by the following example.

Example 8.1. — Let $Z = \mathrm{Spec}(A)$ where $A = \mathbb{Z}[x]$ with an indeterminates x. Let $p > 1$ be a prime number. Let $B = \mathbb{Z}[p^{1/p}]$ and $W = \mathrm{Spec}(B)$. We can naturally identify B with $A/(p - x^p)$ and hence we identify W with a closed subscheme of Z. Let $E = \big(x(p - x^p)A, 2\big)$ be the given idealistic exponent on Z. We have

$$\mathrm{DRed}_W(E) = ((0)B, 2) \cap \big(x(-px^{p-1}A)/(p - x^p)A, 1\big) = (p^2 B, 1)$$

By the blowing-up with center $(p, x)A$ the transform of E has empty singular locus while the transform of $\mathrm{DRed}_W(E)$ is the idealistic exponent $\big((px^{-1}), 1\big) = \big((pp^{-(p-1)/p}), 1\big)$ on W whose singular locus is not empty. This proves that $\mathrm{DRed}_W(E)$ is *not* an ambient reduction of E in the sense of Def. 8.1.

This example appears to stand against the notion of Def. 8.3. However, it will be shown that $\mathrm{DRed}_W(E)$ is indeed an ambient reduction in many important cases, including all algebro-geometric cases over perfect fields of any characteristics. The investigation in these cases will be done under a little more broader framework called *formally retractible cases*, which will be defined and studied later. At any rate, a significant point of Def. 8.3 is that the expression is constructively universal for all regular subschemes W of Z. For instance, if W' is a regular subscheme of W, then $\mathrm{DRed}_W(E)$ induces $\mathrm{DRed}_{W'}(E)$ directly by the application of the natural homomorphism $\mathcal{O}_W \to \mathcal{O}_{W'}$.

9. Retractible Cases

We now proceed to define the notion of *formal retractions* which are mainly of technical use especially to show that $\mathrm{DRed}_W(E)$ can indeed be an ambient reduction in the sense of Def. 8.1 in a wide range of important cases.

Definition 9.1. — Let W be an irreducible regular subscheme of a regular scheme Z. A *retraction from* Z *to* W is a morphism $\mathfrak{i}_{Z/W} : Z \to W$ such that $\mathfrak{i}_{Z/W} \circ \mathfrak{n}_{W/Z} = id_W$

where $\mathfrak{n}_{W/Z}$ denotes the inclusion morphism and id_W is the identity. We then have a homomorphism

$$\iota : \mathcal{O}_W \longrightarrow \mathcal{O}_Z|_W \quad \text{such that } n \circ \iota = \mathrm{id} \tag{9.1}$$

where $|_W$ denotes the point-set restriction having the same stalks, ι is associated with $\mathfrak{i}_{Z/W}$ and n is the natural epimorphism associated with the inclusion $\mathfrak{n}_{W/Z}$. Locally for each point $\eta \in W$ we have a local monomorphism

$$\iota_\eta : \mathcal{O}_{W,\eta} \longrightarrow \mathcal{O}_{Z,\eta} \quad \text{such that } n_\eta \circ \iota_\eta = \mathrm{id} \tag{9.2}$$

where ι_η is induced by ι and n_η is by n.

Definition 9.2. — A retraction $\mathfrak{i}_{Z/W}$ is said to be *diff-regular* if for every $\eta \in W$ there exist $y = (y_1, \ldots, y_d)$ and $\partial_\alpha, \forall\, \alpha \in \mathbb{Z}_0^d$, where $d = \operatorname{codim}(W \subset Z)$, which are subject to the following conditions:

(1) y is a minimal base of $I_{W/Z,\eta}$, the ideal of W in $\mathcal{O}_{Z,\eta}$, and
(2) the $\mathcal{O}_{Z,\eta}$-module $\mathrm{Diff}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ is freely generated by the ∂_α,

such that

$$\partial_\alpha(y^\beta) = \begin{cases} \binom{\beta}{\alpha}\, y^{\beta-\alpha} & \textit{if } \beta \in \alpha + \mathbb{Z}_0^d \\ 0 & \textit{if otherwise} \end{cases} \tag{9.3}$$

Note that $\partial_\alpha, \forall\, \alpha \neq (0)$, are all identically zero in $\mathrm{Im}(\iota_\eta)$ by definition.

Remark 9.1. — Assuming a retraction $\iota_{Z/W}$ of Def. 9.1 is given, the following conditions are all mutually equivalent:

(1) For all $\nu \geqslant 1$, $\Omega^{(\nu)}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ is a finite $\mathcal{O}_{Z,\eta}$-module.
(2) For all $\nu \geqslant 1$, $\Omega^{(\nu)}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ is a free $\mathcal{O}_{Z,\eta}$-module of rank $\binom{\nu+d}{d}$.
(3) For all $\nu \geqslant 1$, the images of $\delta(y_j) = 1 \otimes y_j - y_j \otimes 1, 1 \leqslant j \leqslant d$, generate $\Omega^{(\nu)}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ as $\mathcal{O}_{Z,\eta}$-module, where $(y_1, \ldots, y_d)$ is any one minimal base of the ideal $I_{Z/W,\eta}$.

Moreover, any one of the above conditions implies the diff-regularity of Def. 9.2.

Proof. — The first condition implies that

$$\Omega_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)} \otimes_{\mathrm{Im}(\iota_\eta)} \mathrm{Im}(\iota_\eta)[[y]] \simeq \Omega_{\mathrm{Im}(\iota_\eta)[[y]]/\,\mathrm{Im}(\iota_\eta)}$$

where the right hand side is freely generated by $\delta y_j, 1 \leqslant j \leqslant d$. It is therefore clearly equivalent to any one of the other two conditions. The diff-regularity of Def. 9.2 follows. Here the point is the completion of a local ring is faithfully flat. $\square$

Definition 9.3. — Assume that we are given a retraction $\mathfrak{i}_{Z/W} : Z \to W$. Let $E = (J, b)$ be an idealistic exponent on Z. Then, with respect to the specific $\mathfrak{i}_{Z/W}$, we define *ambient* $\mathfrak{i}$*-reduction of* E *to* W, denoted by $\mathfrak{i}\,\mathrm{Red}_W(E)$, as follows: Pick any point $\eta \in W$ and any minimal ideal base y of W in $\mathcal{O}_{Z,\eta}$. Then write each element $f \in J_\eta$ in the form $\sum_{\alpha\in\mathbb{Z}_0^d} f_\alpha y^\alpha$ with $f_\alpha \in \mathrm{Im}(\iota_\eta), \forall\, \alpha$, and define

$$\mathfrak{i}\,\mathrm{Red}_W(E) = \bigcap_{j=0}^{b-1} (K_j, b-j) \quad \text{where } (K_j)_\eta = \sum_{\substack{\forall\,\alpha \text{ with } |\alpha|\leqslant j \\ \forall\, f\in J_\eta}} f_\alpha \mathcal{O}_{W,\eta} \tag{9.4}$$

Remark 9.2. — The above Def. 9.3 makes a good sense because of the following two facts:

1) For each point $\eta \in W$ and every $j, 0 \leqslant j \leqslant b-1$, the local ideal $(K_j)_\eta$ is independent of the choice of a minimal ideal base y of W in $\mathcal{O}_{Z,\eta}$.

2) A coherent ideal sheaf K_j on W is uniquely determined by saying that its stalks are $(K_j)_\eta, \forall\, \eta \in W$.

Proof. — To prove 1), if $z = (z_1, \dots, z_d)$ is any other minimal ideal base of W in $\mathcal{O}_{Z,\eta}$, then we can write

$$y^\alpha = \sum_{\substack{\beta\in\mathbb{Z}^d \\ |\beta|\geqslant|\alpha|}} a_{\alpha\beta} z^\beta \quad \text{with } a_{\alpha\beta} \in \mathrm{Im}(\iota_\eta), \forall\,(\alpha,\beta)$$

Hence if we write

$$f = \sum_{\alpha\in\mathbb{Z}^d} f_\alpha y^\alpha = \sum_{\gamma\in\mathbb{Z}^d} g_\gamma z^\gamma \quad \text{with } g_\gamma \in \mathrm{Im}(\iota_\eta)$$

then we get

$$g_\gamma = \sum_{\alpha, |\alpha|\leqslant|\gamma|} f_\alpha a_{\alpha\gamma}$$

which belong to $(K_j)_\eta$ for $j = |\gamma|$ and for all $f \in J_\eta$. In this argument, y and z are exchangeable and therefore follows the independence of K_j. Next, as for the assertion 2), the same y stays to be a minimal ideal base of W in $\mathcal{O}_{Z,\xi}$ for all ξ within a neighborhood of $\eta \in W$. Hence, $(K_j)_\eta$ extends to a coherent ideal sheaf K_j with its stalks $(K_j)_\xi$ for all ξ within the same neighborhood of $\eta \in W$. $\square$

If $\mathfrak{i}_{Z/W}$ is diff-regular in the sense of Def. 9.1, then a more intrinsic formula is obtained for $\mathfrak{i}\,\mathrm{Red}_W(E)$ of Def. 9.3 as follows.

Remark 9.3. — Assume that $\mathfrak{i}_{Z/W}$ is diff-regular. Then we have

$$\mathfrak{i}\,\mathrm{Red}_W(E) = \bigcap_{j=0}^{b-1} \Big(\big(\mathrm{Diff}^{(j)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J\big)\mathcal{O}_W, b-j \Big) \tag{9.5}$$

where $\iota : \mathcal{O}_W \to \mathcal{O}_Z$ is the monomorphism associated with $\mathfrak{i}_{Z/W}$.

Proof. — The question of the equality being local, we pick $\eta \in W$ and y as in Def. 9.3. Then we have differential operators $\partial_\alpha \in \mathrm{Diff}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ as was described in Def. 9.1. Here $\partial_\alpha, |\alpha| \leqslant j$, form a free base of $\mathrm{Diff}^{(j)}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)}$ as $\mathcal{O}_Z$-module within a neighborhood of $\eta \in W$. Hence, when $f \in J_\eta$ is written as $f = \sum_{\alpha\in\mathbb{Z}^d} f_\alpha y^\alpha$ with $f_\alpha \in \mathrm{Im}(\iota_\eta), \forall\,\alpha$, we have

$$\big(\mathrm{Diff}^{(j)}_{\mathcal{O}_{Z,\eta}/\,\mathrm{Im}(\iota_\eta)} f\big)\mathcal{O}_{W,\eta} = \{f_\alpha \mid |\alpha| \leqslant j\}\mathcal{O}_{W,\eta}$$

from which our remark follows immediately in view of Def. 9.3. □

Proposition 9.1. — *Assume that we have an ambient retraction* $\mathfrak{i}_{Z/W}$ *which is diff-regular in the sense of Def. 9.1. Then we have the following equivalence:*

$$\mathfrak{i}\,\mathrm{Red}_W(E) \sim \mathrm{DRed}_W(E)$$

in the sense of Def. 2.7, where $\mathrm{DRed}_W(E)$ *is defined by Def. 8.3.*

Proof. — With reference to Remark 9.3, we note that

$$\begin{aligned}\big(\mathrm{Diff}^{(j)}_Z J\big)\mathcal{O}_W &= \Big(\sum_{k+l=j} \mathrm{Diff}^{(k)}_Z \,\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J\Big)\mathcal{O}_W\\ &= \sum_{k+l=j} \mathrm{Diff}^{(k)}_W \Big(\big(\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J\big)\mathcal{O}_W\Big)\end{aligned}$$

Therefore

$$\begin{aligned}\mathfrak{i}\,\mathrm{Red}_W(E) &= \bigcap_{j=0}^{b-1}\Big(\big(\mathrm{Diff}^{(j)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J\big)\mathcal{O}_W, b-j\Big)\\ &\sim \bigcap_{j=0}^{b-1}\bigcap_{a=0}^{j-1}\Big(\mathrm{Diff}^{(a)}_W\big((\mathrm{Diff}^{(j)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J)\mathcal{O}_W\big), b-j-a\Big)\\ &\sim \bigcap_{j=0}^{b-1}\bigcap_{k+l=j}\Big(\mathrm{Diff}^{(k)}_W\big((\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J)\mathcal{O}_W\big), b-j\Big)\\ &\sim \bigcap_{j=0}^{b-1}\Big(\sum_{k+l=j}\mathrm{Diff}^{(k)}_W\big((\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)} J)\mathcal{O}_W\big), b-j\Big)\\ &\sim \bigcap_{j=0}^{b-1}\Big(\big(\mathrm{Diff}^{(j)}_Z J\big)\mathcal{O}_W, b-j\Big) = \mathrm{DRed}_W(E)\end{aligned}$$

□

Definition 9.4. — Let $W \subset Z$ be the same as above. For a point $\eta \in W$, we say that Z is *locally formally retractible to* W *at* η if there exists a monomorphism of $\Bbbk$-algebras

$$\iota_\eta : \widehat{\mathcal{O}}_{W,\eta} \longrightarrow \widehat{\mathcal{O}}_{Z,\eta} \quad \text{such that } \widehat{n}_\eta \circ \iota_\eta = \mathrm{id} \tag{9.6}$$

where $\widehat{\ }$ denotes the max()-adic completion of local rings, $\widehat{n}_\eta$ does the natural homomorphism $\widehat{\mathcal{O}}_{Z,\eta} \to \widehat{\mathcal{O}}_{W,\eta}$ and *id* does the identity endomorphism. Let $\widehat{Z}_\eta = \mathrm{Spec}(\widehat{\mathcal{O}}_{Z,\eta})$ and $\widehat{W}_\eta = \mathrm{Spec}(\widehat{\mathcal{O}}_{W,\eta})$. We then have the retraction from $\widehat{Z}_\eta$ to $\widehat{W}_\eta$ in the sense of Def. 9.1

$$\mathfrak{i}_{\widehat{Z}_\eta/\widehat{W}_\eta} : \widehat{Z}_\eta \longrightarrow \widehat{W}_\eta$$

which is associated with ι_η. The morphism $\mathfrak{i}_{\widehat{Z}_\eta/\widehat{W}_\eta}$, and the homomorphism ι_η, will be called a *local formal retraction from* Z *to* W *at* η. If such ι_η exists at $\eta \in W$, we simply say that Z is *locally formally retractible to* W *at* $\eta \in W$.

Remark 9.4. — We have natural morphisms

$$\mathrm{j}_{Z,\eta} : \widehat{Z}_\eta \longrightarrow Z \quad \text{and} \quad \mathrm{j}_{W,\eta} : \widehat{W}_\eta \longrightarrow W$$

which are compatible with the inclusions $W \subset Z$ and $\widehat{W}_\eta \subset \widehat{Z}_\eta$. Here an important point is that, if Z is excellent (in particular, if it is any scheme of finite type over $\Bbbk$), then the both morphisms $\mathrm{j}_{Z,\eta}$ and $\mathrm{j}_{W,\eta}$ are *universally regular* in the sense of Def. 7.1, so that the Theorem 7.1 is applicable to problems about them.

Example 9.1. — Let us take the example of $W \subset Z$ which was taken in Ex 8.1. Let $R = \mathcal{O}_{Z,\eta}$ with $\eta = (p, x)A$. Then we claim that there exists no formal retraction of Z to W at the point η. In fact, if there existed any, then we should have a derivation ∂ in the completion $\widehat{R}$ such that $\partial(p - x^p)$ is a unit in $\widehat{R}$, which is clearly absurd.

The following lemma asserts that local formal retractions are ubiquitous in the algebraic geometry over a field.

Lemma 9.2. — *Assume that $W \subset Z$ be as in Def. 9.1 and that the base ring $\Bbbk$ is a field $\mathbf{k}$ of any characteristics, Then there exists a formal local retraction from Z to W at every closed point η of W.*

Proof. — Let $w = (w_1, \ldots, w_r)$ be any minimal base of the ideal of W in the local ring $\mathcal{O}_{Z,\eta}$ which is extended to a regular system of parameters (w, y) of $\mathcal{O}_{Z,\eta}$. Let $\widehat{R}$ be the completion of $\mathcal{O}_{Z,\eta}$. Because $\mathcal{O}_{Z,\eta}$ contains a field $\mathbf{k}$, $\widehat{R}$ contains a field K and is isomorphic to its residue field, thanks to the structure theorem. Then we have a natural isomorphism $\widehat{R} \sim K[[w, y]]$. Since y induces a regular system of parameters of $\mathcal{O}_{W,\eta}$, the completion $\widehat{S}$ of this local ring is naturally isomorphic to $K[[y]]$. This yields an obvious injection $\iota_\eta : \widehat{S} \to \widehat{R}$ and we have $n_{\widehat{\eta}} \circ \iota_\eta = id$ with the natural homomorphism $n_{\widehat{\eta}} : \widehat{R} \to \widehat{S}$. This ι_η is the wanted formal retraction in the sense of Def. 9.4. □

Remark 9.5. — If $\mathbf{k}$ is perfect in Lemma 9.2, ι_η of Def. 9.4 can be chosen to be $\mathbf{k}$-homomorphism, *i.e.*, we can choose $K \supset \mathbf{k}$. This is not always possible if $\mathbf{k}$ is not perfect.

Remark 9.6. — With the notation of Def. 9.4, if Z is of finite type over any base ring $\Bbbk$ and if there exists a local formal retraction from Z to W at every closed point of W, then we will later prove that $\mathrm{DRed}_W(E)$ for any E on Z is in fact an ambient retraction of E from Z to W.

10. Ambient Reduction Theorems

We now come to the last of the three key techniques. Namely we will prove theorems named *Ambient Reduction* which are useful for cutting down the dimension of the ambient scheme. For the theory of ambient reductions, we need to restrict our attention to the case of an ambient scheme Z which is of finite type over the base ring $\Bbbk$. As was in the previous section, let W be an irreducible closed regular subscheme of a regular scheme Z. The first ambient reduction theorem, Th. 10.5 below, is for the case when we have local formal retractions. We first prove a few lemmas needed for the proof of the theorem.

Lemma 10.1. — *Let us assume that we have a retraction $\mathrm{i}_{Z/W} : Z \to W$ which is diff-regular in the sense of Def. 9.1. Let $E = (J, b)$ be an idealistic exponent on Z. We have the ambient $\mathrm{i}_{Z/W}$-reduction of E, $\mathrm{i}\,\mathrm{Red}_W(E)$, which is an idealistic exponent on W defined by Def. 9.3. We then have*

$$\mathrm{Sing}\big(\mathrm{i}\,\mathrm{Red}_W(E)\big) = \mathrm{Sing}(E) \cap W$$

Proof. — The question being local, pick $\eta \in W$. Then

$$\begin{aligned} \mathrm{ord}_\eta(E) \geqslant 1 &\iff \mathrm{ord}_\eta(J) \geqslant b \\ &\iff \mathrm{ord}_\eta(\mathrm{Diff}^{(j)}_{\mathcal{O}_{Z,\eta}/\iota_\eta(\mathcal{O}_{W,\eta})} J) \geqslant b - j, \quad \forall j, 0 \leqslant j \leqslant b-1 \\ &\iff \mathrm{ord}_\eta(\mathrm{i}\,\mathrm{Red}_W(E)) \geqslant 1 \end{aligned}$$

where the second $\Longrightarrow$ is by Th. 3.4 and Th. 5.1. □

Lemma 10.2. — *The assumption being the same as above, let t be any finite system of indeterminates. We then have*

$$\big(\mathrm{i}\,\mathrm{Red}_W(E)\big)[t] = \mathrm{i}\,\mathrm{Red}_{W[t]}(E[t])$$

where the right hand side is with respect to the naturally extended retraction $\mathrm{i}_{Z[t]/W[t]}$. This will mean that any assertion about $\mathrm{i}\,\mathrm{Red}_W(E)$ automatically extends to $\mathrm{i}\,\mathrm{Red}_W(E))[t]$ for every t.

Lemma 10.3. — *Let C be any irreducible closed regular nowhere dense subscheme of W. Let us denote the blowing-ups with the same center C as follows:*

$$\pi_Z : Z' \longrightarrow Z \quad \textit{and} \quad \pi_W : W' \longrightarrow W$$

Then π_Z is permissible for E if and only π_W is permissible for $\mathrm{i}\,\mathrm{Red}_W(E)$. Moreover there exists a retraction $\mathrm{i}_{Z'/W'} : Z' \to W'$ uniquely determined by the following commutative diagram:

$$\begin{array}{ccccc} Z' & \xrightarrow{\mathrm{i}_{Z'/W'}} & W' & \xrightarrow{\textit{inclusion}} & Z' \\ \Big\downarrow{\scriptstyle \pi_Z} & & \Big\downarrow{\scriptstyle \pi_W} & & \Big\downarrow{\scriptstyle \pi_Z} \\ Z & \xrightarrow{\mathrm{i}_{Z/W}} & W & \xrightarrow{\textit{inclusion}} & Z \end{array} \tag{10.1}$$

Moreover if E' is the transform of E by π_Z then the transform of $\mathrm{i}\,\mathrm{Red}_W(E)$ by π_W is the ambient i-reduction $\mathrm{i}\,\mathrm{Red}_{W'}(E')$ of E' by $\mathrm{i}_{Z'/W'} : Z' \to W'$.

Proof. — The permissibility assertion is immediate from Lemma 10.1. As for the existence of $\mathfrak{i}_{Z'/W'}$, its uniqueness is even locally certain from the diagram (10.1) because C is nowhere dense and the blowing-ups are isomorphic outside C. Therefore the existence can be checked locally. So we may assume that Z is affine, say $\mathrm{Spec}(A)$, and there exist $y = (y_1, \ldots, y_d)$ and $z = (z_1, \ldots, z_c)$ such that yA is the ideal of W and $(y, z)A$ is the ideal of C, where $d = \mathrm{codim}(W \subset Z)$ and $c = \mathrm{codim}(C \subset W)$. Here we can choose z inside $\iota(A/yA)$ thanks to the retraction $\mathfrak{i}_{Z/W}$. Then $W' \subset Z'$ is covered by one of the affine scheme $\mathrm{Spec}(A_a)$ with $A_a = A[z_a^{(-1)}y, z_a^{(-1)}z]$, $1 \leqslant a \leqslant c$. Pick any one a. Then the inclusion $A/yA \to A$ associated with $\mathfrak{i}_{Z/W}$ is extendable to $A_a/(z_a^{(-1)}y)A_a \to A_a$ which is the one associated with $\mathfrak{i}_{Z'/W'}$ within the open affine $\mathrm{Spec}(A_a) \subset Z'$. The diagram (10.1) is thus established. From now on, we let $a = 1$ for simplicity. There will be no loss of generality. We shall denote $z' = (z_1, z_2/z_1, \ldots, z_c/z_1)$, $y' = y/z_1$ and $\iota' : \mathcal{O}_{W'} \to \mathcal{O}_{Z'}$ for the monomorphism associated with the new retraction $\mathfrak{i}_{Z'/W'}$. As for the assertion on the transforms, we first note that the differential operators in $\mathrm{Diff}_{\mathcal{O}_Z/\,\mathrm{Im}(\iota)}$ act trivially on elements of $\iota(\mathcal{O}_W)$ and in particular on z_1. In other words, the multiplication and division by z_1 commute with the differential operators. Hence, for instance.

$$z_1^{-(b-k)}\,\mathrm{Diff}^{(k)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\,J = \Big(z_1^k\,\mathrm{Diff}^{(k)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\Big)\big(z_1^{-b}J\big)$$

On the other hand, we have

$$\mathrm{Diff}^{(k)}_{\mathcal{O}_{Z'}/\iota'(\mathcal{O}_{W'})} = \sum_{l=0}^{k} z_1^l\,\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)} \tag{10.2}$$

because $\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}$ is freely generated by $\{\partial_\alpha \mid \alpha \in \mathbb{Z}_0^d, |\alpha| \leqslant l\}$ as was described in Def. 9.1 while $\mathrm{Diff}^{(k)}_{\mathcal{O}_{Z'}/\iota'(\mathcal{O}_{W'})}$ is so by $\{z_1^{|\alpha|}\partial_\alpha, |\alpha| \leqslant k\}$. Incidentally this last generation is easily proven by the following calculation which is straight forward from (9.3) of Def. 9.2:

$$(z_1^{|\alpha|}\partial_\alpha)(y'^{\beta}) = \begin{cases} \binom{\beta}{\alpha}\,y'^{\beta-\alpha} & \textit{if } \beta \in \alpha + \mathbb{Z}_0^d \\ 0 & \textit{if otherwise} \end{cases}$$

We write $E' = (J', b)$ for the transform of E by π_Z. Now, by the above (10.2), we get

$$\begin{aligned} \bigcap_{k=0}^{j}\Big(\mathrm{Diff}^{(k)}_{\mathcal{O}_{Z'}/\iota'(\mathcal{O}_{W'})}\,J', b-k\Big) &= \bigcap_{k=0}^{j}\Big(\sum_{l=0}^{k} z_1^{-(b-l)}\,\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\,J, b-k\Big) \\ &= \bigcap_{0\leqslant l\leqslant k\leqslant j}\Big(z_1^{-(b-l)}\,\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\,J, b-k\Big) \\ &= \bigcap_{l=0}^{j}\Big(z_1^{-(b-l)}\,\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\,J, b-l\Big) \end{aligned} \tag{10.3}$$

where the last equivalence is simply because $b-k \geqslant b-l$. Here the first term is the $\mathfrak{i}\,\mathrm{Red}_{W'}(E')$ with respect to $\mathfrak{i}_{Z'/W'}$ while the last is the transform of

$$\mathfrak{i}\,\mathrm{Red}_W(E) = \bigcap_{l=0}^{b-1}\Big(\mathrm{Diff}^{(l)}_{\mathcal{O}_Z/\iota(\mathcal{O}_W)}\,J, b-l\Big)$$

The proof of our lemma is complete with (10.3). □

Corollary 10.4. — *Pick any LSB over Z in the sense of Def. 2.2 and assume that all of its centers are contained in the strict transforms of W by the blowing-ups in it so that the LSB on Z induces an LSB over W. Then the LSB on Z is permissible for E if and only if the induced LSB on W is permissible for* $\mathfrak{i}\,\mathrm{Red}_W(E)$.

***Theorem 10.5* (Ambient Reduction Theorem/retractible case)**

Let Z be a regular scheme of finite type over the base ring $\Bbbk$ and W an irreducible closed regular subscheme of Z. Assume that Z is formally retractible to W everywhere in the sense of Def. 9.4. Then the ambient Diff-reduction $\mathrm{DRed}_W(E)$ *defined by Def. 8.3 is an ambient reduction of E to W in the sense of Def. 8.1. In particular, thanks to Lemma 9.2, this is always true if $\Bbbk$ is a perfect field.*

Remark 10.1. — The assumption of the theorem was that Z is finite type over $\Bbbk$. All we need from this assumption is that the canonical homomorphism

$$\widehat{\mathcal{O}}_{Z,\eta} \otimes_{\mathcal{O}_{Z,\eta}} (\mathrm{Diff}_Z)_\eta \longrightarrow \mathrm{Diff}_{\widehat{\mathcal{O}}_{Z,\eta}/\Bbbk}$$

is bijective for every closed point $\eta \in Z$. It means that every formal differential operator at any point is generated by algebraic differential operators.

Proof. — (Proof of the theorem) The criterion for an ambient reduction in terms of LSB's, described in Def. 8.1, can be checked locally. So pick any closed point $\eta \in W$ and let $\widehat{Z}_\eta = \mathrm{Spec}(\widehat{\mathcal{O}}_{Z,\eta})$ where $\widehat{\ }$ means completion. Letting I be the ideal of $W \subset Z$, we define the subscheme $\widehat{W}_\eta$ of $\widehat{Z}_\eta$ defined by the ideal $I_\eta\widehat{\mathcal{O}}_{Z,\eta}$. Then the canonical morphisms $\mathfrak{j}_Z : \widehat{Z}_\eta \to Z$ and $\mathfrak{j}_W : \widehat{W}_\eta \to W$ are both universally regular and moreover we have a diff-regular retraction

$$\mathfrak{i}_{\widehat{Z}_\eta/\widehat{W}_\eta} : \widehat{Z}_\eta \longrightarrow \widehat{W}_\eta$$

Now pick call L any one LSB over Z in the sense of Def. 2.2 such that its centers are all contained in the strict transforms of W. Let L^* denote the the LSB induced by the L from Z to W. Our end is to prove that

$$L^* \textit{ is permissible for } \mathrm{DRed}_W(E) \Longleftrightarrow L \textit{ is so for } E \tag{10.4}$$

This certainly enough for the proof of our theorem because the situation for $E[t]$ and $\mathrm{DRed}_W(E)[t] = \mathrm{DRed}_W[t](E[t])$ for any t is completely similar to that of E and $\mathrm{DRed}_W(E)$. Now we write $\widehat{L}^*$ for the pull-back of L^* from W to $\widehat{W}_\eta$ and $\widehat{L}$ for the pull-back of L from Z to $\widehat{Z}_\eta$. Also, from Z to $\widehat{Z}_\eta$, we have extensions $\widehat{E}_\eta$ of E and

$\widehat{L}_\eta$ of L, while from W to $\widehat{W}_\eta$, $\widehat{\mathrm{DRed}}_W(E)_\eta$ of $\mathrm{DRed}_W(E)$ and $\widehat{L}^*_\eta$ of L^*. To be more explicit,

$$(10.5)\quad\text{and}\qquad \begin{aligned}\widehat{\mathrm{DRed}}_W(E)_\eta = \widehat{\mathcal{O}}_{Z,\eta}\otimes_{\mathcal{O}_{Z,\eta}}\Big(\mathrm{DRed}_W(E)\Big)_\eta = \bigcap_{j=0}^{b-1}\big(\mathrm{Diff}^{(j)}_{\widehat{Z}_\eta}\widehat{J}_\eta, b-j\big)\\ \widehat{E}_\eta = \widehat{\mathcal{O}}_{Z,\eta}\otimes_{\mathcal{O}_{Z,\eta}} E_\eta = \big(\widehat{J}_\eta, b\big)\quad \text{where } \widehat{J}_\eta = J_\eta\widehat{\mathcal{O}}_{Z,\eta}\end{aligned}$$

where $()_\eta$ means to localize ideals at η and $\otimes$ means to take the tensor product with localized ideals. Here it should be noted that

$$(10.6)\qquad \widehat{\mathrm{DRed}}_W(E)_\eta = \mathrm{DRed}_{\widehat{W}_\eta}(\widehat{E}_\eta)$$

Now we have

$$\begin{aligned}
&L^* \textit{ is permissible for } \mathrm{DRed}_Z(E) \textit{ locally at } \eta && \text{(by Th. 7.1)}\\
\Longleftrightarrow\ &\widehat{L}^* \textit{ is permissible for } \widehat{\mathrm{DRed}}_Z(E) = \mathrm{DRed}_{\widehat{W}_\eta}(\widehat{E}_\eta) && \text{(by (10.6))}\\
\Longleftrightarrow\ &\widehat{L}^* \textit{ is permissible for } \mathfrak{i}\,\mathrm{Red}_{\widehat{W}_\eta}(\widehat{E}_\eta) && \text{(by Prop. 9.1)}\\
\Longleftrightarrow\ &\widehat{L} \textit{ is permissible for } \widehat{E}_\eta && \text{(by Cor. 10.4)}\\
\Longleftrightarrow\ &L \textit{ is permissible for } E \textit{ locally at } \eta && \text{(by Th. 7.1)}
\end{aligned}$$

Thus (10.4) is proven and the proof of our theorem is complete. □

11. "finite presentation"

We are now ready to state the final theorem of this paper. Let Z be a smooth algebraic scheme over a perfect field $\mathbf{k}$. We will assume that Z is connected and hence irreducible, because we loose no generality by doing so within our current interest. Given an idealistic exponent $E = (J, b)$ on Z, we define a graded $\mathcal{O}_Z$-algebra

$$\wp(E) = \sum_{a\geqslant 0} J_{\max}(a)\,T^a$$

to be the sheaf of graded $\mathcal{O}_Z$-algebras on Z associated with:

$$U \longmapsto \wp_U(E) = \sum_{0\leqslant a<\infty} J_{\max}(a)_U T^a \quad \text{for each affine open subset } U \text{ of } Z.$$

where T is an indeterminate and $J_{\max}(a)_U$ is an ideal in the affine ring A_U of $Z|U$, satisfying the following conditions:

For every integer $a \geqslant 0$ and for an ideal I in A_U, we have

$$(I, a) \supset E_U \iff I \subset J_{\max}(a)_U$$

where $E_U = (J_U, b)$ denotes the restriction of E to U and the first inclusion is in the sense of Def.1-c while the second in the set-theoretical sense.

It should be noted that if there are two ideals $I_i, i = 1, 2$, in A_U such that $(I_i, a) \supset E_U$ for both $i = 1, 2$, then we have $(I_1 + I_2, a) \supset E_U$ by [**2**]. Therefore, for each $a \geqslant 0$, we always have the unique largest ideal $J_{\max}(a)_U$ having the above property. Moreover, the property implies that $(J_{\max}(a)_U, a) \supset E_U$ for every a and $(J_{\max}(b\mu)_U, b\mu) \sim \big((J_U)^\mu, b\mu\big) \sim E_U$ for every integer $\mu > 0$ in particular. The algebra $\wp_U(E)$ is uniquely determined by the given E. It should be understood that $J_{\max}(0)_U$ is the unit ideal, irrespective of U. Now we claim:

Finite Presentation Theorem. — *The ideal sheaves $J_{\max}(a)$ are all coherent on Z for all integers $a \geqslant 0$ and $\wp(E)$ is locally finitely generated as $\mathcal{O}_Z$-algebra. Therefore, on each affine open subset of the ambient scheme Z, $\wp(E)$ is finitely generated as* $\mathbf{k}$-*algebra.*

Note that if we take an affine open subset $U = \operatorname{Spec}(A_U)$ of Z, then we have $\wp(E)(U) = \sum_a J_{\max}(a)(U)T^a$. This is a finitely generated as $\mathbf{k}$-algebra if and only if it is so as A_U-algebra because A_U itself is a finitely generated $\mathbf{k}$-algebra.

The rest of the paper is devoted to a proof of this theorem. For this purpose, we need some definitions and technical lemmas.

Definition 11.1. — For an idealistic exponent $F = (H, b)$ on Z, we say that F is *Diff-generated* by a system of idealistic exponents $G_j = (I_j, b_j), 0 \leqslant j \leqslant r$, if for every positive integer μ and for every integer i, $0 \leqslant i < b\mu$, we have

$$\operatorname{Diff}_Z^{(i)} H^\mu \subset \sum_{\sum_j e_j b_j \geqslant b\mu - i} \Big(\prod_j I_j^{e_j}\Big)$$

Here the big summation is taken for all systems $(e_0, \dots, e_r)$ with integers $e_j \geqslant 0$, $\forall\, j$, subject to the condition $\sum_{0 \leqslant j \leqslant r} e_j b_j \geqslant b\mu - i$.

Definition 11.2. — We say that $F = (H, b)$ is *Diff-full* if for every integer i, $0 \leqslant i < b$,

$$\big(\operatorname{Diff}_Z^{(i)} H\big)^b \text{ is contained in the integral closure of } H^{b-i}$$

which is equivalent to saying that if $\phi : \widetilde{Z} \to Z$ is the normalized blowing-up of the ideal sheaf H then $\big(\operatorname{Diff}_Z^{(i)} H\big)^b \mathcal{O}_{\widetilde{Z}} \subset H^{b-i}\mathcal{O}_{\widetilde{Z}}$.

Lemma 11.1. — *Assume that $F = (H, b)$ is Diff-generated by a system of idealistic exponents $G_j = (I_j, b_j)$, $0 \leqslant j \leqslant r$, in the sense of Def. 11.1 and that $G_j \supset F$ in the sense of Def. 2.6 for all j, $0 \leqslant j \leqslant r$. Then, for every positive integer μ, we have*

(1) $F \sim (H^\mu, b\mu) \sim \bigcap_{\sum_j e_j b_j \geqslant b\mu} \Big(\prod_j I_j^{e_j},\, b\mu\Big)$

and moreover for every smooth subscheme W of Z we have

(2) $\operatorname{Red}_W(H^\mu, b\mu) \sim \bigcap_{0 \leqslant j < r} (I_j \mathcal{O}_W,\, b_j)$.

Proof. — For every integer $\mu \geqslant 0$,

$$
\begin{aligned}
(H^\mu, b\mu) &\sim \bigcap_{\left(\sum_j e_j b_j\right)\geqslant b\mu} \Big(\prod_j H^{b_j e_j}, b\big(\sum_j e_j b_j\big)\Big) \\
&\subset \bigcap_{\left(\sum_j e_j b_j\right)\geqslant b\mu} \Big(\prod_j I_j^{be_j}, b\big(\sum_j e_j b_j\big)\Big) \sim \bigcap_{\left(\sum_j e_j b_j\right)\geqslant b\mu} \Big(\prod_j I_j^{e_j},\ \big(\sum_j e_j b_j\big)\Big) \\
&\qquad\subset \bigcap_{\left(\sum_j e_j b_j\right)\geqslant b\mu} \Big(\prod_j I_j^{e_j}, b\mu\Big)
\end{aligned}
$$

where the first inclusion is by the second assertion of [F3] and the last by [F4]. Next by the *Diff-generation* assumption, we have

$$
H^\mu \subset \sum_{\left(\sum_j e_j b_j\right)\geqslant b\mu} \Big(\prod_j I_j^{e_j}\Big)
$$

which implies the reversed inclusion of the corresponding idealistic exponents paired by the same number $b\mu$. We thus obtain (1). Now for (2), we have

$$
\begin{aligned}
\mathrm{Red}_W(H^\mu, b\mu) &= \bigcap_{0\leqslant i<b\mu} \Big(\mathrm{Diff}_Z^{(i)}(H^\mu)\mathcal{O}_W,\ b\mu - i\Big) \\
&\supset \bigcap_{0\leqslant i<b\mu} \Big(\big(\sum_{(\sum_j e_j b_j)\geqslant b\mu - i} \prod_j I_j^{e_j}\big)\mathcal{O}_W,\ b\mu - i\Big) \\
&= \bigcap_{0\leqslant i<b\mu} \Big(\sum_{(\sum_j e_j b_j)\geqslant b\mu - i} \big(\prod_j I_j^{e_j}\mathcal{O}_W\big),\ b\mu - i\Big) \\
&\sim \bigcap_{\substack{0\leqslant i<b\mu,\\ (\sum_j e_j b_j)\geqslant b\mu - i}} \Big(\prod_j I_j^{e_j}\mathcal{O}_W,\ b\mu - i\Big) \\
&\supset \bigcap_{\substack{0\leqslant i<b\mu,\\ (\sum_j e_j b_j)\geqslant b\mu - i}} \Big(\prod_j I_j^{e_j}\mathcal{O}_W,\ \sum_j e_j b_j\Big) \\
&\supset \bigcap_{\substack{0\leqslant i<b\mu,\\ (\sum_j e_j b_j)\geqslant b\mu - i}} \Big(\bigcap_j \big(I_j^{e_j}\mathcal{O}_W,\ e_j b_j\big)\Big) \\
&\sim \bigcap_{\substack{0\leqslant i<b\mu,\\ (\sum_j e_j b_j)\geqslant b\mu - i}} \Big(\bigcap_j \big(I_j\mathcal{O}_W,\ b_j\big)\Big) \sim \bigcap_j \big(I_j\mathcal{O}_W,\ b_j\big).
\end{aligned}
$$

where the first inclusion between idealistic exponents is due to the reversed inclusion of ideals by the *Diff-generation* while the last by $[F3]$. On the other hand, since $F \subset G_j$ we get $\mathrm{Red}_W(F) \subset \mathrm{Red}_W(G_j)$ for every j by the *Ambient Reduction Theorem* and

Def. 1-c. Hence we have

$$\begin{aligned}\mathrm{Red}_W(F) &\subset \bigcap_j \mathrm{Red}_W(G_j)\\ &= \bigcap_j \bigcap_{0\leqslant k<b_j} \big((\mathrm{Diff}_Z^{(k)} I_j)\mathcal{O}_W, b_j-k\big)\\ &\subset \bigcap_j (I_j\mathcal{O}_W, b_j)\end{aligned}$$

which shows the converse to the preceding inclusion and (2) is proven. □

Lemma 11.2. — *Any given idealistic exponent $E = \big(J, b\big)$ on Z is Diff-generated by the following system of idealistic exponents:*

$$\big\{\big(\,\mathrm{Diff}_Z^{(i)} J,\ b_j\big), 0 \leqslant i < b\big\}$$

where $b_i = b - i$. Moreover, define

$$E^{\#} = (J^{\#}, b^{\#}) \quad \textit{with } J^{\#} = \sum_{0\leqslant i<b} \Big(\mathrm{Diff}_Z^{(i)} J\Big)^{b!/(b-i)} \textit{ and } b^{\#} = b!$$

and we claim that $E^{\#}$ is Diff-full.

Proof. — As for the first claim, the problem is local. Namely, it is enough to prove the inclusion of the type of Def. 11.2 *locally* at every point $\xi \in \mathrm{Sing}(E)$. It should be noted that at a point outside $\mathrm{Sing}(E)$ one of the ideals $\mathrm{Diff}_Z^{(i)} J$, $0 \leqslant i < b$, is the unit ideal and the claim is trivial. Let us pick a regular system of parameters $x = (x_1, \ldots, x_n)$ in the local ring of Z at ξ and define the elementary differential operators $\partial_\alpha, \alpha \in \mathbf{Z}_0^n$, where $n = \dim_\xi Z$, by the conditions:

$$\partial_\alpha x^\beta = \begin{cases}\binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \in \alpha + \mathbf{Z}_0^n\\ 0 & \text{if } \beta \notin \alpha + \mathbf{Z}_0^n\end{cases}$$

Let μ be any positive integer. For every integer $i, 0 \leqslant i < b\mu$, pick any one ∂_α with $|\alpha| = i$. Then

$$\partial_\alpha J_\xi^\mu \subset \sum_{\substack{\alpha=\sum_{1\leqslant k\leqslant\mu}\alpha_k\\ \alpha_k\in\mathbf{Z}_0^n}} \Big(\prod_{1\leqslant k\leqslant\mu} \partial_{\alpha_k} J_\xi\Big)$$

and for each $(\alpha_1, \ldots, \alpha_\mu)$ we have

$$\prod_{1\leqslant k\leqslant\mu} \partial_{\alpha_k} J_\xi \subset \prod_{i\geqslant j\geqslant 0} \big(\,\mathrm{Diff}_Z^{(j)} J_\xi\big)^{e_j} \subset \prod_{\min(i,b-1)\geqslant j\geqslant 0} \big(\,\mathrm{Diff}_Z^{(j)} J_\xi\big)^{e_j}$$

where e_j is the number of those α_k such that $|\alpha_k| = j$. Here an important point is that

$$\begin{aligned}\sum_{\min(i,b-1)\geqslant j\geqslant 0} e_j b_j &= \sum_{\min(i,b-1)\geqslant j\geqslant 0} e_j(b-j) \geqslant \sum_{i\geqslant j\geqslant 0} e_j(b-j)\\ &= \Big(\sum_{i\geqslant j\geqslant 0} e_j\Big) b - \sum_{i\geqslant j\geqslant 0} e_j j = b\mu - |\alpha| = b\mu - i.\end{aligned}$$

This inequality and the product are unaffected when we define $e_j = 0$ for $j > i$ and extend the range of j just in case $i < b-1$. Therefore

$$\prod_{1\leqslant k\leqslant \mu} \partial_{\alpha_k} J_\xi \quad \subset \sum_{(\sum_{0\leqslant j<b} e_j b_j)\geqslant b\mu - i} \Big(\prod_{0\leqslant j<b} \big(\operatorname{Diff}_Z^{(j)} J_\xi\big)^{e_j}\Big)$$

This being true for all α and $(\alpha_1,\dots,\alpha_\mu)$ as above, we conclude that

$$\operatorname{Diff}_Z^{(i)} J_\xi^\mu \quad \subset \sum_{(\sum_{0\leqslant j<b} e_j b_j)\geqslant b\mu - i} \Big(\prod_{0\leqslant j<b} \big(\operatorname{Diff}_Z^{(j)} J_\xi\big)^{e_j}\Big)$$

This being true for every $\xi \in \operatorname{Sing}(E)$, the same inclusion holds when the suffix ξ is dropped. The first assertion of the lemma is now proven. Next, to prove the second assertion, let $\rho : \widetilde{Z} \to Z$ be the normalized blowing-up of the ideal sheaf $J^\#$. Let us pick any point $\widetilde{\zeta} \in \widetilde{Z}$ and let $\zeta = \rho(\widetilde{\zeta}) \in Z$. Since the pull back $J^\#\mathcal{O}_{\widetilde{Z}}$ is non-zero principal, there exists an index ι such that

(1) $(\operatorname{Diff}_Z^{(\iota)} J)^{b!/(b-i)}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$ *is non-zero principal, say* $= (h_{\widetilde{\zeta}})\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$

(2) $(J^\#\mathcal{O}_{\widetilde{Z}})_{\widetilde{\zeta}} = (h_{\widetilde{\zeta}})\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$ *and*

(3) $h_{\widetilde{\zeta}}$ *divides* $(\operatorname{Diff}_Z^{(k)} J)^{b!/(b-k)}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$, $\forall\, k,\ 0 \leqslant k < b$.

Now, by the definition of $J^\#$, for $0 \leqslant m < b!$ we have

$$\begin{aligned}
\Big(\operatorname{Diff}_Z^{(m)} J^\#\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}} &\subset \sum_{0\leqslant j<b} \Big(\operatorname{Diff}_Z^{(m)} \big(\operatorname{Diff}_Z^{(j)} J\big)^{b!/(b-j)}\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}\\
&= \sum_{0\leqslant j<b}\Big(\sum_{\big(\sum_{1\leqslant k\leqslant b!/(b-j)} m_k\big)=m}\ \prod_{1\leqslant k\leqslant b!/(b-j)} \operatorname{Diff}_Z^{(m_k)}\operatorname{Diff}_Z^{(j)} J\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}\\
&= \sum_{0\leqslant j<b}\Big(\sum_{\big(\sum_{1\leqslant k\leqslant b!/(b-j)} m_k\big)=m}\ \prod_{1\leqslant k\leqslant b!/(b-j)} \operatorname{Diff}_Z^{(m_k+j)} J\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}\\
&\subset \sum_{0\leqslant j<b}\Big(\Big\{\sum\prod\Big\}\operatorname{Diff}_Z^{(m_k+j)} J\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}
\end{aligned}$$

where the sum inside $(\ )\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$ ranges over all the systems $(m_1,\dots,m_{b!/(b-j)})$ with integers $m_k \geqslant 0$ such that $\sum_k m_k = m$, and $\big\{\sum\prod\big\}$ means the same type of sum-product subject to a modification that each term $\operatorname{Diff}_Z^{(m_k+j)} J$ is replaced by the unit ideal if and only if $m_k + j \geqslant b$. Note that for any j and for any system $(m_1,\dots,m_{b!/(b-j)})$ the sum $m_k + j$ cannot be all $\geqslant b$ for if otherwise we would have $m = \sum_k m_k \geqslant \sum_k (b-j) = \big(b!/(b-j)\big)(b-j) = b!$ which is against our condition $m < b!$. Moreover note that for every $(m_1,\dots,m_{b!/(b-j)})$, say $= (m)$, appearing in the above sum-product we have $b! - m = \big(b!/(b-j)\big)(b-j) - \sum_k m_k = \sum_k \big((b-j) - m_k\big) = \sum_k \big(b - (m_k+j)\big) \leqslant \sum_{k:m_k+j<b} \big(b-(m_k+j)\big)$. Call the last number $B_{(m)}$. Then, thanks to the above (3), the corresponding summand

$$\Big(\prod_{k:m_k+j<b} \operatorname{Diff}_Z^{(m_k+j)} J\Big)\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$$

in the above $(\{\ \})_{\widetilde{\zeta}}$ is divisible by $h_{\widetilde{\zeta}}^{B_{(m)}/b!}$ which means, more rigorously,

$$\Big(\prod_{k:m_k+j<b}\operatorname{Diff}_Z^{(m_k+j)} J\Big)^{b!}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$$

is divisible by $h_{\widetilde{\zeta}}^{B_{(m)}}$ and hence by $h_{\widetilde{\zeta}}^{b!-m}$. We thus conclude that

$$(\operatorname{Diff}_Z^{(m)} J^{\#})^{b!}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}} \quad \text{is divisible by } h_{\widetilde{\zeta}}^{b!-m}$$

But this $h_{\widetilde{\zeta}}^{b!-m}$ is the generator of $(J^{\#}\mathcal{O}_{\widetilde{Z}})_{\widetilde{\zeta}}^{b!-m}$ by (2). Namely, with $b^{\#}=b!$,

$$(\operatorname{Diff}_Z^{(m)} J^{\#})^{b^{\#}}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}} \subset (J^{\#})^{b^{\#}-m}\mathcal{O}_{\widetilde{Z},\widetilde{\zeta}}$$

This being true for all m and for all $\widetilde{\zeta}$, we conclude that $\big(\operatorname{Diff}_Z^{(m)} J^{\#}\big)$ is contained in the integral closure of $(J^{\#})^{b^{\#}-m}$ for all m, *i.e.*, $E^{\#}=(J^{\#},b^{\#})$ is *Diff-full*. The proof of the lemma is now all done. ☐

Lemma 11.3. — *Given $E=(J,b)$ on Z, let $E^{\#}=(J^{\#},b^{\#})$ be the same as in Lemma 11.2. Then for every discrete valuation ring R of rank one with $\widehat{\zeta}=\max(R)$ and for every morphism $\phi:\operatorname{Spec}(R)\to Z$ such that the pull back $J^{\#}R$ of $J^{\#}$ by ϕ is not the unit ideal of R, we have $\phi(\widehat{\zeta})\in\operatorname{Sing}(E)$ and $\operatorname{ord}_{\widehat{\zeta}}(J^{\#}R)\geqslant\operatorname{ord}_{\phi(\widehat{\zeta})}(J^{\#})\geqslant b$.*

Proof. — Let C be the center of the morphism ϕ, *i.e.*, the closed irreducible reduced subscheme whose generic point is $\zeta=\phi(\widehat{\zeta})$. Let $\pi:Z_1\to Z$ be the blowing-up with center C. We then have a morphism $\psi:\operatorname{Spec}(R)\to Z_1$ such that $\phi=\pi\circ\psi$. Let η be the generic point of $\pi^{-1}(\zeta)$, *i.e.*, that of the exceptional divisor for π, and let $\overline{\eta}=\psi(\widehat{\zeta})\in Z_1$. Note that $\overline{\eta}$ is a smooth point of Z_1 and of the exceptional divisor. The local ring $\mathcal{O}_{Z_1,\eta}$ is a discrete valuation ring of rank one and $\operatorname{ord}_\eta(h)=\operatorname{ord}_\zeta(h),\forall\, h\in\mathcal{O}_{Z,\zeta}$. We thus have

$$\operatorname{ord}_{\widehat{\zeta}}(J^{\#}R)\geqslant\operatorname{ord}_{\overline{\eta}}(J^{\#}\mathcal{O}_{Z_1})\geqslant\operatorname{ord}_{\eta}(J^{\#}\mathcal{O}_{Z_1})=\operatorname{ord}_\zeta(J^{\#})$$

So it is enough to prove that if $\operatorname{ord}_\zeta(J^{\#})>0$ then $\operatorname{ord}_\zeta(J^{\#})\geqslant b!$, *i.e.*, $\zeta\in\operatorname{Sing}(E^{\#})$. Assume $\operatorname{ord}_\zeta(J^{\#})<b!$. We then have $\operatorname{ord}_\zeta J<b$ for if otherwise we would have $\operatorname{ord}_\zeta(\operatorname{Diff}_Z^{(i)} J)\geqslant b-i,\forall\, i$ and $\operatorname{ord}_\zeta(J^{\#})\geqslant b!$. Let $e=\operatorname{ord}_\zeta J$. Then $\operatorname{ord}_\zeta(\operatorname{Diff}_Z^{(e)} J)=0$, *i.e.*, $\operatorname{Diff}_Z^{(e)} J_\zeta$ must be the unit ideal in $\mathcal{O}_{Z,\zeta}$. It follows that $\operatorname{ord}_\zeta(J^{\#})=0$. This proves the lemma. ☐

Lemma 11.4. — *If $E=(J,b)$ is Diff-full, then for every smooth subscheme $W\subset Z$ we have $\operatorname{Red}_W(E)\sim(J\mathcal{O}_W,b)$.*

Proof. — By the *Diff-fullness*, $(\operatorname{Diff}_Z^{(j)} J)^b$ is integrally dependent upon J^{b-j}. It follows that $(\operatorname{Diff}_Z^{(j)} J)^b\mathcal{O}_W$ is integrally dependent upon $J^{b-j}\mathcal{O}_W$. Hence

$$\begin{aligned}\operatorname{Red}_W(E)=\bigcap_{0\leqslant j\leqslant b-1}\big((\operatorname{Diff}_Z^{(j)} J)\mathcal{O}_W,b-j\big)&\sim\bigcap_{0\leqslant j\leqslant b-1}\big((\operatorname{Diff}_Z^{(j)} J)^b\mathcal{O}_W,b(b-j)\big)\\ &\supset\bigcap_{0\leqslant j\leqslant b-1}\big(J^{b-j}\mathcal{O}_W,b(b-j)\big)\sim(J\mathcal{O}_W,b)\end{aligned}$$

where the equality is the definition and the inclusion is by the integral dependence. But the definition has the term with $j = 0$, that is $(J\mathcal{O}_W, b)$. Hence the reversed inclusion is also true. The proof is done. □

12. Proof of the Finite Presentation Theorem

First of all we remark that $E = (J, b)$ may be replaced by any other idealistic exponent $F = (K, c) \sim E$, because we have the implication $E \sim F \Rightarrow \wp(E) = \wp(F)$. Hence we may replace E by $E^{\#}$ of Lemma 11.2 because $E^{\#} \sim E$ by the *Diff Theorem* and by Def. 2.7. Thus, we may and will assume:

(i) *E itself is Diff-full in the sense of Def. 11.2.* (The *Diff-fullness* of $E^{\#}$ is by Lemma 11.2.)

(ii) *There exist $G_j = (I_j, b_j) \supset E, 1 \leqslant j \leqslant r$, such that E is Diff-generated by the system G_j, $1 \leqslant j \leqslant r$, in the sense of Def. 11.1.* (The *Diff-generation* is by Lemma 11.2, where $G_j = \big(\mathrm{Diff}_Z^{(j)} J, b - j\big)$, $0 \leqslant j < b$, and the inclusions are by *Diff Theorem*.)

(iii) *For every discrete rank one valuation ring R and for every morphism $\phi : \mathrm{Spec}(R) \to Z$ with $JR \neq R$ we have that $\mathrm{Im}(\phi) \cap \mathrm{Sing}(E)$ is not empty.* (This is by Lemma 11.3 for $E^{\#}$.)

Let $\rho : \widetilde{Z} \to Z$ be the normalized blowing-up of the ideal sheaf J, so that $\widetilde{Z}$ is normal and $J\mathcal{O}_{\widetilde{Z}}$ is locally non-zero principal. For brevity, we will write $D_j = \mathrm{Diff}_Z^{(j)} J$, $0 \leqslant j < b$. Since E is *Diff-full*, we have $D_j^b \mathcal{O}_{\widetilde{Z}} \subset J^{b-j}\mathcal{O}_{\widetilde{Z}}$ and hence $D_j^b \mathcal{O}_{\widetilde{Z}}$ is divisible by $J^{b-j}\mathcal{O}_{\widetilde{Z}}$ because the last ideal is locally non-zero principal. Let us make clear what we want to prove under the assumptions (i), (ii) and (iii). Following the notation in the definition of $\wp(E)$, we want to prove:

$$(\flat) \qquad J_{\max}(b\mu) = \rho_*(J^\mu \mathcal{O}_{\widetilde{Z}}) \text{ for every integer } \mu \geqslant 0,$$

Before going to prove ($\flat$), let us first see that if it is proven then the main theorem follows. This implication is seen as follows. The question is local in Z and we will assume that Z is affine, say $Z = \mathrm{Spec}(A)$. We have $J_{\max}(a)^b \subset J_{\max}(ba)$ by their definition and we know that $\rho_*(J^a \mathcal{O}_{\widetilde{Z}})$ is *integral* over J^a in the sense of the *ideal theory*. If ($\flat$) is proven, then for $\forall\, g \in J_{\max}(a)$, g^b is *integral* over J^a in the sense of the *ideal theory*. This is equivalent to saying that $(gT^a)^b$ is *integral* over the graded algebra $\sum_{\mu \geqslant 0} J^\mu T^{b\mu}$ in the sense of the *ring theory*. Let $P(E) = \sum_{\mu \geqslant 0} J^\mu T^{b\mu}$. In view of [F5], we can conclude that

> *$\wp(E)$ is equal to the integral closure of the k-algebra $P(E)$ in the field of fractions $\mathbf{K}$ of $A[T]$.*

Here, since $\mathbf{K}$ is finitely generated as a *field* over k and $P(E)$ is finitely generated as *k-algebra*, it follows from the general theory of commutative algebra that the integral closure $\wp(E)$ of $P(E)$ in $\mathbf{K}$ is a finite $P(E)$-module and hence $\wp(E)$ is finitely generated as k-algebra. Thus ($\flat$) is all that remains to be proven.

Let us now proceed to prove ($\flat$). Let $\widetilde{\eta}_i, 1 \leqslant i \leqslant s$ be the generic points of the subscheme of $\widetilde{Z}$ defined by the ideal $J\mathcal{O}_{\widetilde{Z}}$. Since $\widetilde{Z}$ is normal and $J\mathcal{O}_{\widetilde{Z}}$ is locally principal, they are all smooth points of $\widetilde{Z}$. We can find an open affine subscheme $\widetilde{U} = \mathrm{Spec}(\widetilde{A})$ of $\widetilde{Z}$ such that we have $\widetilde{\eta}_i \in \widetilde{U}, \forall\, i$, and $\widetilde{U}$ is smooth. Since $\widetilde{A}$ is finitely generated as k-algebra, we can choose a finite set of indeterminates $t = (t_1, \ldots, t_r)$ such that there exists a surjective k-algebra homomorphism $\lambda : k[t] \to \widetilde{A}$. Combined with the canonical inclusion $A \hookrightarrow \widetilde{A}$, λ naturally extends to a surjective homomorphism $\Lambda : A[t] \to \widetilde{A}$. Let B be the kernel of Λ and let $W = \mathrm{Spec}(A[t]/B)$, which is a smooth subscheme of $Z[t]$. It is naturally isomorphic to $\widetilde{U}$. By definition, we have

$$\begin{aligned}
\mathrm{Red}_W(E[t]) &= \bigcap_{0\leqslant j<b} \big((\mathrm{Diff}^{(j)}_{Z[t]} J[t])\mathcal{O}_W, b-j\big)\\
&= \bigcap_{0\leqslant j<b} \Big(\big((\mathrm{Diff}^{(j)}_{Z} J)[t]\big)\mathcal{O}_W, b-j\Big)\\
&= \bigcap_{0\leqslant j<b} \Big((\mathrm{Diff}^{(j)}_{Z} J)\mathcal{O}_{\widetilde{U}}, b-j\Big)
\end{aligned}$$

where the last equality is by the isomorphism $A[t]/B \simeq \widetilde{A}$. By (iii), the images $\eta_i = \rho(\widetilde{\eta}_i)$ are all in $\mathrm{Sing}(E)$ and hence $\mathrm{ord}_{\widetilde{\eta}_i}(J\mathcal{O}_{\widetilde{U}}) \geqslant \mathrm{ord}_{\eta_i}(J) \geqslant b, \forall\, i$. Viewing $\widetilde{\eta}_i$ as points of $W \subset Z[t]$ as well as of $\widetilde{U}$, we see that

$$\begin{aligned}
\mathrm{ord}_{\widetilde{\eta}_i}(\mathrm{Red}_W(E[t])) &= \min_{0\leqslant j<b}\{\mathrm{ord}_{\widetilde{\eta}_i}\big((\mathrm{Diff}^{(j)}_{Z} J)\mathcal{O}_{\widetilde{U}}\big)/(b-j)\}\\
&\geqslant \min_{0\leqslant j<b}\{\mathrm{ord}_{\eta_i}(\mathrm{Diff}^{(j)}_{Z} J)/(b-j)\} \geqslant 1
\end{aligned}$$

and hence we have $\widetilde{\eta}_i \in \mathrm{Sing}(\mathrm{Red}_W(E[t]))$, $\forall\, i$. Now pick any idealistic exponent $F = (H, c) \sim E$ on Z. We then have

$$\begin{aligned}
\mathrm{ord}_{\widetilde{\eta}_i}(H\mathcal{O}_{\widetilde{U}})/c = \mathrm{ord}_{\widetilde{\eta}_i}(H[t]\mathcal{O}_W)/c &\geqslant \mathrm{ord}_{\widetilde{\eta}_i}\big(\mathrm{Red}_W(F[t])\big)\\
&= \mathrm{ord}_{\widetilde{\eta}_i}\big(\mathrm{Red}_W(E[t])\big) = \mathrm{ord}_{\widetilde{\eta}_i}(J\mathcal{O}_{\widetilde{U}})/b, \quad \forall\, i
\end{aligned}$$

Here the first equality is by $A[t]/B \simeq \widetilde{A}$ and the second inequality is by the definition of Red_W expressed as an intersection of idealistic exponents including $(H[t], c)$ itself. The equality before the last, follows $F \sim E$ by the *Numerical Exponent Theorem* and the *Ambient Reduction Theorem.* Finally the last equality is by Lemma 11.4 thanks to the assumption (i). Now, apply the above inequality to the case of $F = (J_{\max}(b\mu), b\mu), \mu > 0$, and we get

$$\mathrm{ord}_{\widetilde{\eta}_i}(J_{\max}(b\mu)\mathcal{O}_{\widetilde{U}})/b\mu \geqslant \mathrm{ord}_{\widetilde{\eta}_i}(J\mathcal{O}_{\widetilde{U}})/b, \quad \forall\, i$$

which implies $J_{\max}(b\mu)\mathcal{O}_{\widetilde{U},\widetilde{\eta}_i}$ is divisible by $J^{\mu}\mathcal{O}_{\widetilde{U},\widetilde{\eta}_i}, \forall\, i$, because the local ring is a discrete rank one valuation ring. Since $J^{\mu}\mathcal{O}_{\widetilde{U}}$ is locally non-zero principal everywhere on a normal scheme and the $\{\widetilde{\eta}_i\}$ are all the generic points of $\mathrm{Spec}(\mathcal{O}_{\widetilde{Z}}/J\mathcal{O}_{\widetilde{Z}})$, it follows that $J_{\max}(b\mu)\mathcal{O}_{\widetilde{Z}}$ is divisible by $J^{\mu}\mathcal{O}_{\widetilde{Z}}$. In particular, we have

$$J_{\max}(b\mu)\mathcal{O}_{\widetilde{Z}} \subset J^{\mu}\mathcal{O}_{\widetilde{Z}}, \quad \forall\, \mu \geqslant 1$$

However, by the maximality of $J_{\max}$, we have

$$J_{\max}(b\mu) \supset J^{\mu} \quad \text{and} \quad J_{\max}(b\mu) = \rho_*(J_{\max}(b\mu)\mathcal{O}_{\widetilde{Z}}), \quad \forall\, \mu \geqslant 1$$

and hence the above converse inclusion implies

$$J_{\max}(b\mu)\mathcal{O}_{\widetilde{Z}} = J^{\mu}\mathcal{O}_{\widetilde{Z}} \quad \text{and} \quad J_{\max}(b\mu) = \rho_*(J^{\mu}\mathcal{O}_{\widetilde{Z}}), \quad \forall\, \mu \geqslant 1$$

This proves ($\flat$). We complete the proof of the theorem with an additional remark which shows the coherency of $J_{\max}(a), \forall\, a$. The replacement of (J, b) by $(J^{\#}, b^{\#})$, called $\#$-*operation*, is compatible with any localization of the affine ring A, that is with the restriction from an open affine set of Z to any smaller one. Moreover, we saw that $\wp(E)$ is the integral closure of $P(E)$ in the function field $\mathbf{K}$ of the scheme Z. The *integral closure* is also compatible with any localization. The coherency is clear. The proof of the theorem is now completed.

References

[1] S. Abhyankar – Desingularization of plane curves, in *Singularities (Arcata, 1981)*, Proc. Sympos. Pure Math., vol. 40, American Mathematical Society, 1983.

[2] V. Cossart, C. Galindo & O. Piltant – Un example effectif de gradué non noethérien associé à une valuation divisorielle, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 1, p. 105–112.

[3] J. Giraud – Sur la théorie du contact maximal, *Math. Z.* **137** (1974), p. 286–310.

[4] ———, Contact maximal en caractéristique positive, *Ann. scient. Éc. Norm. Sup.* 4e *série* **8** (1975), p. 201–234.

[5] H. Hironaka – Gardening of infinitely near singularities, in *Proc. Nordic Summer School in Math. (Oslo, 1970)*, Sijthoff and Noordhoff, Alphen aan den Rijn, p. 315–332.

[6] ———, *Introduction to the theory of infinitely near singular points*, Mem de Mat, vol. 28, Inst Jorge Juan, Madrid, 1974.

[7] ———, Idealistic exponents of singularity, in *Algebraic Geometry, J.J. Sylvester Symposium (Johns Hopkins Univ., 1976)*, Johns Hopkins Univ. Press, Baltimore, Md., 1977, p. 52–125.

[8] B. Youssin – *Newton Polyhedra without coordinates*, Mem. Amer. Math. Soc., vol. 433, American Mathematical Society, 1990.

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