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Splines and Radial Functions

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ABOUT THE DEFICIENT SPLINE COLLOCATION METHOD FOR PARTICULAR DIFFERENTIAL AND INTEGRAL EQUATIONS WITH DELAY

Abstract. The aim of this paper is to present the application of a particular collocation method (recently developed by the authors) to numerically solve some differential and Volterra integral equations with constant delay. The unknown function is approximated by using deficient spline functions. The existence and uniqueness of the numerical solution are studied; some aspects of the problem related to the estimation of the errors as well as the convergence properties are presented. Numerical examples are provided.

1. Introduction

In recent years a great deal of dynamical processes has been described and investigated by differential and integral equations with deviating arguments. It is well known that the versatility of such equations in modelling processes in various applications, especially in physics, engineering, biomathematics, medical sciences, economics, etc., provides the best, and sometimes the only, realistic simulation of observed phenomena.

Since solutions of such equations in general are not found explicitly, methods for their approximate solutions reveal very useful.

Recently we have proposed a deficient spline collocation method to approximate the solution of the first and second order delay differential equations (DDEs) [2] also in the neutral case (NDDEs) [3], [4], [5] and the solution of Volterra integral equations with delay (VDIEs) [6].

More precisely, we deal with the numerical solutions by combining two classic Numerical Analysis methodologies: approximation through the spline functional class and determination of the approximating function by a collocation method. In literature the two techniques are frequently used separately, but they are rarely combined to solve delay differential and integral equations. For instance in [1] they are applied in the numerical solution of first order delay differential equation, in [8] they are extended in the numerical solution of second order differential equations with delay, in [10] they are proposed in the case of Volterra integral equations. In all those works some advantages of that technique are outlined.

In any case, from those works one can draw the conclusion that spline methods are characterized by a large application spectrum, thanks to their weak convergence requirements, but they are affected by serious stability problems when their order increases. This explains why spline collocation techniques are not so often used.

In our works [2], [3], [4], [5], [6] taking into account that the phenomena described by the delay equations are very irregular, we proposed the following ideas:

i) the use of low order splines, in order to guarantee stability;

ii) the weakness of the continuity requirements at connecting points, so that lowly regular functions can be satisfactorily dealt with.

Therefore we propose the collocation using deficient splines (as defined in the next section), namely splines pertaining to class $\mathcal{C}^{m-2}$ (deficiency 1), where $m \in \mathbb{N}$, $m \geq 2$, is the spline degree.

Consequently we can use the advantages of the two (collocation and deficient spline) aspects.

The collocation methods provide the global spline expression, therefore they are selected:

i) in the case of DDE and NDDE, to eliminate the problems due to high-order interpolation, in the continuous extension

ii) in the case of VDIE, to use the expression of the spline in the evaluation of integrals in intervals preceding the current one

iii) to allow the use of variable intervals and spline degrees

iv) to state numerical models such that existence and unicity of the solution can be proved

v) to implement a simple and efficient algorithm.

About the deficient spline of polynomial degree $m \geq 2$:

i) we choose a classical convenient expression of the spline

ii) we choose low polynomial degree spline to maintain stability of the method and to deal with weakly regular solution

iii) weakening the spline regularity in the linking points, we can adapt the continuity class of the spline approximating to solutions at very low regularity.

As the equations with delay argument concerning the modelling processes are very often linear and with constant delay, in this paper we study the application of the numerical method proposed to these cases. We refer to works [2] to [6] for non linear cases.

In the second section we study the numerical model both for differential and Volterra integral equations. In the third section we give some numerical examples.

2. The description of spline collocation method

In this section we study the application of the numerical method to some linear DDE, NDDE and VDIE with constant delay.

2.1. The case of delay differential equations

We consider the following second order delay differential equation (DDE):

$$(1)\qquad \begin{array}{l} y''(x) = k_1 y(x) + k_2 y'(x) + f(x, y(g(x)), y'(g(x))), \quad x \in [a, b] \\ y(x) = \varphi(x), y'(x) = \varphi'(x), \quad x \in [\alpha, a], \quad \alpha \le a, \quad \alpha = \underset{x\in[a,b]}{Inf}(g(x)) \end{array}$$

$\alpha \le g(x) \le x, \quad x \in [\alpha, b], \quad \varphi \in \mathcal{C}^{m-2}[\alpha, a], m > 2, \ k_1, k_2 \in \mathbb{R}$
$f : [a, b] \times \mathcal{C}^1[\alpha, b] \times \mathcal{C}[\alpha, b] \to \mathbb{R}.$

We suppose verified the hypotheses so that the problem (1) has a unique solution $y \in \mathcal{C}^2[a, b] \cap \mathcal{C}^1[\alpha, b]$ (see [7]).

As it is known (see [7]) jump discontinuities can occur in various higher order derivatives of the solution y even if f, g, φ are analytic in their arguments. Such jump discontinuities are caused by the delay function g and propagate from the point a, moving ahead with the increasing order of derivatives.

If we denote the jump discontinuities by $\{\xi_j\}$, it is also known that ξ_j are the roots of equation $g(\xi_j) = \xi_{j-1}$ [7]; $\xi_0 = a$ is a jump discontinuity of φ (or of its derivatives). Since in (1) the delay function g does not depend on y (no state depending argument), we can consider the jump discontinuities for sufficiently high order derivatives to be such that $\xi_0 < \xi_1 < \ldots < \xi_{k-1} < \xi_k < \ldots < \xi_M$.

In the following we will consider $g(x) := x - \tau$ ($\tau \in \mathbb{R}, \tau > 0$) so that $\xi_j = a + j\tau$ ($j = 0, 1, \ldots, M$) and $\alpha = a - \tau$.

We shall construct for the problem (1) a deficient polynomial spline approximating function of degree $m \ge 3$, denoted by $s : [a, b] \to \mathbb{R}$, $s \in \mathcal{S}_m$, $s \in \mathcal{C}^{m-2}$, which will be defined on each interval $[\xi_j, \xi_{j+1}]$ ($j = 0, 1, \ldots, M-1$). For this construction we shall use successively the collocation methods as in [8]. Let us consider the first interval $[\xi_0, \xi_1]$ which is $[a, \xi_1]$. Let us define a uniform partition $\xi_0 = t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots < t_N = \xi_1$ where $t_j := t_0 + jh$ ($j = 0, 1, \ldots, N$), $h = (\xi_1 - \xi_0)/N$. On the first interval $[t_0, t_1]$ the spline component is defined by

$$(2)\qquad \begin{aligned} s_0(t) \ :\ &= \varphi(t_0) + \varphi'(t_0)(t - t_0) + \varphi''(t_0)(t - t_0)^2/2 + \ldots + \\ &+\varphi^{(m-2)}(t_0)(t - t_0)^{m-2}/(m-2)! + \\ &+a_0/(m-1)!(t - t_0)^{m-1} + b_0/m!(t - t_0)^m \end{aligned}$$

with a_0, b_0 to be determined by the following system of collocation conditions:

$$\begin{cases} s_0''(t_0 + h/2) = k_1 s_0(t_0 + h/2) + k_2 s_0'(t_0 + h/2) + \\ \qquad\quad + f(t_0 + h/2, \varphi(t_0 + h/2 - \tau), \varphi'(t_0 + h/2 - \tau)) \\ s_0''(t_1) = k_1 s_0(t_1) + k_2 s_0'(t_1) + f(t_1, \varphi(t_1 - \tau), \varphi'(t_1 - \tau)) \end{cases}$$

Once determined the polynomial (2), on the next interval $[t_1, t_2]$, we define

$$(3) \quad s_1(t) := \sum_{j=0}^{m-2} s_0^{(j)}(t_1)(t-t_1)^j/j! + a_1/(m-1)!(t-t_1)^{m-1} + b_1/m!(t-t_1)^m$$

where $s_0^{(j)}(t_1),\ 0 \le j \le m-2$, are left-hand limits of derivative as $t \to t_1$ of the segment of s defined on $[t_0, t_1]$ and a_1, b_1 are determined from the following collocation conditions:

$$\begin{cases} s_1''(t_1+h/2) = k_1 s_1(t_1+h/2) + k_2 s_1'(t_1+h/2) + \\ \qquad + f(t_1+h/2, \varphi(t_1+h/2-\tau), \varphi'(t_1+h/2-\tau)) \\ s_1''(t_2) = k_1 s_1(t_2) + k_2 s_1'(t_2) + f(t_2, \varphi(t_2-\tau), \varphi'(t_2-\tau)) \end{cases}$$

We remark that the peculiarity of these collocation conditions is the fact that they take into account the historical behaviour of the approximating spline, which is relevant for the delay nature of the considered equation.

Analogously for $t \in [t_k, t_{k+1}]$ we have

$$(4) \quad s_k(t) := \sum_{j=0}^{m-2} s_{k-1}^{(j)}(t_k)(t-t_k)^j/j! + a_k/(m-1)!(t-t_k)^{m-1} + b_k/m!(t-t_k)^m$$

where $s_{k-1}^{(j)}(t_k) = \lim_{t\to t_k} s_{k-1}^{(j)}(t)$, $t \in [t_{k-1}, t_k]$ and a_k, b_k are determined from

$$(5) \qquad \begin{cases} s_k''(t_k+\frac{h}{2}) = k_1 s_k(t_k+h/2) + k_2 s_k'(t_k+h/2) + \\ \qquad + f(t_k+\frac{h}{2}, \varphi(t_k+\frac{h}{2}-\tau), \varphi'(t_k+\frac{h}{2}-\tau)) \\ s_k''(t_{k+1}) = k_1 s_k(t_{k+1}) + k_2 s_k'(t_{k+1}) + \\ \qquad + f(t_{k+1}, \varphi(t_{k+1}-\tau), \varphi'(t_{k+1}-\tau)) \end{cases}$$

In general the spline function $s : [a, b] \to \mathbb{R}$, ($s \in \mathcal{S}_m$, $s \in \mathcal{C}^{m-2}$) approximating the solution of (1) on the interval $I_i := [\xi_i, \xi_{i+1}]$ ($i = 0, 1, ..., M-1$) is defined in $[t_k, t_{k+1}]$ where $t_k := t_0 + kh,\ k = 0, 1, ..., N-1;\ t_0 := \xi_i,\ t_N = \xi_{i+1},\ h := \frac{\xi_{i+1}-\xi_i}{N}$ as:

$$(6) \quad s_{k/I_i}(t) := \sum_{j=0}^{m-2} s_{k-1/I_i}^{(j)}(t_k)(t-t_k)^j/j! + \frac{a_k}{(m-1)!}(t-t_k)^{m-1} + \frac{b_k}{m!}(t-t_k)^m$$

with $a_k,\ b_k$ determined, as in (5) by

$$(7) \qquad \begin{cases} s_{k/I_i}''(t_k+\frac{h}{2}) = k_1 s_{k/I_i}(t_k+h/2) + k_2 s_{k/I_i}'(t_k+h/2) + \\ \qquad + f(t_k+\frac{h}{2}, s_{I_{i-1}}(t_k+\frac{h}{2}-\tau), s_{I_{i-1}}'(t_k+\frac{h}{2}-\tau)) \\ s_{k/I_i}''(t_{k+1}) = k_1 s_{k/I_i}(t_{k+1}) + k_2 s_{k/I_i}'(t_{k+1}) + \\ \qquad + f(t_{k+1}, s_{I_{i-1}}(t_{k+1}-\tau), s_{I_{i-1}}'(t_{k+1}-\tau)) \end{cases}$$

where $s_{I_{i-1}} \in \mathcal{S}_m$, $s_{I_{i-1}} \in \mathcal{C}^{m-2}$ is the spline approximating the solution of (1) on the interval I_{i-1}.

In the following, to simplify the notations, the theoretical results will be related only to (5); their generalization to (7) is immediate.

If we set

$$A_k(t) = \sum_{j=0}^{m-2} s_{k-1}^{(j)}(t_k)(t-t_k)^j/j! \tag{8}$$

then (5) becomes:

$$\begin{cases} \frac{a_k}{(m-3)!}\left(1-\frac{h}{2(m-2)}\left(k_1\frac{h}{2(m-1)}+k_2\right)\right)\left(\frac{h}{2}\right)^{m-3}+ \\ \quad +\frac{b_k}{(m-2)!}\left(1-\frac{h}{2(m-1)}\left(k_1\frac{h}{2m}+k_2\right)\right)\left(\frac{h}{2}\right)^{m-2}= \\ \quad -A_k''(t_k+\frac{h}{2})+k_1A_k(t_k+\frac{h}{2})+k_2A_k'(t_k+\frac{h}{2})+ \\ \quad +f(t_k+\frac{h}{2},\varphi(t_k+\frac{h}{2}-\tau),\varphi'(t_k+\frac{h}{2}-\tau)) \\ \\ \frac{a_k}{(m-3)!}\left(1-\frac{h}{m-2}\left(k_1\frac{h}{m-1}+k_2\right)\right)h^{m-3}+ \\ \quad +\frac{b_k}{(m-2)!}\left(1-\frac{h}{m-1}\left(k_1\frac{h}{m}+k_2\right)\right)h^{m-2}= \\ \quad -A_k''(t_{k+1})+k_1A_k(t_{k+1})+k_2A_k'(t_{k+1})+ \\ \quad +f(t_{k+1},\varphi(t_{k+1}-\tau),\varphi'(t_{k+1}-\tau)) \end{cases} \tag{9}$$

It remains to find under what conditions on h, the parameters $a_k, b_k,\ 0 \le k \le N-1$ can be uniquely determined from (9).

It is easy to prove the following:

THEOREM 1. *Let us consider the delay differential problems in (1). Under the hypotheses of existence and uniqueness of the analytic solution, there exists a unique spline approximation solution $s : [a,b] \to \mathbb{R}$, ($s \in \mathcal{S}_m$, $s \in \mathcal{C}^{m-2}$) of (1) given by the above construction for $h \neq 0$ if and only if the following condition is satisfied:*

$$\begin{vmatrix} 1-\frac{h}{2(m-2)}\left(k_1\frac{h}{2(m-1)}+k_2\right) & \frac{1}{2}\left(1-\frac{h}{2(m-1)}\left(k_1\frac{h}{2m}+k_2\right)\right) \\ 1-\frac{h}{m-2}\left(k_1\frac{h}{m-1}+k_2\right) & 1-\frac{h}{m-1}\left(k_1\frac{h}{m}+k_2\right) \end{vmatrix} \neq 0$$

COROLLARY 1. *If $k_1 = k_2 = 0$ and $m \geq 3$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

COROLLARY 2. *If $k_1 = 0$, $k_2 \neq 0$ and $3 \le m < 10$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

COROLLARY 3. *If $k_1 \neq 0$, $k_2 = 0$ and $3 \le m < 10$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

We can tackle by the same method also the following neutral delay differential equation (NDDE):

$$\begin{gathered} y'(x) = k_1 y(x) + f(x, y(g(x)), y'(g(x))), \quad x \in [a,b] \\ y(x) = \varphi(x), \quad x \in [\alpha, a], \quad \alpha \le a, \quad \alpha = \mathop{Inf}_{x\in[a,b]}(g(x)) \\ \alpha \le g(x) \le x\ , \quad x \in [\alpha, b] \end{gathered} \tag{10}$$

Let us assume that: $f : [a, b] \times \mathcal{C}^1[\alpha, b] \times \mathcal{C}[\alpha, b] \to \mathbb{R}$, $g \in \mathcal{C}[\alpha, b]$, $\alpha \le g(x) \le x$, $x \in [\alpha, b]$, $\varphi \in \mathcal{C}^{m-1}[\alpha, a]$, $m \ge 1$, $m \in N$, $k_1 \in \mathbb{R}$.

We suppose verified the hypotheses so that the problem (10) has a unique solution $y \in \mathcal{C}^1[a, b] \cap \mathcal{C}[\alpha, b]$ (see [7]).

Analogously to (1), we consider $g(x) := x - \tau$ ($\tau \in \mathbb{R}, \tau > 0$) and the jump discontinuities $\xi_j = a + j\tau$ $(j = 0, 1, ..., M)$, $\alpha = a - \tau$. In each interval $I_i = [\xi_i, \xi_{i+1}]$ $(i = 0, 1, ..., M - 1)$ we shall construct for the problem (10) a polynomial spline approximating function (6) of degree $m \ge 2$ and deficiency 1 and we determine the coefficients a_k, b_k through the following collocation system:

$$\begin{cases} s'_{k/I_i}(t_k + \frac{h}{2}) = k_1 s_{k/I_i}(t_k + h/2) + \\ \qquad + f(t_k + \frac{h}{2}, s_{I_{i-1}}(t_k + \frac{h}{2} - \tau), s'_{I_{i-1}}(t_k + \frac{h}{2} - \tau)) \\ s'_{k/I_i}(t_{k+1}) = k_1 s_{k/I_i}(t_{k+1}) + \\ \qquad + f(t_{k+1}, s_{I_{i-1}}(t_{k+1} - \tau), s'_{I_{i-1}}(t_{k+1} - \tau)) \end{cases}$$

It follows that in the first interval $[\xi_0, \xi_1]$ (the generalization to I_i, $i = 1, ..., M-1$ is immediate) assuming $A_k(t)$ as in (8):

$$\begin{cases} \frac{a_k}{(m-2)!}\left(1 - k_1\frac{h}{2(m-1)}\right)\left(\frac{h}{2}\right)^{m-2} + \frac{b_k}{(m-1)!}\left(1 - k_1\frac{h}{2m}\right)\left(\frac{h}{2}\right)^{m-1} = \\ \qquad -A'_k(t_k + \frac{h}{2}) + k_1 A_k(t_k + \frac{h}{2}) + \\ \qquad + f(t_k + \frac{h}{2}, \varphi(t_k + \frac{h}{2} - \tau), \varphi'(t_k + \frac{h}{2} - \tau)) \\ \\ \frac{a_k}{(m-2)!}\left(1 - k_1\frac{h}{m-1}\right)h^{m-2} + \frac{b_k}{(m-1)!}\left(1 - k_1\frac{h}{m}\right)h^{m-1} = \\ \qquad -A'_k(t_{k+1}) + k_1 A_k(t_{k+1}) + \\ \qquad + f(t_{k+1}, \varphi(t_{k+1} - \tau), \varphi'(t_{k+1} - \tau)) \end{cases} \tag{11}$$

It is easy to prove the following:

THEOREM 2. *Let us consider the delay neutral differential problems in (10). Under the hypotheses of existence and uniqueness of the analytic solution, there exists a unique spline approximation solution $s : [a, b] \to \mathbb{R}$, ($s \in \mathcal{S}_m$, $s \in \mathcal{C}^{m-2}$) of (10) given by the above construction for $h \neq 0$, if and only if the following condition is satisfied:*

$$\begin{vmatrix} 1 - k_1\frac{h}{2(m-1)} & \frac{1}{2}\left(1 - k_1\frac{h}{2m}\right) \\ 1 - k_1\frac{h}{m-1} & 1 - k_1\frac{h}{m} \end{vmatrix} \neq 0$$

COROLLARY 4. *If $k_1 = 0$ and $m \ge 2$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

COROLLARY 5. *If $k_1 \neq 0$ and $2 \le m < 9$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

REMARK 1. As the condition of the Theorem provides the non singularity of the coefficient matrix of system (11), its extension to a linear system of n delay differential equations of first order is immediate.

REMARK 2. For the consistency and convergence of the numerical solutions of (1) and (10) we can take into account the results obtained in more general cases. In [2] , [5] it is shown that the spline collocation method appears as (m–1)step-method. Consequently for $m = 3$ and $m = 4$ the cubic and quartic approximating splines yield the same values of the solution of (1) in the knots as discrete 2-step and 3-step method respectively. Analogously for $m = 2$ and $m = 3$ the trapezoidal and the Simpson's rule give the same discrete solutions of (10) as quadratic and cubic spline respectively. Consequently it is possible to prove consistency and convergence of the method. *The numerical stability of the method is not guaranteed (see [2], [5]) when $m > 4$ for (1) and when $m > 3$ for (10).*

2.2. The case of Volterra integral equations

Let us use the same method for the following Volterra integral equation with positive and constant delay (VDIE):

$$y(x) = \int_0^x k_1\, y(t)dt + \int_0^{x-\tau} K_2(x, t, y(t))dt + g(x),\ x \in J = [0, T] \tag{12}$$

with $k_1 \in \mathbb{R}$, the delay $\tau \in \mathbb{R},\ \tau > 0,\ y(x) = \phi(x)$ for $x \in [-\tau, 0)$.

We assume that the given functions $\phi : [-\tau, 0] \to \mathbb{R}$, $g : J \to \mathbb{R}$, $K_2 : \Omega_\tau \times \mathbb{R} \to \mathbb{R}$ ($\Omega_\tau := J \times [-\tau, T-\tau]$) are at least continuous on their domains such that (12) possesses a unique solution $y \in \mathcal{C}(J)$.

If $K_2 = 0$ equation (12) reduces to Volterra integral equation (VIE).

We suppose that $T = M\tau$ for some $M \in \mathbb{N}$. For $N \in \mathbb{N}$ (which satisfies $N/M \in \mathbb{N}$), let $h = T/N$ and $r = \tau/h \in \mathbb{N}$.

Chosen $t_i = ih$ ($i = -r, ..., 0, 1, ..., N;\ t_{-r} = -\tau, t_N = T$), the coefficients $a_k,\ b_k$ of $s_k(t)$ defined in $[t_k, t_{k+1}]$ ($k = 0, ..., N-1$) with $\tau \le t_k < T$ are determined through the following collocation system:

$$\begin{cases} s_k(t_k + \frac{h}{2}) = \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} k_1\, s_j(t)dt + \int_{kh}^{kh+\frac{h}{2}} k_1 s_k(t)dt + \\ \qquad + \sum_{j=0}^{k-1-r} \int_{jh}^{(j+1)h} K_2(t_k + \frac{h}{2}, t, s_j(t))dt + \\ \qquad + \int_{(k-r)h}^{(k-r)h+\frac{h}{2}} K_2(t_k + \frac{h}{2}, t, s_{k-r}(t))dt + g(t_k + \frac{h}{2}) \\ s_k(t_{k+1}) = \sum_{j=0}^{k} \int_{jh}^{(j+1)h} k_1\, s_j(t)dt + \\ \qquad + \sum_{j=0}^{k-r} \int_{jh}^{(j+1)h} K_2(t_{k+1}, t, s_j(t))dt + g(t_{k+1}) \end{cases} \tag{13}$$

and if $0 \le t_k < \tau$ from

$$\begin{cases} s_k(t_k + \frac{h}{2}) = \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} k_1\, s_j(t)dt + \\ \qquad + \int_{kh}^{kh+\frac{h}{2}} k_1 s_k(t)dt + \sum_{j=k-r}^{-1} \int_{jh}^{(j+1)h} K_2(t_k + \frac{h}{2}, t, s_j(t))dt + \\ \qquad - \int_{(k-r)h}^{(k-r)h+\frac{h}{2}} K_2(t_k + \frac{h}{2}, t, s_{k-r}(t))dt + g(t_k + \frac{h}{2}) \\ s_k(t_{k+1}) = \sum_{j=0}^{k} \int_{jh}^{(j+1)h} k_1\, s_j(t)dt + \\ \qquad + \sum_{j=k-r+1}^{-1} \int_{jh}^{(j+1)h} K_2(t_{k+1}, t, s_j(t))dt + g(t_{k+1}) \end{cases} \tag{14}$$

provided that $s_k(t) = \phi(t)$ in $[t_k, t_{k+1}]$ $(k = -r, ..., -1)$.

Consequently (13), with $A_k(t)$ as in (8), becomes:

$$\begin{cases} \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1}\left(1 - k_1\frac{h}{2m}\right) + \frac{b_k}{m!}(\frac{h}{2})^m\left(1 - k_1\frac{h}{2(m+1)}\right) = \\ \qquad -A_k(t_k + \frac{h}{2}) + \int_{kh}^{kh+\frac{h}{2}} k_1 A_k(t)dt + \sum_{j=0}^{k-1}\int_{jh}^{(j+1)h} k_1(A_j(t) \\ \qquad +\frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad + \sum_{j=0}^{k-1-r}\int_{jh}^{(j+1)h} K_2(t_k + \frac{h}{2}, t, A_j(t) + \frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \\ \qquad +\frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad + \int_{(k-r)h}^{(k-r)h+\frac{h}{2}} K_2(t_k + \frac{h}{2}, t, A_{k-r}(t) + \frac{a_{k-r}}{(m-1)!}(t - t_{k-r})^{m-1} + \\ \qquad +\frac{b_{k-r}}{m!}(t - t_{k-r})^m)dt + g(t_k + \frac{h}{2}) \\ \\ \frac{a_k}{(m-1)!}h^{m-1}\left(1 - k_1\frac{h}{m}\right) + \frac{b_k}{m!}h^m\left(1 - k_1\frac{h}{m+1}\right) = \\ \qquad -A_k(t_{k+1}) + \int_{kh}^{(k+1)h} k_1 A_k(t)dt + \sum_{j=0}^{k-1}\int_{jh}^{(j+1)h} k_1(A_j(t) + \\ \qquad +\frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad + \sum_{j=0}^{k-r}\int_{jh}^{(j+1)h} K_2(t_{k+1}, t, A_j(t) + \frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \\ \qquad +\frac{b_j}{m!}(t - t_j)^m)dt + g(t_{k+1}) \end{cases}$$

Analogously (14) becomes:

$$\begin{cases} \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1}\left(1 - k_1\frac{h}{2m}\right) + \frac{b_k}{m!}(\frac{h}{2})^m\left(1 - k_1\frac{h}{2(m+1)}\right) = \\ \qquad -A_k(t_k + \frac{h}{2}) + \int_{kh}^{kh+\frac{h}{2}} k_1 A_k(t)dt + \\ \qquad + \sum_{j=0}^{k-1}\int_{jh}^{(j+1)h} k_1(A_j(t) + \frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad + \sum_{j=k-r}^{-1}\int_{jh}^{(j+1)h} K_2(t_k + \frac{h}{2}, t, A_j(t) + \frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \\ \qquad +\frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad - \int_{(k-r)h}^{(k-r)h+\frac{h}{2}} K_2(t_k + \frac{h}{2}, t, A_{k-r}(t) + \frac{a_{k-r}}{(m-1)!}(t - t_{k-r})^{m-1} + \\ \qquad +\frac{b_{k-r}}{m!}(t - t_{k-r})^m)dt + g(t_k + \frac{h}{2}) \\ \\ \frac{a_k}{(m-1)!}h^{m-1}\left(1 - k_1\frac{h}{m}\right) + \frac{b_k}{m!}h^m\left(1 - k_1\frac{h}{m+1}\right) = \\ \qquad -A_k(t_{k+1}) + \int_{kh}^{(k+1)h} k_1 A_k(t)dt + \sum_{j=0}^{k-1}\int_{jh}^{(j+1)h} k_1(A_j(t) + \\ \qquad +\frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \frac{b_j}{m!}(t - t_j)^m)dt + \\ \qquad + \sum_{j=k-r+1}^{-1}\int_{jh}^{(j+1)h} K_2(t_{k+1}, t, A_j(t) + \frac{a_j}{(m-1)!}(t - t_j)^{m-1} + \\ \qquad +\frac{b_j}{m!}(t - t_j)^m)dt + g(t_{k+1}) \end{cases}$$

It is easy to prove the following:

THEOREM 3. *Let us consider equation (12). Under the hypotheses of existence and uniqueness of the analytic solution, there exists a unique spline approximation solution* $s : [0, T] \to \mathbb{R}$, *(*$s \in \mathcal{S}_m$, $s \in \mathcal{C}^{m-2}$*) of (12) given by the above construction* *for $h \neq 0$ if and only if the following condition is satisfied:*

$$\begin{vmatrix} 1 - k_1 \frac{h}{2m} & \frac{1}{2}(1 - k_1 \frac{h}{2(m+1)}) \\ 1 - k_1 \frac{h}{m} & 1 - k_1 \frac{h}{m+1} \end{vmatrix} \neq 0$$

COROLLARY 6. *If $k_1 = 0$ and $m \geq 2$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

COROLLARY 7. *If $k_1 \neq 0$ and $2 \leq m < 8$ the condition is satisfied $\forall h$ $(h \neq 0)$.*

REMARK 3. About the convergence and the numerical stability of the method applied to (12) we refer to [9].

3. Numerical examples

In the following we present some numerical results to enlighten the features of the presented numerical method. We emphasize that we will show examples just for cases with exact solutions belonging to a low regularity class, because our method is dedicated just to these cases. In all the examples the existence and uniqueness of the numerical solution is guaranteed for any value of the integration step h.

Our computer programs are written in MATLAB5.3, which has a machine precision $\varepsilon \simeq 10^{-16}$.

Our first example refers to the following second order DDE

$$y''(t) = \left| t - \frac{1}{2} \right| + y'(t-1)$$

which is to be solved on $[0, 1]$ with history $y(t) = 1$ for $t \leq 0$.
The analytical solution is $y(t) = -\frac{1}{6}t^3 + \frac{1}{4}t^2 + 1$ in $[0, 1/2]$, and $y(t) = \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t + \frac{23}{24}$ in $[1/2, 1]$; therefore the solution $y(t) \in \mathcal{C}^2$, so using $m = 4$, our approximating deficient spline function belongs exactly to the same class of regularity of the analytical solution. We remark that this problem is smooth, as at $t = 1/2$ the left third derivative slightly differs from the right third derivative. We chose a quite large value $h = 0.1$ and we get a comparison with an analogous collocation method using classical splines. The following Table 1 reports the errors es_d and es_c we obtained respectively using deficient spline $s_d \in \mathcal{S}_4$, $s_d \in \mathcal{C}^2$ and classical spline $s_c \in \mathcal{S}_4$,

$s_c \in \mathcal{C}^3$.

t	es_d	es_c
0.1	1.0E-4	1.4E-4
0.2	3.1E-5	9.4E-5
0.3	2.7E-5	2.9E-4
0.4	6.8E-5	1.0E-3
0.5	9.1E-5	2.0E-3
0.6	1.0E-4	3.3E-3
0.7	1.4E-4	4.3E-3
0.8	1.9E-4	5.0E-3
0.9	2.5E-4	5.4E-3
1.0	3.3E-4	5.5E-3

Table 1

It is clear that deficient spline behaves better than classical spline, as it exhibits the same class of regularity as the analytical solution, even when large integration steps are used.

As a second example, we consider the following NDDE:

$$y'(t) = -500\frac{y(t-1)}{y'(t-1)}$$

which is to be solved on $[0, 2]$ with history $y(t) = e^{-t}$ for $t \leq 0$.
The analytical solution is $y(t) = 500t + 1$ in $[0, 1]$, and $y(t) = -250t^2 + 499t + 252$ in $[1, 2]$. Therefore the solution $y(t) \in \mathcal{C}^0[0, 2]$; so using $m = 2$, our approximating deficient spline function belongs exactly to the same class of regularity as the analytical solution. We emphasize that this problem is really rough, as at $t = 1$ the left first derivative and the right first derivative differ significantly: indeed $y'(1)_- = 500$ whereas $y'(1)_+ = -1$. Therefore we could expect some numerical troubles. On the contrary our method deals very well with this kind of problems, as already pointed out. We chose a quite large value $h = 0.25$; at $t = 1$ we obtain numerically the exact value and at $t = 2$ the final absolute error is $2.6E - 4$. This suffices to show how our method is accurate and efficient and cheap. Figure 1 reports the behavior of the analytical solution (solid line) together with the numerical solution (rectangles) for the case $h = 0.25$.

As a third example we consider the following system of first order DDE's suggested as Example 1 in [11] . The equations

$$\begin{aligned} y_1'(t) &= y_1(t-1) \\ y_2'(t) &= y_1(t-1) + y_2(t-0.2) \\ y_3'(t) &= y_2(t-1) \end{aligned}$$

are to be solved on $[0, 1]$ with history $y_1(t) = 1$, $y_2(t) = 1$, $y_3(t) = 1$ for $t \leq 0$.

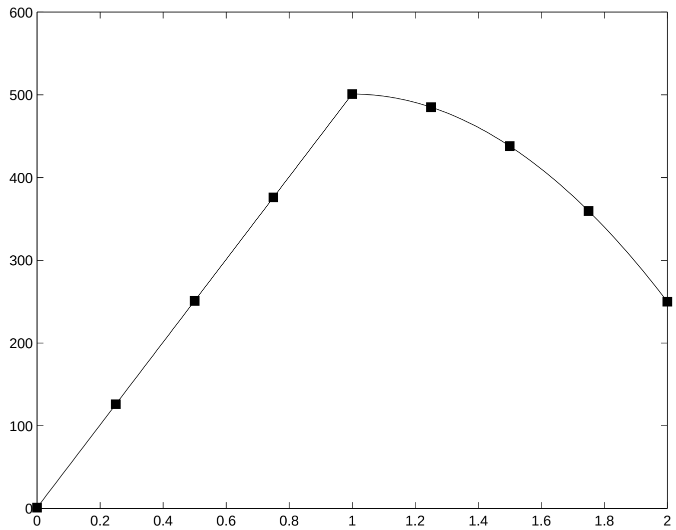

Figure 1: h=0.25

A comparison between the solutions computed by means of **dde23** (see [11]**)** and by our method (with $h = 0.01$) show that the three solution curves coincide and the number of required flops has the same order of magnitude; in this case using our method no advantages occur, because the solutions are very regular. However this example is interesting in order to show that *our method works efficiently also for systems of equations and moreover that different delays are allowed and can be conveniently handled.* We remark that in this case the linear system to be solved has M equations and M unknowns, where $M = 2n$ with n equal to the number of given first order equations; in this case $M = 6$.

About the integral equations, at first we consider the following Volterra integral equation without delay argument. This example is reported just to show that even in this case our method works really well, when solution exhibits low regularity.

$$\begin{aligned} y(x) &= g(x) + \int_0^x y(s)ds \\ g(x) &= \begin{cases} \frac{x^3}{3} - x^2 + \frac{1-x}{4} & 0 \leq x \leq \frac{1}{2} \\ -\left(\frac{x^3}{3} - x^2 + \frac{1-x}{4}\right) - \frac{1}{6} & \frac{1}{2} \leq x \leq 1 \end{cases} \end{aligned}$$

The exact solution is:

$$y(x) = \left| x^2 - \frac{1}{4} \right|$$

We computed our solution in $x = 1$. Using $m = 2$, we built splines $s \in S_2$, $s \in$

$\mathcal{C}^0[0, 1]$, that is of the same class of regularity of the analytical solution. Even in this case, we obtain very good numerical results; in particular at $t = 1$ our error is comparable with the machine precision, even when large integration steps are used.

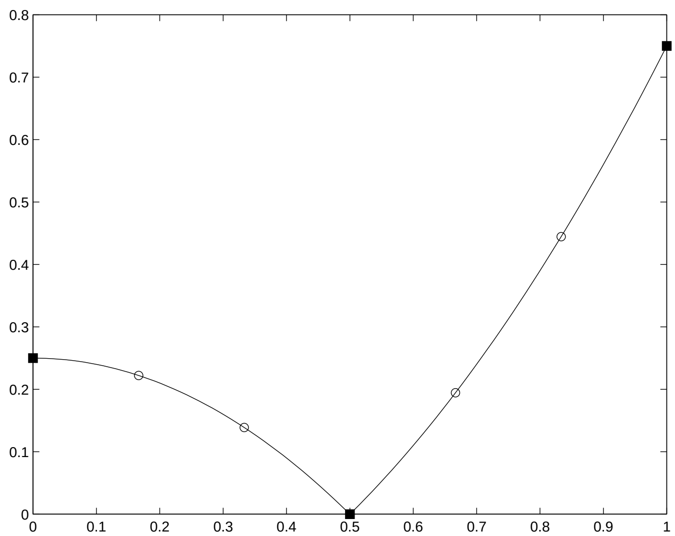

Figure 2: h=0.5

Figure 2 refers just to the case $h = 0.5$; there solid line shows the exact solution in $[0, 1]$; rectangles show the integration points and circles show intermediate points of our numerical solution computed by means of spline analytical expression relating to each integration interval. It is evident that even when a large integration step is used, our numerical solution coincides with the analytical one.

At last we consider the following integral equation with delay arguments:

$$\begin{aligned} y(x) &= g(x) + \int_0^x y(s)ds - \int_0^{x-\tau} y(s)ds \\ \tau &= 1, \qquad y(x) = 0 \quad \text{for} \quad x \in [-1, 0] \\ g(x) &= \begin{cases} 100x - 50x^2 & \text{for } x \in [0, 1/2] \\ -400(x-1)^3 + 100(x-1)^4 - 75/4 & \text{for } x \in [1/2, 1] \end{cases} \end{aligned}$$

The exact solution is:

$$y(x) = \begin{cases} 100x & \text{for } x \in [0, 1/2] \\ -400(x-1)^3 & \text{for } x \in [1/2, 1] \end{cases}$$

Even in this case the solution $y(x)$ to be approximated belongs to class $\mathcal{C}^0[0, 1]$. We used a large integration step $h_1 = 0.5$ in $[0, 1/2]$ and a shorter step h_2 in $[1/2, 1]$, where the solution is not linear.

Figure 3 refers to the case $h_1 = 0.5$ and $h_2 = 0.125$; there solid line shows the exact solution in $[0, 1]$ together with the history in $[-1, 0]$; rectangles show the numerical solution in the integration points and circles show the numerical solution in the intermediate points (computed by means of the analytical expression of spline).

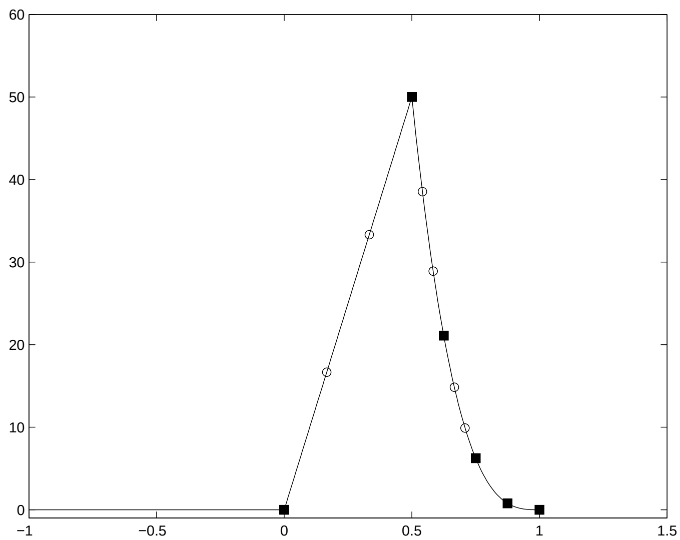

Figure 3: $h_1 = 0.5$, $h_2 = 0.125$

It is evident that even in this case results are very satisfactory.

In more details, the numerical solution in $x = 1$ is computed with an error equal to $1.0E - 2$ when $h_2 = 0.25$ and with an error equal to $6.6E - 4$ when $h_2 = 0.125$.

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