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INTEGRABILITY OF SOME FUNCTIONS ON SEMI-ANALYTIC SETS

by

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***Abstract.* —** Using the properties of Lipschitz stratification we show that some functions on a semi-analytic sets, in particular the invariant polynomials of curvature form, are locally integrable. The result holds as well for subanalytic sets.

***Résumé* (Intégrabilité de certaines fonctions sur les ensembles semi-analytiques)**
En utilisant les propriétés des stratifications lipschitziennes on montre l'intégrabilité locale d'une classe de fonctions définies sur les ensembles semi-analytiques. Cette classe contient les polynômes invariants de la courbure. Le résultat est vrai aussi pour les ensembles sous-analytiques.

I wrote this paper as an appendix to [**7**] back in 1988. It contains the proof of integrability of curvature of the regular part of a semi-analytic set, Proposition 1 below. This result can be proven in a simpler way using the functoriality of curvature form as for instance shown in [**1**] and that is why back in 1988 I put this appendix to a drawer. On the other hand the proof presented below is quite different than the standard one and uses techniques that can be useful, see for instance [**6**].

The proof presented in this paper follows to a big extend the ideas of the proof of a similar statement in the complex domain given by T. Mostowski in [**5**]. It is based by a direct estimate of curvature in terms of second derivatives and consequently, thanks to techniques developped in [**7**], in terms of the distances to strata of a Lipschitz stratification. Let us now outline the main points of the proof. Let $X \subset \mathbb{R}^n$ be semi-analytic and let $k = \dim X \leqslant n-1$. Decomposing X into finitely many pieces we may suppose that it is the graph of a semi-analytic mapping $\overline{U} \to \mathbb{R}^{n-k}$, with $U \subset \mathbb{R}^k$ open and semi-analytic. The integrability of the curvature forms on X reduces to the integrability on U of some combinations of the partial derivatives of F of the first and second order. The former we may suppose bounded by a more precise decomposition of X (we use the so called decomposition into L-regular sets). The second order derivatives are then bounded by the first order ones divided the distances to the strata

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of a stratification of $\overline{U}$. This follows from an inequality (12) that plays an important rôle in the proof of the existence of Lipschitz stratification of semi- and subanalytic sets, see Lemma 4.5 of [**7**] and Proposition 3.1 of [**9**]. Therefore the integrability of curvature is reduced to the integrability on $\overline{U}$ of functions of the form

$$A(x) = \frac{(d_s(x))^\gamma}{\prod_{j=0}^{n-1} d_j(x)},$$

where d_j denote the distances to the j dimensional strata. These functions are generally not integrable since the direct integration gives logarithms. A more delicate analysis in Lemma 4 below, shows that $A(x)$ is integrable on some "horn neighbourhoods" of strata, where the distance to a fixed stratum is dominated by the distances to the smaller strata, and as we show in Lemma 7 this is precisely what we need for the integrability of curvature. Finally Lemma 4 follows fairly easily by induction on dimension thanks to Lemmas 2 and 3 below which relate the distance to a semi-analytic set and the distances to its projections and to its sections. Note that Lemmas 2-4 follows from the regular projections theorem, see [**5**], Proposition 2.1 of [**7**], and [**9**] section 5, and do not require the use of Lipschitz stratifications. In particular Lemma 4 holds for any stratification, not necessarily Lipschitz.

The paper is presented below virtually in its original form. Only the evident misprints and orthographic and gramatical errors were corrected. Since 1988 the theory of Lipschitz stratification was further developed by T. Mostowski and myself. The reader may consult [**8**] for an account of this development. In particular the regular projection theorem and the existence of Lipschitz stratification was proven for subanalytic sets [**9**], and hence all the results of this paper hold as well in the subanalytic set-up. As follows from [**9**], it is easy to bound the number of regular projections in Proposition 2.1 of [**7**]. In particular, in lemmas 2 and 3 we may take $N = n+1$ and any generic $(n+1)$-tuple of vectors $\xi_1, \dots, \xi_{n+1}$ from $\mathbb{R}^n$ satisfies the statements.

For the reader convenience, we recall briefly Dubson's argument [**1**]. Let X be a k-dimensional subanalytic subset of an n dimensional real analytic manifold M with a riemannian tensor. Let $\mathrm{G}_k(TM)$ denote the k-Grassmann bundle of TM whose fibre of $x \in M$ is the Grassmannian of k-dimensional subspaces of T_xM. We denote by T the tautological k bundle on $\mathrm{G}_k(TM)$. Note that the metric tensor on M induces a metric tensor on T. Let X_{reg} denote the regular (k-dimensional) part of X. The Nash blowing-up $\widetilde{X}$ of X is the closure in $\mathrm{G}_k(TM)$ of

$$\{(x, \xi) \in \mathrm{G}_k(TM) \mid x \in X_{\mathrm{reg}} \text{ and } \xi = T_x X_{\mathrm{reg}}\}.$$

It is known that $\widetilde{X}$ is subanalytic. Let $\pi : \widetilde{X} \to X$ denote the projection. Then, clearly, $\pi^* TX|_{X_{\mathrm{reg}}}$ coincides with $T|_{\pi^{-1}(X_{\mathrm{reg}})}$ and hence extends on $\widetilde{X}$. As a consequence the pull-back of the curvature form Ω of X_{reg} coincides, on $\pi^{-1}(X_{\mathrm{reg}})$, with the curvature form Ω_T of T. Let P be an invariant homogeneous polynomial of degree k. Then $P(\Omega)$ is integrable on each relatively compact subset Y of X_{reg}. Indeed, since π is proper $\widetilde{Y} = \pi^{-1}(Y)$ is relatively compact. Moreover, being subanalytic, $\widetilde{Y}$ has finite k-volume. On the other hand $\pi^* P(\Omega) = P(\pi^*\Omega) = p(\Omega_T)$ and the latter is integrable on $\widetilde{Y}$.

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The aim of this paper is to prove the following proposition.

Proposition 1. — *Let M be a real analytic manifold with a given metric tensor. Let $X \subset M$ be a compact k-dimensional semi-analytic set and let Ω be the curvature form on the set X_{reg} of regular points of X of the induced metric tensor. Then, for every invariant homogeneous polynomial P of degree k, the k-form $P(\Omega)$ is integrable on X_{reg}. If X_{reg} is oriented, then* $\mathrm{Pf}(\Omega)$ *is integrable. (see, for exemple,* [**4**] *for the definition of the Pfaffian* Pf*).*

First we investigate the function of distance to a semi-analytic set. Let $X \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be a compact semi-analytic set. For a given $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi(\xi) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection parallel to $(\xi, 1)$ and by $\mathrm{dist}_\xi(x, X)$ the distance from x to X in $(\xi, 1)$ direction

$$\mathrm{dist}_\xi(x, X) := \mathrm{dist}(x, X \cap (\pi(\xi))^{-1}(\pi(\xi)(x))).$$

Of course $\mathrm{dist}_\xi(x, X) \geqslant \mathrm{dist}(x, X)$. It is a well-known fact, see [**3**], that $\mathrm{dist}(x, X)$ is a subanalytic, but not necessarily semi-analytic, function. Note that for any ξ, $\mathrm{dist}_\xi(x, X)$ is also subanalytic.

Lemma 2. — *Let X be a compact semi-analytic subset of $\mathbb{R}^n$. Then there are a finite number of vectors $\xi_1, \dots, \xi_N \in \mathbb{R}^n$, a positive constant C, and a semi-analytic subset $Y \subset X$ such that* $\dim Y < n-1$ *and*

$$\min\{\min_j \mathrm{dist}_{\xi_j}(x, X), \mathrm{dist}(x, Y)\} \leqslant C\,\mathrm{dist}(x, X), \tag{1}$$

for all $x \in \mathbb{R}^n$.

Proof. — Since X is compact, it is sufficient to prove the lemma locally in a neighbourhood of every $x_0 \in \mathbb{R}^n$. If $x_0 \notin \mathrm{Fr}(X) = X \setminus \mathrm{Int}(X)$, putting $\xi = 0$ we obtain (1) with $C = 1$.

Let $x_0 \in \mathrm{Fr}(X)$. It suffices to prove the lemma for $\mathrm{Fr}(X)$ instead of X, so we can assume that $\dim X \leqslant n-1$. We complexify $\mathbb{R}^n$ and consider a complex hypersurface $\widetilde{X}$ in an open neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X \cap \widetilde{U} \subset \widetilde{X}$. Take constants $C, \varepsilon > 0$ and vectors $\xi_1, \dots, \xi_N$ satisfying the assertion of Corollary 2.4 of [**7**] for $(\widetilde{X}, x_0)$. In particular, for every x close to x_0 there exists $\xi \in \{\xi_1, \dots, \xi_N\}$ such that the intersection of the open cone

$$S_\varepsilon(x,\xi) = \{x + \lambda(\eta, 1) \mid |\eta - \xi| < \varepsilon, \lambda \in \mathbb{C}^*\}$$

with X is of the form given in (8) of [**7**]. We recall for the reader's convenience that it means that

$$S_\varepsilon(x,\xi) \cap X = \bigcup_i \{x + \lambda_i(\eta)(\eta, 1) \mid |\eta - \xi| < \varepsilon\},$$

where λ_i, $i = 1, \dots, r$, are real analytic functions defined on $|\eta - \xi| < \varepsilon$ and satisfying $\lambda_i(\eta) \neq \lambda_j(\eta)$ for $i \neq j$ and all η, and $|D\lambda_i| \leqslant C|\lambda_i|$. Furthermore we may assume that for each j, $\pi(\xi_j)|_X$ is a branched analytic covering and let $B(\xi_j)$ be its critical locus. Put $Y = \bigcup_j \pi(\xi_j)^{-1}(B(\xi_j)) \cap X$. Clearly Y is semi-analytic and $\dim Y < n-1$.

Let U be a sufficiently small neighbourhood of x_0 such that $U \subset \widetilde{U} \cap \mathbb{R}^n$. Let $x \in U$ and we assume that the regular projection corresponding to x is standard $\mathbb{R}^n \to \mathbb{R}^{n-1}$. Let $p \in X$ be one of the points nearest to x. Let $p' = \pi(p)$, $x' = \pi(x)$, and let $U' = \pi(U)$. If $x' = p'$ then $\operatorname{dist}(x, X) = \operatorname{dist}_\xi(x, X)$. So, we assume $x' \neq p'$ and consider the segment $\overline{p'x'}$. Starting from p we lift $\overline{p'x'}$ to a smooth curve γ on X until we reach a point $s \in Y$ or s of the form $(x', \lambda_i(0))$ for some $i = 1, \dots, r$. We denote $\pi(s)$ by s'. It remains to prove that

$$|x - s| \leqslant C|x - p|, \tag{2}$$

for a universal constant C. This follows from Remark 2.5 of [**7**]. More precisely, if $p \in S_{\varepsilon'}(x, 0)$, where ε' is given by Remark 2.5 of [**7**], the length of γ is estimated by $C'|p' - s'|$. Hence

$$|x - s| \leqslant |x - p| + |p - s| \leqslant |x - p| + C'|p' - s'| \leqslant |x - p| + C'|p' - x'|$$

and consequently (2) follows. If $s \notin S_{\varepsilon'}(x, 0)$ then

$$|x - s| \leqslant C|x' - s'| \leqslant C|x' - p'| \leqslant C|x - p|.$$

If $s \in S_{\varepsilon'}(x, 0)$ and $p \notin S_{\varepsilon'}(x, 0)$, then we may find $r \in \gamma \cap \operatorname{Fr}(S_{\varepsilon'}(x, 0))$ and by the above

$$|x - s| \leqslant |x - r| + |r - s| \leqslant C'|x' - r'| \leqslant C'|x' - p'| \leqslant C|x - p|. \qquad \square$$

Lemma 3. — *Let X be a semi-analytic subset of $\mathbb{R}^n$, $\dim X < n - 1$, and $x_0 \in \mathbb{R}^n$. Then there exist a finite number of vectors $\xi_1, \dots, \xi_N \in \mathbb{R}^n$ and constants $C, \varepsilon > 0$ such that for a sufficiently small neighbourhood U of x_0 and every $x \in U$ there is ξ_j such that $X \cap U \subset \mathbb{R}^n \smallsetminus S_\varepsilon(x, \xi_j)$. In particular*

$$\operatorname{dist}(x, X) \leqslant C \max_j \{\operatorname{dist}(\pi(\xi_j)(x), \pi(\xi_j)(X \cap U))\}.$$

Proof. — It is sufficient to prove the lemma for $x_0 \in X$. Complexify $\mathbb{R}^n$ and consider complex hypersurfaces $\widetilde{X}_1, \widetilde{X}_2$ in an open neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X \cap \widetilde{U} \subset \widetilde{X}_1 \cap \widetilde{X}_2$ and $\dim_{\mathbb{C}} \widetilde{X}_1 \cap \widetilde{X}_2 = n - 2$. Then the lemma follows from Proposition 2.1 of [**7**] applied to $\widetilde{X}_1 \cup \widetilde{X}_2$. $\square$

Now we consider the following situation. Let X be a compact semi-analytic subset of $\mathbb{R}^n$ and $\dim X = n$. Let

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n = X$$

be a family of semi-analytic subsets of X such that $\dim X^i \leqslant i$ for each i. For any $N \in \mathbb{N}$, $C > 0$ and $j = 0, \dots, n-1$, consider the following subsets of $U = X \setminus (\mathrm{Fr}(X) \cup X^{n-1})$

$$U_{N,C,j} = \{x \in U \mid d_j(x) < C[d_{j-1}(x)]^N\},$$

where $d_j(x) = \mathrm{dist}(x, X^j)$. (If $X_j = \varnothing$ then we mean $d_j \equiv 1$).

Lemma 4. — *For any $N, N' \geqslant 1$, $C, C' > 0$, $\gamma > 0$, $s = 0, \dots, n-1$, the function*

$$A(x) = \frac{(d_s(x))^\gamma}{\prod_{j=0}^{n-1} d_j(x)}$$

is integrable on $U_{N,C,s} \setminus \bigcup_{j>s} U_{N',C',j}$.

Proof. — Induction on $n = \dim X$.

Since X is compact, it suffices to prove the lemma locally, that is in a neighbourhood of each point of X. Fix $x_0 \in X$. Assume that X is contained in a sufficiently small neighbourhood V of x_0. We apply Lemma 2 to X^{n-1} and Lemma 3 to X^{n-2} at x_0. We can do it simultaneously and obtain a finite number $\xi_1, \dots, \xi_N$ of vectors and a semi-analytic subset Y of X, $\dim Y < n-1$, such that for every $x \in V$ (shrinking V if necessary), either for some ξ_j

$$\mathrm{dist}_{\xi_j}(x, X^{n-1}) \leqslant C d_{n-1}(x) \tag{3}$$

$$\text{and} \quad d_r(x) \leqslant C\,\mathrm{dist}(\pi(\xi_j)(x), \pi(\xi_j)(X^r)), \tag{4}$$

for each $r = 0, 1, \dots, n-2$, or

$$\mathrm{dist}(x, Y) \leqslant C d_{n-1}(x), \tag{5}$$

for some $C > 0$. Indeed, we can find complex hypersurfaces $\widetilde{X}_1, \widetilde{X}_2$ of a neighbourhood $\widetilde{U}$ of x_0 in $\mathbb{C}^n$ such that $X^{n-1} \cap \widetilde{U} \subset \widetilde{X}_1$, $X^{n-2} \cap \widetilde{U} \subset \widetilde{X}_1 \cap \widetilde{X}_2$, $\dim_{\mathbb{C}}(\widetilde{X}_1 \cap \widetilde{X}_2) < n-1$. Then $\xi_1, \dots, \xi_N$ given by Corollary 2.4 of [**7**] applied to $(\widetilde{X}_1 \cup \widetilde{X}_2, x_0)$ satisfies the properties claimed above (see also the proofs of Lemmas 2 and 3).

Apply again Lemma 3 to $Y \cup X^{n-2}$ at x_0 and add the obtained vectors to the set $\xi_1, \dots, \xi_N$. In conclusion, for each $x \in V$ there is ξ_j so that the inequality (4) holds for $r = 0, \dots, n-2$ and

$$\mathrm{dist}(x, Y) \leqslant C\,\mathrm{dist}(\pi(\xi_j)(x), \pi(\xi_j)(Y)) \tag{6}$$

$$S_\varepsilon(x, \xi_j) \cap Y = \varnothing. \tag{7}$$

Furthermore, we may require that for each ξ_j, $\pi(\xi_j))|_{X^{n-1}}$ is finite and $\pi(\xi_j)(Y)$, $\pi(\xi_j)(X^r)$, $r = 0, \dots, n$, are semi-analytic subsets of $\mathbb{R}^{n-1}$ (see [**2**]).

Fix $\xi = \xi_i$ for a moment and assume that $\pi = \pi(\xi_i)$ is the standard projection. Denote $T = \pi(X)$ and $T^r = \pi(X^r)$ for $r < n-1$. Let W be the subanalytic subset of $U_{N,C,s} \smallsetminus \bigcup_{j>s} U_{N',C',j}$ consisting of such x that (3) and (4) hold with $\xi_j = \xi$.

Consider first the case $s < n-2$. Then

$$U' \cap \pi(W) \subset U'_{N,C,s} \smallsetminus \bigcup_{j>s} U'_{N',C',j},$$

where $U', U'_{N,C,s}, \dots$ are constructed in an analogous manner for the family $T^0 \subset T^1 \subset \cdots \subset T^{n-1}$ (the constants C', C, N, N' may be different). Denote $\operatorname{dist}(x', T^r)$ by $d'_r(x')$. If $x \in W$ then

$$\operatorname{dist}_\xi(x, X^{n-1}) \geqslant d_{n-1}(x) \geqslant C[d_s(x)]^{(n-1-s)N'} \geqslant C[d'_s(\pi(\xi)))]^{N''}. \tag{8}$$

Fix $x' \in U'$. The set $W \cap \pi^{-1}(x')$, as subanalytic, consists of a finite union of segments and their number is uniformly bounded. Therefore, by (8),

$$\int_{\pi^{-1}(x')} [\operatorname{dist}_\xi(x, X^{n-1})]^{-1} \leqslant C \max_{x \in \pi^{-1}(x')} |\ln \operatorname{dist}_\xi(x, X^{n-1})| \leqslant C' |\ln d'_s(x')|. \tag{9}$$

Note that, by construction, $\dim(T \smallsetminus U') < n-1$, so $\dim(W \smallsetminus \pi^{-1}(U') < n$ and hence $W \smallsetminus \pi^{-1}(U')$ is of measure 0 (see, for instance, [**2**]). Consequently,

$$\begin{aligned}\int_W A(x) &\leqslant C \int_{\pi(W)\cap U'} \left(\frac{[d'_s(x')]^\gamma}{\prod_{j=0}^{n-2} d'_j(x')} \int_{\pi^{-1}(x')\cap W} [\operatorname{dist}_\xi(x, X^{n-1})]^{-1} \right) \\ &\leqslant C' \int_{\pi(W)\cap U'} \frac{[d'_s(x')]^\gamma |\ln d'_s(x')|}{\prod_{j=0}^{n-2} d'_j(x')}\end{aligned}$$

and the last integral is finite by the inductive hypothesis. A similar situation occurs if we consider the subset W of $U_{N,C,s} \smallsetminus \bigcup_{j>s} U_{N,C,j}$ where (5)-(7) hold. By (7) the entire length of $\pi^{-1}(x') \cap W$ is smaller than $C \operatorname{dist}(x', \pi(Y))$. Consequently

$$\int_{\pi^{-1}(x')\cap W} [\operatorname{dist}_\xi(x, X^{n-1})]^{-1} \leqslant C \int_{\pi^{-1}(x')\cap W} [\operatorname{dist}(x, Y)]^{-1} \leqslant C,$$

and we prove the integrability of A on W in the same way as above.

Consider now the case $s = n-1$. Let W be the subset of $U_{N,C,n-1}$ for which (3), (4) hold. For $x' \in U'$ the set $\pi^{-1}(x') \cap W$ consists of a finite number of segments and their number is uniformly bounded. Consequently

$$\begin{aligned}\int_{\pi^{-1}(x')\cap W} [d_{n-1}(x)]^{\gamma-1} &\leqslant C \int_{\pi^{-1}(x')\cap W} [\operatorname{dist}_\xi(x, X^{n-1})]^{\gamma-1} \\ &\leqslant C \max_{x\in\pi^{-1}(x')} (d_{n-1}(x))^\gamma \leqslant C'(d'_{n-2}(x'))^{N\gamma}.\end{aligned}$$

Hence

$$\int_W A(x) \leqslant C \int_{\pi(W)\cap U'} \left(\prod_{j=0}^{n-2} (d'_j(x'))^{-1} \int_{\pi^{-1}(x')\cap W} (d_{n-1}(x))^{\gamma-1} \right)$$
$$\leqslant C' \int_{\pi(W)\cap U'} \frac{[d'_{n-2}(x')]^{\gamma'}}{\prod_{j=0}^{n-2} d'_j(x')}$$

The last integral is finite on $\bigcup_{s=0}^{n-2}(U'_{N,C,s} \smallsetminus \bigcup_{j>s} U'_{N,C,s})$ since $d'_{n-2}(x') \leqslant d'_s(x')$, for each $s = 0,\dots,n-2$, and by the inductive hypothesis. On $U' \smallsetminus \bigcup_{s=0}^{n-2} U'_{N,C,s}$ it is also finite, since all $(d'_s)^{-1}$ are bounded.

For the subset $W \subset U_{N,C,n-1}$ consisting of the points where (5)-(7) hold, we have

$$\int_{\pi^{-1}(x')\cap W} [d_{n-1}(x)]^{\gamma-1} \leqslant C \int_{\pi^{-1}(x')\cap W} [\operatorname{dist}(x,Y)]^{\gamma-1} \leqslant (\operatorname{dist}(x',\pi(Y)))^{\gamma}.$$

So we must add $\pi(Y)$ to T^{n-2} and repeat the above procedure. This ends the proof. □

Now we assume $X \subset \mathbb{R}^n$, $\dim X = k$, to be L-regular in the sense of Definition 3.2 of [**7**]. In particular, X is the graph of a mapping $F : \overline{U} \to \mathbb{R}^{n-k}$, where $\overline{U}$ is an L-regular subset of $\mathbb{R}^k$, U open in $\mathbb{R}^k$, and the partial derivatives of the first order of F are uniformly bounded on U. The regular part X_{reg} of X equals the graph of F restricted to U. We denote $\partial F_i/\partial x_j$, for $i = 1,\dots,n-k$ and $j = 1,\dots,k$, by F_{ij}. Let $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ be the standard basis of $\mathbb{R}^n$. Then

$$\boldsymbol{f}_j(x,F(x)) = \begin{cases} \boldsymbol{e}_j + DF(x)\boldsymbol{e}_j & j = 1,\dots,k \\ -\operatorname{grad} F_{j+k} + \boldsymbol{e}_j & \text{for } j > k \end{cases}$$

is a basis of $\mathbb{R}^n$ for each $x \in X_{\mathrm{reg}}$. The first k vectors are tangent to X_{reg}. Let ω_{ij} be the connection matrix for this frame. From the structural equation, see [**4**] Appendix C,

$$\Omega_{\alpha\alpha'} = -\sum_{\mu=k+1}^{n} \omega_{\alpha'\mu} \wedge \omega_{\mu\alpha} + \Omega'_{\alpha'\alpha}, \tag{10}$$

where Ω, Ω' are the curvature matrices for X_{reg} and $\mathbb{R}^n$. Given vector $v \in \mathbb{R}^n$, we define a vector field $V(x,F(x)) = (v, DF(x)v)$ on X_{reg}. Then

$$|\omega_{i_1 i_2}(V)| \leqslant C \sum_{j=1}^{k} (|D(\frac{\partial F}{\partial x_j} v| + 1)(|v| + 1), \tag{11}$$

for some constant C.

Our next purpose is to estimate $D(F_{ij})(x)v$ for various vectors v. By Lemma 4.5 of [**7**], or more generally by [**9**] Proposition 3.1, there exists a stratification $\mathcal{S}$ of $\overline{U}$, such that for any Lipschitz vector field w on $\overline{U}$ tangent to the strata of $\mathcal{S}$

$$|DF_{ij}(x)w(x)| \leqslant CL|F_{ij}(x)|, \tag{12}$$

where L is a Lipschitz constant of w and C is a universal constant. Denote $d_i(x) = \operatorname{dist}(x, S^i)$.

Lemma 5. — *Let $\mathcal{S}$ be a Lipschitz stratification of a semi-analytic set X (in the sense of [**7**]). Then for some positive constant C and any $q \in \mathring{S}^j$ there exist Lipschitz $\mathcal{S}$-compatible vector fields $v_0, \dots, v_{j-1}$ on S^j such that*

(1) *v_i has the Lipschitz constant $C[d_i(q)]^{-1}$ for all $i = 1, \dots, j-1$,*
(2) *$v_0(x), \dots, v_{j-1}(x)$ is an orthonormal basis of $T_q\mathring{S}^j$.*

(here we mean $d_r \equiv 1$ for $r < l$, where l satisfies $S^l \neq \varnothing$, $S^{l-1} = \varnothing$.)

Proof. — It is sufficient to show that, for $i = 0, 1, \dots, j-1$, there exists an i-dimensional linear subspace V^i of $T_q\mathring{S}^j$ such that for each $v \in T_q\mathring{S}^j$ one can find a Lipschitz $\mathcal{S}$-compatible vector field w on S^j with the Lipschitz constant $C[d_i(q)]^{-1}|v|$ and $w(q) = v$. We shall show it by induction on j. For $j = l$, it is a simple consequence of [**7**] Proposition 1.5. Assume that the lemma is true for all j smaller than s. Let $q' \in S^{s-1}$ satisfies

$$|q - q'| \leqslant \frac{3}{2} d_{s-1}(q).$$

Let $q' \in \mathring{S}^k$. Then, of course, $k < s$. Take i such that $k \leqslant i < s$. Then as V^i we may chose any i-dimensional subspace of $T_q\mathring{S}^s$. Indeed, take any $v \in T_q\mathring{S}^j$. By (i) of Proposition 1.5 of [**7**] we may construct a Lipschitz $\mathcal{S}$-compatible vector field w on S^s such that $w(q) = v$, $w|_{S^{s-1}} \equiv 0$ and with the Lipschitz constant $C[d_i(q)]^{-1}|v|$.

Let $i < k$. By the inductive hypothesis we can find an i-dimensional vector subspace W^i of $T_{q'}\mathring{S}^k$ with the desired properties for q'. Fix $w \in W'$, $|w| = 1$. Let $\widetilde{w}$ be a $\mathcal{S}$-compatible Lipschitz vector field on S^k, with the Lipschitz constant $C[d_i(q')]^{-1}|w|$, $\widetilde{w}(q') = w$. By (i) of Proposition 1.5 [**7**], we may extend $\widetilde{w}$ on S^s in such a way that $\widetilde{w}(q) = \mathrm{P}_q(\widetilde{w}(q'))$ (and, of course, $\widetilde{w}$ remains Lipschitz and $\mathcal{S}$-compatible with a Lipschitz constant $L = C[d_i(q')]^{-1}|w|$). If additionally $|q - q'| \leqslant \frac{1}{2}L^{-1}$, then

$$|\widetilde{w}(q)| \geqslant |\widetilde{w}(q')| - |\widetilde{w}(q') - \widetilde{w}(q)| \geqslant \tfrac{1}{2}|w| = \tfrac{1}{2},$$

and $d_i(q) \leqslant d_i(q') + |q - q'| \leqslant C d_i(q')$. Hence $V^i = P_q W^i$ has the desired properties.

Assume $|q - q'| \geqslant \frac{1}{2}L^{-1}$. Then

$$d_{s-1}(q) \geqslant C d_i(q'),$$

for a constant $C > 0$ and we may suppose $C < \frac{1}{2}$. Therefore

$$d_{s-1}(q) \geqslant C d_i(q) - C|q - q'|,$$

and consequently

$$d_{s-1}(q) \geqslant \widetilde{C} d_i(q),$$

for some constant $\widetilde{C} > 0$. Hence, as in the case $i \geqslant k$, any i-dimensional subspace of $T_q\mathring{S}^j$ has the desired properties. □

Corollary 6. — *Let $\overline{U}$, F, and $\mathcal{S}$ be as above. Then*

$$|P(\Omega)(x, F(x))| \leqslant C \prod_{j=0}^{k-1} \big(d_j(x, F(x))\big)^{-1}, \tag{13}$$

for all $x \in U$ and some constant C.

Proof. — It follows easily from Lemma 5, (10), and (11). $\square$

Our next step is to use Lemma 4. In order to be able to do it we strengthen the estimate (13).

Lemma 7. — *Let $\overline{U}$, F be as above. Then there exist an L-stratification $\mathcal{S}$ of $\overline{U}$ satisfying* (12) *for all F_{ij} and Lipschitz $\mathcal{S}$-compatible vector fields w, a positive integer N, and constants $C > 0$, $0 \leqslant \delta < 1$, $0 < \gamma < 1$, such that for any $r = 0, \dots, n-1$ and $q \in U_{N,C,r}$ there are $w \in \mathbb{R}^n$, $|w| = 1$, and $q' \in \mathring{S}^r$ which satisfy*

$$|q - q'| \leqslant C d_j(q), \tag{14}$$

$$|\operatorname{P}_{q'} w| < \delta, \tag{15}$$

$$|DF_{ij}(q)w| \leqslant C(d_{r-1}(q))^{\gamma-1}, \tag{16}$$

for $i = 1, \dots, n-k$; $j = 1, \dots, k$ (for $j-1 < l$ we mean $d_{j-1} \equiv 1$).

Proof. — To simplify the notation, we assume $\operatorname{diam} U \leqslant \frac{1}{2}$ and consider only $U_{N,r} = U_{N,1,r}$. Note that $U_{N,r} \supset U_{N',r}$ for $N' > N$.

Let $\mathcal{S}$ be an L-stratification of $\overline{U}$ satisfying Lemma 5 for all F_{ij} and the above additional conditions of Lemma 7 for all $r > s$. We construct an L-stratification $\mathcal{S}'$ of $\overline{U}$ satisfying the conditions of Lemma 5 for all F_{ij} and the above additional conditions for all $r \geqslant s$. The first step is to enlarge S^{s-1} in such a way that the extra conditions hold for $U_{N,s}$. Note that if S^{s-1} is bigger, $U_{N,j}$ is smaller. By [**7**] Proposition 3.5, S^s is the union of L-regular sets X^i defined by $g_i : \overline{V} \to \mathbb{R}^{k-s}$ (as in Definition 3.2 of [**7**]), in some system of coordinates. We add all ∂X_i to S^{s-1}. Fix X_i for a moment. Assume that the associated system of coordinates is standard and $\pi : \mathbb{R}^k \to \mathbb{R}^s$ is the standard projection. Let $U_i = \pi^{-1}(V_i) \cap U'$. Consider on $U_i \times V$, where $V = \{(0, v) \in \{0\} \times \mathbb{R}^{k-s} \subset \mathbb{R}^s \times \mathbb{R}^{k-s} \mid |v| = 1\}$, the semi-analytic function

$$\beta(x, v) = \sum_{i=1}^{n-k} \sum_{r=1}^{k-s} |v_r F_{ir}(x)|^2 \, |x - (\pi(x), g_i(\pi(x))|^2.$$

The graph of β is not only a semi-analytic subset of $\overline{U_i} \times V \times \mathbb{R}$, see [**7**] Lemma 2.3, but also it is semi-algebraic in direction $V \times \mathbb{R}$. By Łojasiewicz's version of Tarski-Seindenberg Theorem [**3**], the graph of

$$\alpha(x) = \min_v \beta(x, v)$$

is semi-analytic and semi-algebraic in direction $\mathbb{R}$. Let $\pi_k : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ denote the standard projection. We shall prove that the dimension of

$$W = X_i \cap \pi_k(\overline{\operatorname{graph} \alpha} \smallsetminus \mathbb{R}^k \times \{0\})$$

is smaller than s.

Suppose, by contradiction, that $\dim W = s$. The sets

$$X_i(\varepsilon) = \overline{\{x \in U_i \mid \alpha(x) \geqslant \varepsilon\}} \cap X_i$$

are semi-analytic and $\bigcup_{n=1}^{\infty} X_i(1/n) = W$, so by Baire's theorem $\dim X_i(\varepsilon) = s$ for $\varepsilon > 0$ sufficiently small (if $\dim X_i(\varepsilon) < s$ then $X_i(\varepsilon)$ is closed and nowhere dense in X_i). Consider a semi-analytic set $Y = \{x \in U_i\,;\, \alpha(x) > \varepsilon/2\}$. Then $X_i(\varepsilon)$ is contained in the closure of Y. Choose $p = (p', g_i(p')) \in X_i(\varepsilon)$ such that $X_i(\varepsilon)$ is near p a nonsingular s-dimensional analytic manifold. We can assume that the pair $(Y, X_i(\varepsilon))$ satisfies Whitney's conditions near p, see [**3**]. In particular, $p \in \overline{\pi^{-1}(p') \cap Y}$ and therefore by the curve selection lemma there exists an $\mathbb{R}$-analytic curve $\gamma(t) : [0, \delta) \to \overline{\pi^{-1}(p') \cap Y}$ such that $\gamma(0) = p$ and $\gamma(0, \varepsilon) \subset Y$. Replacing eventually t by t^r, for some $r \in \mathbb{N}$, we can assume that $F \circ \gamma$ and all $F_{ij} \circ \gamma$ are analytic. Put $w(t) = \dot\gamma(t)/|\dot\gamma(t)|$. Then, for $f = F_{ij}$,

$$\begin{aligned}\Big| t \frac{d(f\circ\gamma)}{dt}\Big|^2 &= |t|^2 |Df(\gamma(t)) w(t)|^2 |\dot\gamma(t)|^2 \\ &\geqslant |Df(\gamma(t))w(t)|^2 |\gamma(t) - p|^2 = C\beta(\gamma(t), w(t)).\end{aligned}$$

Therefore, $\lim_{t\to 0} \alpha(\gamma(t)) = 0$ and this contradicts our assumption.

So, we have $\dim W < s$. Add W to S^{s-1} and extend α to a continuous function on $U_i \cup (X_i \smallsetminus S^{s-1})$ putting $\alpha|_{X_i \smallsetminus S^{s-1}} \equiv 0$. By Łojasiewicz Inequality, [**3**], there exists $M \in \mathbb{N}$ such that $\alpha(x)(d_{s-1}(x))^M$ can be extended to a continuous function on $U_i \cup X_i$, vanishing on X_i. We apply the Łojasiewicz Inequality again to find constants C, α satisfying

$$\alpha(x)(d_{s-1}(x))^M \leqslant C[\operatorname{dist}(x, X_i)]^\alpha \tag{17}$$

for all $x \in U_i$. Take $q \in U_{N,s}$ (for N sufficiently large, N will be specified later). Let $p \in S^s$ satisfy

$$|q - p| \leqslant \frac{3}{2} d_s(q).$$

If $p \in X_i$ and N sufficiently large, then, since X_i is L-regular, $q \in U_i$ and the point $q' = (\pi(q), g_i(\pi(q)))$ satisfies

$$|q - q'| \leqslant C d_s(q), \tag{18}$$

for some constant C not depending on q, q', X_i. In particular, then

$$\operatorname{dist}(q, X_i) \leqslant C d_s(q).$$

Therefore, (17) follows

$$\alpha(q)(d_{s-1}(q))^M \leqslant C(d_s(q))^\alpha. \tag{19}$$

By the assumption that $\operatorname{diam} \overline{U} \leqslant \frac{1}{2}$, we can assume $C = 1$ in (19). We also require $N > 2M/\alpha$. Then, because $q \in U_{N,s}$, (19) gives

$$\alpha(q) \leqslant (d_s(q))^{\alpha/2}.$$

This and (18) give (14) and (16), for some w satisfying (15) (If N is large, then $q' \in \mathring{S}^s$). To complete the proof we find an L-stratification of S^{s-1} compatible with the initial stratification. □

Corollary 8. — *Let F, $\overline{U}$, and $\mathcal{S}$ be as in Lemma 7. Then, for some $N \in \mathbb{N}$, $\gamma, C > 0$, and each $j = 0, \ldots, n-1$*

$$|P(\Omega)(q, F(q))| \leqslant C[d_j(q)]^\gamma \prod_{j=0}^{k-1} (d_j(q))^{-1},$$

for all $q \in U_{N,C,j}$.

Proof. — Assume, as above, that $\operatorname{diam} X \leqslant \frac{1}{2}$ and consider only the sets $U_{N,j} = U_{N,1,j}$. Fix N satisfying the assertion of Lemma 7. Let $q \in U_{N,s}$ and q', w, δ be given by Lemma 7. Let $v_0, \ldots, v_{k-1}$ be the Lipschitz vector fields given by Lemma 5 for $\mathcal{S}$ and q. If v is any combination of $v_0, \ldots, v_{s-1}$ and $|v(q)| = 1$, then

$$|\operatorname{P}_{q'}^\perp v(q)| \leqslant= |\operatorname{P}_{q'}^\perp (v(q) - v(q'))| \leqslant C\frac{|q-q'|}{d_{s-1}(q)}.$$

If N is sufficiently large then

$$|\operatorname{P}_{q'}^\perp v(q)| \leqslant \sqrt{\frac{1-\delta}{2}}.$$

So the angle between W and the space generated by $v_0(q), \ldots, v_{s-1}(q)$ is grater than some small but positive constant. This, Lemma 5, (10) and (11) give the desired result. □

Proof of Proposition 1. — Because X is compact it is, by [**7**] Propostion 3.5 a union of L-regular sets. Thus we may assume that X is L-regular and the proposition follows from Corollary 8 and Lemma 4. □

References

[1] A. Dubson – Calcul des invariantes numériques des singularités et applications, Preprint, Bonn, 1981.

[2] H. Hironaka – *Introduction to real-analytic sets and real-analytic maps*, Quaderni dei Gruppi di Ricerca Matematica del Consiglio Nazionale delle Ricerche, Istituto Matematico L. Tonelli dell'Università di Pisa, 1973.

[3] S. Łojasiewicz – Ensembles semi-analytiques, preprint, I.H.E.S., 1965.

[4] J. Milnor & J. Stasheff – *Characteristic Classes*, Annals of Mathematics Studies, vol. 6, Princeton University Press, Princeton, NJ, 1974.

[5] T. Mostowski – *Lipschitz equisingularity*, Dissertationes Math., vol. 243, PWN, Warsaw, 1985.

[6] ______, Lipschitz stratifications and Lipschitz isotopies, in *Geometric Singularity Theory* (S. Janeczko & S. Łojasiewicz, eds.), vol. 65, Banach Center Publications, Warszawa, 2004, p. 179–210.

[7] A. Parusiński – Lipschitz properties of semianalytic sets, *Ann. Inst. Fourier (Grenoble)* **38** (1988), p. 189–213.

[8] ______, Lipschitz stratification, in *Global Analysis in Modern Mathematics, Proceedings of a symposium in Honor of Richard Palais' Sixties Birthday* (K. Uhlenbeck, ed.), Publish or Perish, Houston, 1993, p. 73–91.

[9] ______, Lipschitz stratifications of subanalytic sets, *Ann. scient. Éc. Norm. Sup.* 4^e *série* **27** (1994), p. 661–696.

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