Bernstein- and Markov-type inequalities

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Abstract

This survey discusses the classical Bernstein and Markov inequalities for the derivatives of polynomials, as well as some of their extensions to general sets.

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Contents

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1 The original Bernstein and Markov inequalities

In 1912 S. N. Bernstein proved in [6] his famous inequality that now takes the form

$$|T_n'(\theta)| \le n \sup_t |T_n(t)|, \qquad \theta \in \mathbf{R}, \tag{1}$$

where

$$T_n(t) = a_0 + (a_1 \cos t + b_1 \sin t) + \cdots + (a_n \cos nt + b_n \sin nt)$$

is an arbitrary trigonometric polynomial of degree at most n. In (1) equality occurs for example for $T_n(t) = \sin nt$ and $\theta = 0$. Bernstein stated and proved his inequality in this form only for even or odd trigonometric polynomials, and by decomposition into even and odd parts, for arbitrary trigonometric polynomials he had $2n$ on the right. However the improved version with the correct factor was soon found by E. Landau and M. Riesz [18], and L. Fejér observed that the odd case actually implies (1) in its full generality.[1]

Let us rewrite (1) in the form

$$\|T_n'\| \le n\|T_n\|,$$

where $\|T_n\| := \sup_t |T_n(t)|$ is the supremum norm over the whole real line. In general, the supremum norm on a set E is defined as

$$\|f\|_E := \sup_{t\in E} |f(t)|.$$

If

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

is an algebraic polynomial of degree at most $n = 1, 2, \ldots$, then $P_n(\cos t)$ is a trigonometric polynomial of degree at most n, and for it we obtain from (1)

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \qquad x \in (-1, 1), \tag{2}$$

which is the "Bernstein's inequality" for algebraic polynomials.

The right-hand side blows up as $x \to \pm 1$, so (1) does not give information on how large the norm of P_n' can be in terms of the norm of P_n. This question was answered by the following estimate due to A. A. Markov [13] from 1890:

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}. \tag{3}$$

The polynomial inequalities we have been discussing have various applications. In approximation theory they are fundamental in establishing converse results, i.e., when one deduces smoothness from a given rate of approximation. For their applications in other areas see [29].

The inequalities (2) and (3) are sharp and can be applied on any interval instead of $[-1, 1]$. On more general sets they also give some information, e.g., if $E = \cup[a_i, b_i]$ consists of finitely many intervals, then (3) yields (by applying (3) to each subinterval separately) that

$$\|P_n'\|_E \le n^2 2(\min_i (b_i - a_i))^{-1} \|P_n\|_E,$$

[1] Indeed, it is sufficient to consider $\theta = 0$, and if we apply Bernstein's theorem for the odd polynomial $(T_n(x) - T_n(-x))/2$ at $\theta = 0$, then we get (1).

but here the "Markov factor" $2(\min_i(b_i - a_i))^{-1}$ on the right is not precise, it can be replaced by a smaller quantity.

In this paper we shall be interested in the form of the Bernstein and Markov inequalities on general sets E. The primary concern will be to identify the best (or asymptotically best) Bernstein and Markov factors which are connected with geometric (more precisely, potential theoretic) properties of the underlying set E. In this respect we mention that until ca. 2000 the analogue of (2) or (3) was known only in a few special cases, e.g., for two intervals of equal length, which can be reduced to the single interval case by the $x \to x^2$ substitution (see [7]). We shall focus on the supremum norm—analogous results in other norms are scarce, but we shall mention one in the last section. Some open problems will also be stated.

There are some interesting local variants of the Bernstein inequality by V. Andrievskii [3] as well as their connection with Bernstein's and Vasiliev's theorem on approximation of $|x|$ by polynomials ([1], [2], [30]), but we shall not discuss them for they need the concept of Green's functions which we want to avoid in this note.

2 Equilibrium measures

Many extensions and generalizations of the original Bernstein and Markov inequalities have been found in the last 130 years. We mention here only one, namely in 1960 V. S. Videnskii [32] proved the analogue of (1) on intervals shorter than the whole period: if $\beta \in (0, \pi)$, then for $\theta \in (-\beta, \beta)$ we have

$$|T_n'(\theta)| \leq n \frac{\cos \theta/2}{\sqrt{\sin^2 \beta/2 - \sin^2 \theta/2}} \|T_n\|_{[-\beta,\beta]}. \tag{4}$$

This inequality of Videnskii was sort of a curiosity for almost half of a century because the nature of the factor on the right was hidden—we shall see that it comes from an equilibrium density. Until recently it was unknown what the analogue of the classical inequalities on two (or more intervals), and even less on general sets, are, and we shall see that the correct forms are related to some equilibrium densities.

To formulate the appropriate statements, we need to introduce a few notions from potential theory. For a general reference to logarithmic potential theory see [17].

Let $E \subset \mathbf{C}$ be a compact subset of the plane. Think of E as a conductor, and put a unit charge on E, which can freely move in E. After a while the charge settles, it reaches an equilibrium state where its internal energy is minimal. The mathematical formulation is the following (on the plane, Coulomb's law takes the form that the repelling force between charged particles is proportional to the reciprocal of the distance): except for pathological cases, there is a unique probability measure μ_E on E, called the equilibrium measure of E, that minimizes the energy integral

$$\int \int \log \frac{1}{|z-t|} d\mu(z) d\mu(t). \tag{5}$$

This μ_E certainly exists in all the cases we are considering in this paper.

When $E \subset \mathbf{R}$ we shall denote by $\omega_E(t)$ the density (Radon-Nykodim derivative) of μ_E with respect to Lebesgue measure wherever it exists. It certainly exists in the (one dimensional) interior of E. For example,

$$\omega_{[-1,1]}(t) = \frac{1}{\pi\sqrt{1-t^2}}, \qquad t \in (-1, 1),$$

is just the well-known Chebyshev distribution. More generally, if E consists of finitely many intervals on the real line, say

$$E = \bigcup_{i=1}^{m} [a_{2i-1}, a_{2i}], \qquad a_1 < a_2 < \cdots < a_{2m},$$

then (see [22])

$$\omega_E(t) = \frac{1}{\pi} \frac{\prod_{i=1}^{m-1} |t - \xi_i|}{\sqrt{\prod_{i=1}^{2m} |t - a_i|}}, \qquad t \in E, \tag{6}$$

where the $\xi_j \in (a_{2j}, a_{2j+1})$, $j = 1, \ldots, m-1$, are the unique solutions of the system of equations

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{i=1}^{m-1} (u - \xi_i)}{\sqrt{\prod_{i=1}^{2m} |u - a_i|}} \, du = 0, \quad j = 1, \ldots, m-1. \tag{7}$$

In a similar fashion, if E consists of disjoint smooth Jordan curves and arcs with arc measure s_E, then we set $d\mu_E := \omega_E ds_E$, and this ω_E is then called the equilibrium density on E. For example, if E is a circle of radius r then $\omega_E(z) \equiv 1/(2\pi r)$ on E. As another example, consider a lemniscate

$$\sigma := \{z : \ |T_N(z)| = 1\},$$

where T_N is an algebraic polynomial of degree N, in which case[2]

$$\omega_\sigma(z) = \frac{|T_N'(z)|}{2\pi N}.$$

If E has only one component, then the equilibrium measure is closely related to the conformal map of its unbounded domain. In fact, let E be a smooth Jordan curve (homeomorphic image of a circle) or arc (homeomorphic image of a segment), and Φ a conformal map from the exterior of E onto the exterior of the unit circle that leaves infinity invariant. This Φ can be extended to E as a continuously differentiable function (with the exception of the endpoints of E when E is a Jordan arc). If E is a Jordan curve, then simply

$$\omega_E(z) = \frac{1}{2\pi} |\Phi'(z)|.$$

If, however, E is a Jordan arc, then it has two sides, say positive and negative sides, and every point $z \in E$ different from the endpoints of E is considered to belong to both sides, where they represent different points $z_\pm$ (with different Φ-images). In this case

$$\omega_E(z) = \frac{1}{2\pi} (|\Phi'(z_+)| + |\Phi'(z_-)|).$$

For example, if E is the arc of the unit circle that runs from $\mathrm{e}^{-\mathrm{i}\beta}$ to $\mathrm{e}^{\mathrm{i}\beta}$ counterclockwise, then

$$\omega_E(\mathrm{e}^{\mathrm{i}t}) = \frac{1}{2\pi} \frac{\cos t/2}{\sqrt{\sin^2 \beta/2 - \sin^2 t/2}}, \qquad t \in (-\beta, \beta). \tag{8}$$

[2]To be precise, σ consists of Jordan curves only if $T_N' \neq 0$ on σ, but the formula given for ω_E is true without this assumption (excluding double points where the density can be considered to be 0).

3 The general Bernstein inequality

3.1 Trigonometric polynomials

The form (8) indicates the nature of the factor in Videnskii's inequality (4): if $[\beta, \beta] \subset (-\pi, \pi)$, then one must consider the arc

$$\Gamma := \{e^{it} : \ t \in [-\beta, \beta]\}$$

on the unit circle, and the Videnskii factor at a point $\theta \in (-\beta, \beta)$ is 2π times $\omega_\Gamma(e^{i\theta})$. It turns out that this is true in general as is shown by the following result of A. Lukashov from [12] (see also [25]). For a 2π-periodic closed set $E \subset \mathbf{R}$ let

$$\Gamma_E := \{e^{it} : \ t \in E\} \tag{9}$$

be its image when we identify $\mathbf{R}/(\text{mod } 2\pi)$ with the unit circle. Then, for any trigonometric polynomial T_n of degree at most $n = 1, 2, \ldots$, we have (considering the one-dimensional interior $\mathrm{Int}(E)$ of E)

$$|T_n'(\theta)| \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E, \qquad \theta \in \mathrm{Int}(E), \tag{10}$$

where ω_{Γ_E} denotes the equilibrium density of Γ_E.

The result is sharp (see [25]): if $\theta \in E$ is an interior point of E, then there are trigonometric polynomials $T_n \not\equiv 0$ of degree at most $n = 1, 2, \ldots$ such that

$$|T_n'(\theta)| \ge (1 - o(1)) n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E, \tag{11}$$

where $o(1)$ tends to 0 as $n \to \infty$.

3.2 Algebraic polynomials on the real line

The algebraic version (proved in M. Baran's paper [5] and independently in [22]) reads as follows. If $E \subset \mathbf{R}$ is a compact set, then for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$, we have

$$|P_n'(x)| \le n\pi \omega_E(x) \|P_n\|_E, \qquad x \in \mathrm{Int}(E). \tag{12}$$

This is sharp again: if $x_0 \in \mathrm{Int}(E)$ is arbitrary, then there are polynomials P_n of degree at most $n = 1, 2, \ldots$ such that

$$|P_n'(x_0)| \ge (1 - o(1)) n\pi \omega_E(x_0) \|P_n\|_E.$$

We mention that (12) can also be deduced from (10) via a suitable linear transformation and the substitution $x = \cos t$.

Note that in the special case $E = [-1, 1]$ the inequality (12) gives back the original Bernstein inequality (2) because $\omega_{[-1,1]}(x) = 1/\pi\sqrt{1 - x^2}$.

Actually, for real polynomials more than (12) is true (see [24]):

$$\left(\frac{P_n'(x)}{\pi \omega_E(x)} \right)^2 + n^2 P_n(x)^2 \le n^2 \|P_n\|_E^2, \qquad x \in \mathrm{Int}(E), \tag{13}$$

which is the analogue of the beautiful inequality

$$\left(P_n'(x) \sqrt{1 - x^2} \right)^2 + n^2 P_n(x)^2 \le n^2 \|P_n\|_{[-1,1]}^2, \quad x \in [-1, 1], \tag{14}$$

of G. Szegő [21] and G. Schaake and J. G. van der Corput [19].

3.3 Algebraic polynomials on a circular set

The complete analogue of (12) is known for closed subsets E of the unit circle, see [16]. Indeed, if E is such a set and $z \in E$ is an inner point of E (i.e. an inner point of a subarc of E), then for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(z)| \leq \frac{n}{2}(1 + 2\pi\omega_E(z))\|P_n\|_E. \tag{15}$$

Furthermore, this is sharp: for an inner point $z \in E$ there are polynomials $P_n \not\equiv 0$ of degree $n = 1, 2, \ldots$ for which

$$|P_n'(z)| \geq (1 - o(1))\frac{n}{2}(1 + 2\pi\omega_E(z))\|P_n\|_E.$$

We shall discuss later why there is a difference in the Bernstein factors in (12) and (15).

4 The general Markov inequality

4.1 Intervals on the real line

In the sense of the preceding section, what is the form of the Markov inequality (3) on more general sets than an interval, say on a set consisting of finitely many intervals? Let $E = \cup_{i=1}^m [a_{2i-1}, a_{2i}]$, $a_1 < a_2 < \cdots < a_{2m}$, be such a set. When we consider the analogue of the Markov inequality for E, we actually have to talk about a Markov-type local inequality around every endpoint of E. Indeed, away from the endpoints (12) is true, therefore there the derivative is bounded by a constant times n times the norm of the polynomial, so the n^2 factor is needed only close to the endpoints (as in the single interval case). It is also clear that different endpoints play different roles. So let a_j be an endpoint of E, and let E_j be the part of E that lies closer to a_j than to any other endpoint. Let M_j be the smallest constant for which

$$\|P_n'\|_{E_j} \leq (1 + o(1))M_j n^2 \|P_n\|_E, \qquad \deg(P_n) \leq n, \; n = 1, 2, \ldots, \tag{16}$$

holds, where $o(1)$ tends to 0 (uniformly in the polynomials P_n) as n tends to infinity. This M_j depends on what endpoint a_j we are considering, and it gives the asymptotically best constant in the corresponding local Markov inequality. Its value can be expressed in terms of the equilibrium density ω_E. Indeed it is clear from (6) that at a_j the limit

$$\Omega_j := \lim_{t \to a_j, \; t \in E} \sqrt{|t - a_j|}\,\omega_E(t) = \frac{1}{\pi} \frac{\prod_{i=1}^{m-1} |a_j - \xi_i|}{\sqrt{\prod_{i \neq j} |a_j - a_i|}} \tag{17}$$

exists, where ξ_i are the numbers from (7). For example, if $E = [-1, 1]$, $a_1 = -1$, $a_2 = 1$, then $\Omega_{1,2} = 1/\pi\sqrt{2}$. With this Ω_j the asymptotic Markov factors M_j can be expressed (see [22]) as

$$M_j = 2\pi^2 \Omega_j^2, \qquad j = 1, \ldots, 2m. \tag{18}$$

From here the global Markov inequality easily follows:

$$\|P_n'\|_E \leq (1 + o(1))n^2 \left(\max_{1 \leq j \leq 2m} 2\pi^2 \Omega_j^2 \right) \|P_n\|_E, \qquad \deg(P_n) \leq n. \tag{19}$$

On the right the $o(1)$ term tends to 0 uniformly in the polynomials P_n as $n \to \infty$, and, in general, this term cannot be dropped.

While the inequalities (10) and (12) give the best possible results for all n (in both for each given n the equality is attained at some points), the estimates in (16) and (19) are sharp only in an asymptotic sense because of the term $(1+o(1))$ on the right. Here $o(1)$ tends to zero independently of the polynomials P_n as $n \to \infty$, and the given inequality may not be true without the $(1+o(1))$ factor. This will be true in all subsequent results containing that factor.

If we call the n-th Markov constant the smallest number $L_n = L_{n,E}$ for which

$$\|P_n'\|_E \le n^2 L_n \|P_n\|_E \tag{20}$$

is true for all polynomials P_n of degree at most n, then the determination of L_n seems to be a very difficult problem.

Problem 1 *For a given set E consisting of finitely many intervals and for a given degree n find the n-th Markov constant L_n.*

Analogous questions can be raised in connection with all subsequent results that contain the $(1+o(1))$ factor on the right. We shall not mention those problems separately.

4.2 Markov's inequality on a system of arcs on a circle

Suppose now that E consists of finitely many circular arcs, say

$$E = \bigcup_{k=1}^{m} \{\mathrm{e}^{\mathrm{i}t} : \ t \in [\alpha_{2k-1}, \alpha_{2k}]\},$$

where $-\pi \le \alpha_1 < \alpha_2 < \cdots < \alpha_{2m} < \pi$. In this case an explicit form similar to the one in (6) is known for the equilibrium measure (see e.g., [9]). We define for an endpoint $A_j = \mathrm{e}^{\mathrm{i}\alpha_j}$ of a subarc of E the quantity Ω_j as

$$\Omega_j := \lim_{z \to A_j,\ z \in E} \sqrt{|z - A_j|}\,\omega_E(z). \tag{21}$$

With this, we have the analogue of (16)–(18) with sharp constant:

$$\|P_n'\|_{E_j} \le (1+o(1)) 2\pi^2 \Omega_j^2 n^2 \|P_n\|_E, \qquad \deg(P_n) \le n, \ n = 1, 2, \ldots, \tag{22}$$

where E_j is the part of E that lies closer to A_j than to any other endpoint in E.

4.3 Markov's inequality for trigonometric polynomials

We have already mentioned Videnskii's inequality (10). However, if one considers derivatives of trigonometric polynomials on an interval (or system of intervals) shorter than 2π, then a Markov-type estimate also emerges since the factor in (10) blows up around the endpoints. Already the original paper [32] of Videnskii contained that if T_n is a trigonometric polynomial of degree at most n and $0 < \beta < \pi$, then

$$\|T_n'\|_{[-\beta,\beta]} \le (1+o(1)) n^2 2\cot(\beta/2) \|T_n\|_{[-\beta,\beta]}.$$

When

$$E = \bigcup_{k=1}^{m} [\alpha_{2k-1}, \alpha_{2k}], \qquad -\pi \le \alpha_1 < \alpha_2 < \cdots < \alpha_{2m} < \pi,$$

we should again consider the set

$$\Gamma_E = \{\mathrm{e}^{\mathrm{i}t} : \ t \in E\}$$

(see (9)) and the corresponding expressions

$$\Omega_j^{\Gamma_E} = \lim_{z \to A_j,\ z \in \Gamma_E} \sqrt{|z - A_j|}\omega_{\Gamma_E}(z), \qquad A_j = \mathrm{e}^{\mathrm{i}\alpha_j}, \tag{23}$$

from (21). Now if E_j is the part of E that is closer to the endpoint α_j than to any other of the endpoint of a subinterval of E, then (see [9])

$$\|T_n'\|_{E_j} \le (1 + o(1))n^2 8\pi^2 (\Omega_j^{\Gamma_E})^2 \|T_n\|_E, \tag{24}$$

and here the constant on the right is sharp.

Note that in this estimate $\pi^2(\Omega_j^{\Gamma_E})^2$ is multiplied by 8 and not by 2 as in the polynomials cases up to now.

5 Jordan curves and arcs

Let $C_1 = \{z : \ |z| = 1\}$ be the unit circle. If P_n an algebraic polynomial of degree at most n, then $P_n(\mathrm{e}^{\mathrm{i}t})$ is a trigonometric polynomial of degree at most n, so by Bernstein's inequality (1), we have

$$\left| \frac{dP_n(\mathrm{e}^{\mathrm{i}t})}{dt} \right| \le n \max |P_n|.$$

The left-hand side is $|P_n'(\mathrm{e}^{\mathrm{i}t})||\mathrm{i}\mathrm{e}^{\mathrm{i}t}| = |P_n'(\mathrm{e}^{\mathrm{i}t})|$, and we obtain

$$|P_n'(z)| \le n\|P_n\|_{C_1}, \qquad z \in C_1. \tag{25}$$

This inequality is due to M. Riesz [18] (although it can be easily derived from (1), remember that (1) was originally given with a factor $2n$ on the right, so [18] contains the first correct proof of (25)).

5.1 Unions of Jordan curves

Riesz' inequality was extended to Jordan curves and families of Jordan curves in [15]: if E is a finite union of disjoint C^2-smooth Jordan curves (homeomorphic images of circles) lying exterior to each other, then for polynomials P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(z)| \le (1 + o(1))n2\pi\omega_E(z)\|P_n\|_E, \qquad z \in E. \tag{26}$$

Furthermore, (26) is best possible: if $z_0 \in E$, then there are polynomials $P_n \not\equiv 0$ of degree at most $n = 1, 2, \ldots$ for which

$$|P_n'(z_0)| \ge (1 - o(1))n2\pi\omega_E(z_0)\|P_n\|_E.$$

Note that if E is the unit circle, then $\omega_E \equiv 1/2\pi$, so (26) gives back the original inequality (25) of M. Riesz modulo the $(1 + o(1))$ factor which, in general, cannot be dropped in the Jordan curve case.

If E is the union of C^2-smooth Jordan curves, then (26) implies the Markov-type norm inequality

$$\|P_n'\|_E \leq (1+o(1))n\Big(\max_{z\in E}\omega_E(z)\Big)\|P_n\|_E, \tag{27}$$

which is sharp again in the sense that on the right-hand side no smaller constant can be written than $\max\omega_E(z)$.

For the inequality (26) at a given point $z \in E$ one does not need the C^2-smoothness of E, it is sufficient that E is C^2-smooth in a neighborhood of z. Hence, if E is the union of piecewise C^2-smooth Jordan curves, then (26) holds at any point of E which is not a corner point, i.e., where smooth subarcs of E meet. At corner points the situation is different: if two subarcs of E meet at z_0 at an external angle $2\pi\alpha$, $0 < \alpha < 1$, then

$$|P_n'(z_0)| \leq Cn^\alpha\|P_n\|_E, \qquad \deg(P_n) \leq n,$$

see [20], and here the order n^α is best possible (explaining also why in Bernstein's inequality the order is n while in Markov's it is n^2).

Problem 2 *Determine the smallest C for which*

$$|P_n'(z_0)| \leq (1+o(1))Cn^\alpha\|P_n\|_E, \qquad \deg(P_n) \leq n.$$

The solution of this problem would be interesting even in such a simple case when E is the unit square.

By the maximum modulus theorem both (26) and (27) hold true if E is the union of the (closed) domains enclosed by finitely many C^2 Jordan curves (in which case the equilibrium measure μ_E is supported on the boundary of E, and ω_E denotes the density of μ_E with respect to the arclength measure on that boundary). Actually, (26) is true under much more general assumptions on the compact set E. It is sufficient that E coincides with the closure of its interior, and its boundary is a C^2-smooth Jordan arc in a neighborhood of the point z where we consider (26). That this is *not true* when the closure assumption is not satisfied is shown by (12) (note that there we have $\pi\omega_E(z)$ while in (26) the correct factor is $2\pi\omega_E(z)$) and even more dramatically by the Jordan arc case to be discussed below.

5.2 Bernstein's inequality on a Jordan arc

The preceding results satisfactorily answer the form of the Bernstein and Markov inequalities on unions of smooth Jordan curves. It has turned out that the case of Jordan arcs is different and much more difficult, and actually we have the precise results only for one Jordan arc. To explain why arcs are different than curves one needs to say a few words about the proof of (26). Using inverse images under polynomial maps one can deduce (26) from (25) for lemniscates, i.e., sets of the form $\sigma = \{z : |T_N(z)| = 1\}$, where T_N is a polynomial of fixed degree (this deduction is by far not trivial, but possible using the so-called polynomial inverse image method, see [23])). Note that a lemniscate may have several components, so the splitting of the underlying domain occurs at this stage of the proof. Now smooth Jordan curves, and actually families of Jordan curves, can be approximated from inside and from outside by lemniscates that touch the set at a given point (this is done via the sharp version of Hilbert's lemniscate theorem, see [15]), and that allows one to deduce (26) in its full generality from its validity on lemniscates σ. Since arcs do not have interior

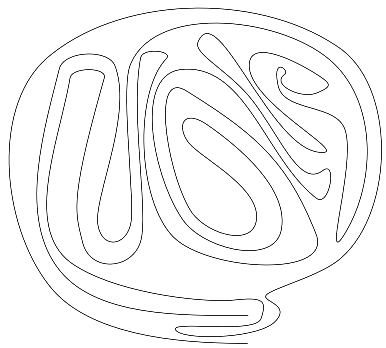

Figure 1: A “wild” Jordan arc

domains, that is not possible for arcs, and, as we shall see, the form of the corresponding result is indeed different.

In the general inequalities we have considered so far, always the equilibrium density ω_E gave the (asymptotically) best Bernstein-factors, and the Markov-factors have also been expressed in terms of them. In some sense this was accidental, it was due to either a symmetry (when $E \subset \mathbf{R}$) or to an absolute lack of symmetry (when E was a Jordan curve for which the two sides of E, the exterior and interior sides of E, play absolutely different roles). This is no longer the case when we consider Jordan arcs, for which the Bernstein factors are not expressible via the equilibrium density.

So let E be a Jordan arc, i.e., a homeomorphic image of a segment. We assume $C^{2+\alpha}$ smoothness of E with some $\alpha > 0$. As has already been discussed in Section 2, E has two sides, and every point $z \in E$ different from the endpoints of E gives rise to two different points $z_\pm$ on the two sides. With these, $\omega_E(z) = (|\Phi'(z_+)| + |\Phi'(z_-)|)/2\pi$, where Φ is a conformal map from the exterior of E onto the exterior of the unit circle leaving infinity invariant.

Now the Bernstein inequality on E for algebraic polynomials takes the form (for $z \in E$ being different from the two endpoints of E)

$$|P_n'(z)| \le (1 + o(1))n \max(|\Phi'(z_+)|, |\Phi'(z_-)|)\|P_n\|_E \tag{28}$$

(see [8] for analytic arcs and [11] for the general case). This is best possible: one cannot write anything smaller than $\max(|\Phi'(z_+)|, |\Phi'(z_+)|)$ on the right. As for the $o(1)$ term in (28), it may depend on the position of z inside E, but it is uniform in z on any closed subarc of E that does not contain the endpoints of E and, as before, and it is uniform in P_n, $n = 1, 2, \ldots$.

To appreciate the strength of (28) (or that of (26)) let us mention that the (smooth) Jordan arc in it can be arbitrary, and a general (smooth) Jordan arc can be pretty complicated, see for example, Figure 1.

Problem 3 *Find the analogue of (28) for E consisting of more than one (smooth) Jordan arc or when E is the union of Jordan curves and arcs.*

We believe that the answer to this problem will be the following. There is a possibly multivalent analytic function Ψ in the unbounded component Ω of $\mathbf{C} \setminus E$ that maps Ω onto the exterior of the unit circle locally conformally. While this Ψ is multivalent, its absolute value $|\Psi|$ is single-valued, and $g_\Omega(z) = \log|\Psi(z)|$ is actually the Green's function of Ω with pole at infinity (there are other definitions of the Green's function, one should take any of them). Now (in the one component case when Ψ is just the conformal map Φ that was considered before) the moduli $|\Phi'_\pm(z)|$ in (28) are precisely the normal derivatives $\partial g_\Omega(z)/\partial \mathbf{n}_\pm$ of g_Ω in the direction of the two normals to E at z, hence (28) can be written as

$$|P_n'(z)| \le (1+o(1))n \max\left(\frac{\partial g_\Omega(z)}{\partial \mathbf{n}_+}, \frac{\partial g_\Omega(z)}{\partial \mathbf{n}_-}\right) \|P_n\|_E, \tag{29}$$

and it is expected that this form remains true not just when E is a single Jordan arc, but also when E is the union of smooth Jordan arcs and curves (if z belongs to a Jordan curve, then the normal derivative in the direction of the inner domain is considered 0).

The conjecture just explained is true in two special cases: when E is a union of real intervals and when E is the union of finitely many arcs on the unit circle. In fact, both in (12), resp. (15), that cover these cases, the Bernstein factors $\pi\omega_E(x)$, resp. $(1+2\pi\omega_E(z))/2$, are precisely the maximum of the normal derivatives (see [16]).

If E is a piecewise smooth Jordan curve which may have "corners", then (28) still holds for points where z is smooth.

Problem 4 *Find the analogue of (28) for a piecewise smooth Jordan arc E at corner points.*

If at a corner point the two connecting subarcs form complementary angles $\alpha 2\pi$ and $(1-\alpha)2\pi$, $0 \le \alpha \le 1/2$, then

$$|P_n'(z)| \le (1+o(1))Cn^{1-\alpha}\|P_n\|_E, \qquad \deg(P_n) \le n,$$

with some constant C, and the problem is to determine the smallest C. This is not known even in such simple cases when E is the union of two perpendicular segments of equal length.

5.3 Markov's inequality on a Jordan arc

As for Markov's inequality, let the endpoints of the Jordan arc E be the points A and B. Consider e.g., the endpoint A, and let $\tilde{E}$ be the part of E that is closer to z than to the other endpoint of E. It turns out that as $z \to A$ the density $\omega_E(z)$ behaves like $1/\sqrt{|z-A|}$, and actually the limit

$$\Omega_A := \lim_{z\to A,\ z\in E} \sqrt{|z-A|}\,\omega_E(z)$$

exists. With it we have the Markov inequality around A (see [26]):

$$\|P_n'\|_{\tilde{E}} \le (1+o(1))n^2 2\pi^2\Omega_A^2\|P_n\|_E, \qquad \deg(P_n) \le n, \tag{30}$$

and this is best possible in the sense that one cannot write a smaller number than $2\pi^2\Omega_A^2$ on the right. From here the global Markov inequality

$$\|P_n'\|_E \le (1+o(1))n^2 2\pi^2(\max(\Omega_A,\Omega_B))^2\|P_n\|_E, \qquad \deg(P_n) \le n, \tag{31}$$

follows immediately, and this is sharp again.

Problem 5 *Prove (30) when E is a union of smooth Jordan arcs.*

That (30) should be the correct form also for a system of arcs is indicated by (18) and (22) which are the special cases when E is the union of finitely many intervals or the union of finitely many arcs on the unit circle.

We also mention that the just discussed results in this section are valid in a suitable form not only for polynomials, but also for rational functions for which the poles stay away from E, see [11].

6 Higher derivatives

For higher derivatives the correct form of the Markov inequality (3) was given in 1892 by V. A. Markov [14], the brother of A. A. Markov: if $k \geq 1$ is a natural number, then

$$\|P_n^{(k)}\|_{[-1,1]} \leq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}\|P_n\|_{[-1,1]}. \tag{32}$$

The equality is attained for the Chebyshev polynomials $P_n(x) = \cos(n \arccos x)$. If we write (32) in the less precise form

$$\|P_n^{(k)}\|_{[-1,1]} \leq \frac{n^{2k}}{(2k-1)!!}\|P_n\|_{[-1,1]}, \tag{33}$$

and compare it with

$$\|P_n^{(k)}\|_{[-1,1]} \leq n^{2k}\|P_n\|_{[-1,1]}$$

which is obtained from the original Markov inequality (3) by iteration, then we can see a mysterious improvement of $1/(2k-1)!!$. It turns out that the same improvement appears in other higher order Markov-type inequalities, as well, but that is not the case for Bernstein-type estimates.

6.1 Higher order Markov inequalities

Indeed, let $E = \cup_{i=1}^m [a_{2i-1}, a_{2i}]$ be a set consisting of finitely many intervals. Then the analogue of of (16)–(18) for higher derivatives is (see [27])

$$\|P_n^{(k)}\|_{E_j} \leq (1+o(1))\frac{2^k\pi^{2k}\Omega_j^{2k}}{(2k-1)!!}\|P_n\|_E \tag{34}$$

with an asymptotically sharp factor on the right. From here the global Markov inequality

$$\|P_n^{(k)}\|_E \leq (1+o(1))\frac{2^k\pi^{2k}(\max_j \Omega_j)^{2k}}{(2k-1)!!}\|P_n\|_E \tag{35}$$

is an easy consequence.

In a similar fashion, if E is a Jordan arc as in (30) with endpoints A and B, then we have (see [11], [26])

$$\|P_n^{(k)}\|_{\tilde{E}} \leq (1+o(1))n^{2k}\frac{2^k\pi^{2k}\Omega_A^{2k}}{(2k-1)!!}\|P_n\|_E, \tag{36}$$

and, as an immediate consequence,

$$\|P_n^{(k)}\|_E \leq (1+o(1))n^{2k}\frac{2^k\pi^{2k}\max(\Omega_A,\Omega_B)^{2k}}{(2k-1)!!}\|P_n\|_E,$$

again with the best constants (i.e., no smaller number can be written on the right).

We do not have an explanation for the factor $1/(2k-1)!!$, but we do know how it appears. Consider e.g., (36), and assume that the endpoint A is at the origin. Then

$$\Gamma := \{z : \ z^2 \in E\}$$

is a Jordan arc symmetric with respect to the origin for which 0 is an "inner" point, and for it the quantities $|\Phi'(0_\pm)|$ from (28) are the same, and can be expressed by Ω_0. Consider $R_{2n}(z) := P_n(z^2)$. For $k \geq 2$ the term $P_n^{(k)}(z^2)$ appears in the $2k$-th derivative of $R_{2n}(z) = P_n(z^2)$ if we use Faá di Bruno's formula for the $2k$-th derivative of composite functions, and $1/(2k-1)!!$ appears as the coefficient of that term (when everything is evaluated at $z = 0$). Now an application of (38) below with $2k$ instead of k for R_{2n} at $z = 0$ yields the bound given in (36) (at least at the endpoint 0).

6.2 Higher order Bernstein inequalities

When we consider Bernstein-type estimates the situation is different, no improvement factor like $1/(2k-1)!!$ appears. Indeed, the higher derivative form of (12) and (28) are

$$|P_n^{(k)}(x)| \leq (1 + o(1))n^k(\pi\omega_E(x))^k \|P_n\|_E, \qquad x \in \mathrm{Int}(E), \tag{37}$$

(when $E \subset \mathbf{R}$), and

$$|P_n^{(k)}(z)| \leq (1 + o(1))n^k \max(|\Phi'(z_+)|, |\Phi'(z_-)|)^k \|P_n\|_E \tag{38}$$

(when E is a Jordan arc), which are best possible. So in these cases the best results are obtained from the estimate on the first derivative by taking formal powers, and there is no improvement of the sort $1/(2k-1)!!$ as opposed to the above-discussed Markov inequalities.

In a similar manner, if E consists of a finite number of smooth Jordan curves, then the Riesz inequalities (26) and (27) for higher derivatives take the best possible forms

$$|P_n^{(k)}(z)| \leq (1 + o(1))n^k(2\pi\omega_E(z))^k \|P_n\|_E, \qquad z \in E, \tag{39}$$

and

$$\|P_n^{(k)}\|_E \leq (1 + o(1))n^k \Big((\max_{z\in E} \omega_E(z))\Big)^k \|P_n\|_E,$$

so there is no improvement again compared to straight iterations.

While (37)–(39) seem to appear as iterations of the $k = 1$ case, no straightforward iteration is possible. However, the proofs still use the $k = 1$ case inductively in combination with a localization technique using so-called fast decreasing polynomials.

We close this section by stating the higher order analogue of (10), i.e., the higher order Bernstein inequality for trigonometric polynomials: if $E \subset \mathbf{R}$ is a 2π-periodic closed set, then

$$|T_n^{(k)}(\theta)| \leq n^k \Big(2\pi\omega_{\Gamma_E}(\mathrm{e}^{\mathrm{i}\theta})\Big)^k \|T_n\|_E, \qquad \theta \in \mathrm{Int}(E), \tag{40}$$

where ω_{Γ_E} denotes the equilibrium density of the set (9). As before, (40) is sharp in the sense that no smaller factor than $(2\pi\omega_{\Gamma_E}(\mathrm{e}^{\mathrm{i}\theta}))^k$ can be written on the right.

The inequality (38) appears in [11], (40) in [10], and the proof of (37) was given in Appendix 2 of [30]. While (39) has not been recorded before, it can be deduced from the $k = 1$ case using the machinery of [9] or [30, Appendix 2] .

The higher order versions of (15), (22) and (24) are also known and follow the above pattern ($1/(2k-1)!!$ improvement in the Markov case and no improvement in the Bernstein case). We refer the reader to [9].

7 L^2-Markov inequalities

The L^p version of the preceding results is much less known. Here we shall consider only the case $p = 2$.

Let ν_κ denote the smallest positive zero of the Bessel function $J_{(\kappa-1)/2}$ of the first kind (see e.g., [33]). It was proved in [4] by A. I. Aptekarev, A. Draux, V. A. Kalyagin and D. N. Tulyakov that for polynomials P_n of degree at most $n = 1, 2, \ldots$

$$\left(\int_{-1}^{1} |P_n'(x)|^2 dx\right)^{1/2} \le (1+o(1))n^2 \frac{1}{2\nu_0} \left(\int_{-1}^{1} |P_n(x)|^2 dx\right)^{1/2}. \tag{41}$$

Furthermore, on the right $1/2\nu_0$ is the smallest possible constant.

If $E = \cup_{i=1}^m [a_{2i-1}, a_{2i}]$ is the union of m intervals, then the extension of (41) to E reads as (see [28])

$$\left(\int_{E} |P_n'(x)|^2 dx\right)^{1/2} \le (1+o(1))n^2 (\max_j \pi^2 \Omega_j^2) \frac{1}{\nu_0} \left(\int_{E} |P_n(x)|^2 dx\right)^{1/2}, \tag{42}$$

where Ω_j are the quantities defined in (17). Furthermore, this estimate is sharp, no smaller constant can be written on the right.

Recall that if $E = [-1, 1]$, $a_1 = -1$, $a_2 = 1$, then $\Omega_{1,2} = 1/\pi\sqrt{2}$, so (42) reduces to (41).

More generally, let $w(x) = (1+x)^\alpha (1-x)^\beta$, $\alpha, \beta > -1$, be a Jacobi weight. Then the sharp L^2-Markov inequality with this weight is

$$\left(\int_{-1}^{1} |P_n'(x)|^2 w(x) dx\right)^{1/2} \le (1+o(1))n^2 \frac{1}{2\nu_{\min(\alpha,\beta)}} \left(\int_{-1}^{1} |P_n(x)|^2 w(x) dx\right)^{1/2} \tag{43}$$

(see [4] for $|\alpha - \beta| \le 4$ and [28] for the other cases).

The analogue of this for several intervals is as follows. Let $E = \cup_{i=1}^m [a_{2i-1}, a_{2i}]$ be the union of m intervals and

$$w(t) = h(t) \prod_{i=1}^{2m} |t - a_i|^{\alpha_i},$$

$\alpha_i > -1$, a generalized Jacobi weight, where h is a positive continuous function on E. Then (see [28])

$$\|P_n'\|_{L^2(w)} \le (1+o(1))n^2 M(E, w) \|P_n\|_{L^2(w)}, \qquad \deg(P_n) \le n, \tag{44}$$

where the smallest possible constant $M(E, w)$ is

$$M(K, w) = \max_{1 \le j \le 2m} \frac{\pi^2 \Omega_j^2}{\nu_{\alpha_j}}. \tag{45}$$

Problem 6 *Find the precise form of these inequalities in other L^p, $1 \le p < \infty$, norms.*

Problem 7 *The Bernstein-type version of (41) is*

$$\left(\int_{-1}^{1} |\sqrt{1-x^2} P_n'(x)|^2 dx\right)^{1/2} \le (1+o(1))nC \left(\int_{-1}^{1} |P_n(x)|^2 dx\right)^{1/2},$$

but the smallest possible C here is not known.

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