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HYPERBOLIC EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

Abstract. The goal of this article is to present new trends in the theory of solutions valued in Sobolev spaces for strictly hyperbolic Cauchy problems of second order with non-Lipschitz coefficients. A very precise relation between oscillating behaviour of coefficients and loss of derivatives of solution is given. Several methods as energy method together with sharp Gårding's inequality and construction of parametrix are used to get optimal results. Counter-examples complete the article.

1. Introduction

In this course we are interested in the Cauchy problem

$$
(1)\qquad \begin{aligned} & u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = 0 \qquad \text{on } (0,T) \times \mathbb{R}^n , \\ & u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \qquad \text{for } x \in \mathbb{R}^n . \end{aligned}
$$

Setting $a(t,x,\xi) := \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l$ we suppose with a positive constant C the strict hyperbolicity assumption

$$
(2)\qquad a(t,x,\xi) \geq C\,|\xi|^2
$$

with $a_{kl} = a_{lk}$, $k,l = 1,\cdots,n$.

DEFINITION 1. *The Cauchy problem (1) is well-posed if we can fix function spaces* A_1, A_2 *for the data* φ, ψ *in such a way that there exists a uniquely determined solution* $u \in C([0,T], B_1) \cap C^1([0,T], B_2)$ *possessing the domain of dependence property.*

The question we will discuss in this course is how the regularity of the coefficients $a_{kl} = a_{kl}(t,x)$ is related to the well-posedness of the Cauchy problem (1).

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2. Low regularity of coefficients

2.1. L_1-property with respect to t

In [10] the authors studied the Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} \partial_{x_k}(a_{kl}(t,x)\partial_{x_l}u) = 0 \quad \text{on } (0,T)\times\Omega\,,$$
$$u(0,x) = \varphi(x)\,,\; u_t(0,x) = \psi(x) \quad \text{on } \Omega\,,$$

where Ω is an arbitrary open set of $\mathbb{R}^n$ and $T > 0$. The coefficients of the elliptic operator in self-adjoint form satisfy the next analyticity assumption:
For any compact set K of Ω and for any multi-index β there exist a constant A_K and a function $\Lambda_K = \Lambda_K(t)$ belonging to $L^1(0,T)$ such that

$$\Big|\sum_{k,l=1}^{n} \partial_x^\beta a_{kl}(t,x)\Big| \le \Lambda_K(t) A_K^{|\beta|}|\beta|!\,.$$

Moreover, the strict hyperbolicity condition

$$\lambda_0|\xi|^2 \le \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l \le \Lambda(t)|\xi|^2$$

is satisfied with $\lambda_0 > 0$ and $\sqrt{\Lambda(t)} \in L^1(0,T)$.

THEOREM 1. *Let us suppose these assumptions. If the data φ and ψ are real analytic on Ω, then there exists a unique solution $u = u(t,x)$ on the conoid $\Gamma_\Omega^T \subset \mathbb{R}^{n+1}$. The conoid is defined by*

$$\Gamma_\Omega^T = \Big\{(t,x):\ \text{dist}\,(x,\mathbb{R}^n\setminus\Omega) > \int_0^T \sqrt{\Lambda(s)}ds\,,\ t\in[0,T]\Big\}\,.$$

The solution is C^1 in t and real analytic in x.

QUESTIONS. The Cauchy problem can be studied for elliptic equations in the case of analytic data. Why do we need the hyperbolicity assumption? What is the difference between the hyperbolic and the elliptic case?

We know from the results of [8] for the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0\,,\quad u(0,x) = \varphi(x)\,,\quad u_t(0,x) = \psi(x)\,,$$

that assumptions like $a \in L^p(0,T)$, $p > 1$, or even $a \in C[0,T]$, don't allow to weaken the analyticity assumption for data φ, ψ to get well-posedness results.

THEOREM 2. *For any class $\mathcal{E}\{M_h\}$ of infinitely differentiable functions which strictly contains the space $\mathcal{A}$ of real analytic functions on $\mathbb{R}$ there exists a coefficient $a = a(t) \in C[0,T],\ \ a(t) \geq \lambda > 0$, such that the above Cauchy problem is not well-posed in $\mathcal{E}\{M_h\}$.*

2.2. C^κ-property with respect to t

Let us start with the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x)\,,$$

where $a \in C^\kappa[0,T],\ \kappa \in (0,1)$. From [5] we have the following result:

THEOREM 3. *If $a \in C^\kappa[0,T]$, then this Cauchy problem is well-posed in Gevrey classes G^s for $s < \frac{1}{1-\kappa}$. To $\varphi,\ \ \psi \in G^s$ we have a uniquely determined solution $u \in C^2([0,T], G^s)$.*

We can use different definitions for G^s (by the behaviour of derivatives on compact subsets, by the behaviour of Fourier transform). If

- $s = \frac{1}{1-\kappa}$, then we should be able to prove local existence in t;
- $s > \frac{1}{1-\kappa}$, then there is no well-posedness in G^s.

The paper [22] is concerned with the strictly hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x)u_{x_k x_l} + \text{ lower order terms } = f(t,x)$$
$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

with coefficients depending Hölderian on t and Gevrey on x. It was proved well-posedness in Gevrey spaces G^s. Here G^s stays for a scale of Banach spaces.

One should understand

- how to define the Gevrey space with respect to x, maybe some suitable dependence on t is reasonable, thus scales of Gevrey spaces appear;
- the difference between $s = \frac{1}{1-\kappa}$ and $s < \frac{1}{1-\kappa}$, in the first case the solution should exist locally, in the second case globally in t if we constructed the right scale of Gevrey spaces.

REMARK 1. In the proof of Theorem 3 we use instead of $a \in C^\kappa[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq A\,\tau^\kappa$ for $\tau \in [0, T/2]$. But then the solution belongs only to $H^{2,1}([0,T], G^s)$.

3. High regularity of coefficients

3.1. Lip-property with respect to t

Let us suppose $a \in C^1[0,T],\ a(t) \geq C > 0$, in the strictly hyperbolic Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0,$$
$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

Using the energy method and Gronwall's Lemma one can prove immediately the well-posedness in Sobolev spaces H^s, that is, if $\varphi \in H^{s+1}(\mathbb{R}^n),\ \psi \in H^s(\mathbb{R}^n)$, then there exists a uniquely determined solution $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)\ \ (s \in N_0)$. A more precise result is given in [20].

THEOREM 4. *If the coefficients* $a_{kl} \in C([0,T], B^s) \cap C^1([0,T], B^0)$ *and* $\varphi \in H^{s+1},\ \psi \in H^s$, *then there exists a uniquely determined solution* $u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$. *Moreover, the energy inequality* $E_k(u)(t) \leq C_k E_k(u)(0)$ *holds for* $0 \leq k \leq s$, *where* $E_k(u)$ *denotes the energy of* k*'th order of the solution* u.

By B^∞ we denote the space of infinitely differentiable functions having bounded derivatives on $\mathbb{R}^n$. Its topology is generated by the family of norms of spaces $B^s,\ s \in \mathbb{N}$, consisting of functions with bounded derivatives up to order s.

REMARK 2. For our starting problem we can suppose instead of $a \in C^1[0,T]$ the condition $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq A\ \tau$ for $\tau \in [0, T/2]$. Then we have the same statement as in Theorem 4. The only difference is that the solution belongs to $C([0,T], H^{s+1}) \cap H^{1,2}([0,T], H^s) \cap H^{2,1}([0,T], H^{s-1})$.

PROBLEM 1. Use the literature to get information about whether one can weaken the assumptions for a_{kl} from Theorem 4 to show the energy estimates $E_k(u)(t) \leq C_k E_k(u)(0)$ for $0 \leq k \leq s$.

All results from this section imply that no loss of derivatives appears, that is, the energy $E_k(u)(t)$ of k-th order can be estimated by the energy $E_k(u)(0)$ of k-th order.

Let us recall some standard arguments:

- If the coefficients have more regularity $C^1([0,T], B^\infty)$, and the data φ and ψ are from H^∞, then the Cauchy problem is H^∞ well-posed, that is, there exists a uniquely determined solution from $C^2([0,T], H^\infty)$.
  This result follows from the energy inequality.
- Together with the domain of dependence property from H^∞ well-posedness we conclude C^∞ well-posedness, that is, to arbitrary data φ and ψ from C^∞ there exists a uniquely determined solution from $C^2([0,T], C^\infty)$.

This result follows from the energy inequality and the domain of dependence property.

Results for domain of dependence property:

THEOREM 5 ([5]). *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t) u_{x_k x_l} = f(t,x),\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)\ .$$

The coefficients $a_{kl} = a_{lk}$ *are real and belong to* $L^1(0,T)$*. Moreover,* $\sum_{k,l=1}^{n} a_{kl}(t)\xi_k\xi_l \geq \lambda_0|\xi|^2$ *with* $\lambda_0 > 0$*. If* $u \in H^{2,1}([0,T],\mathcal{A}')$ *is a solution for given* $\varphi, \psi \in \mathcal{A}'$ *and* $f \in L^1([0,T],\mathcal{A}')$*, then from* $\varphi \equiv \psi \equiv f \equiv 0$ *for* $|x - x_0| < \rho$ *it follows that* $u \equiv 0$ *on the set*

$$\{(t,x) \in [0,T] \times \mathbb{R}^n : |x - x_0| < \rho - \int_0^t \sqrt{|a(s)|} ds\}.$$

Here $|a(t)|$ *denotes the Euclidean matrix norm,* $\mathcal{A}'$ *denotes the space of analytic functionals.*

THEOREM 6 ([20]). *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = f(t,x),\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x).$$

The real coefficients $a_{kl} = a_{lk}$ *satisfy* $a_{kl} \in C^{1+\sigma}([0,T] \times \mathbb{R}^n) \cap C([0,T],B^0)$*. Let us define* $\lambda^2_{\max} := \sup_{|\xi|=1,[0,T]\times\mathbb{R}^n} a_{kl}(t,x)\xi_k\xi_l$*. Then* $\varphi = \psi \equiv 0$ *on* $D \cap \{t = 0\}$ *and* $f \equiv 0$ *on* D *implies* $u \equiv 0$ *on* D*, where* D *denotes the interior for* $t \geq 0$ *of the backward cone* $\{(x,t) : |x - x_0| = \lambda_{\max}(t_0 - t),\ (x_0,t_0) \in (0,T] \times \mathbb{R}^n\}$.

3.2. Finite loss of derivatives

In this section we are interested in weakening the Lip-property for the coefficients $a_{kl} = a_{kl}(t)$ in such a way, that we can prove energy inequalities of the form $E_{s-s_0}(u)(t) \leq E_s(u)(t)$, where $s_0 > 0$. The value s_0 describes the so-called *loss of derivatives*.

Global condition

The next idea goes back to [5]. The authors supposed the so-called LogLip-property, that is, the coefficients a_{kl} satisfy

$$|a_{kl}(t_1) - a_{kl}(t_2)| \leq C|t_1 - t_2|\,|\ln|t_1 - t_2|\,| \quad \text{for all} \quad t_1, t_2 \in [0,T], t_1 \neq t_2\ .$$

More precisely, the authors used the condition

$$\int_0^{T-\tau} |a_{kl}(t+\tau) - a_{kl}(t)|dt \le C\ \tau(|\ln\ \tau| + 1) \quad \text{for} \quad \tau \in (0, T/2]\ .$$

Under this condition well-posedness in C^∞ was proved.

As far as the author knows there is no classification of LogLip-behaviour with respect to the related loss of derivatives. He expects the following classification for solutions of the Cauchy problem $u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$:

Let us suppose $|a(t_1) - a(t_2)| \le C|t_1 - t_2|\,|\ln|t_1 - t_2||^\gamma$ *for all* $t_1, t_2 \in [0, T], t_1 \ne t_2$. *Then the energy estimates* $E_{s-s_0}(u)(t) \le C\ E_s(u)(0)$ *should hold, where*

- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is arbitrary small and positive if* $\gamma \in (0, 1)$,
- s_0 *is positive if* $\gamma = 1$,
- *there is no positive constant* s_0 *if* $\gamma > 1$ *(infinite loss of derivatives).*

The statement for $\gamma = 0$ *can be found in [5]. The counter-example from [9] implies the statement for* $\gamma > 1$.

OPEN PROBLEM 1. Prove the above statement for $\gamma \in (0, 1)$!

OPEN PROBLEM 2. The results of [9] show that $\gamma = 1$ gives a finite loss of derivatives. Do we have a concrete example which shows that the solution has really a finite loss of derivatives?

We already mentioned the paper [9]. In this paper the authors studied strictly hyperbolic Cauchy problems with coefficients of the principal part depending LogLip on spatial and time variables.

- If the principal part is as in (1.1) but with an elliptic operator in divergence form, then the authors derive energy estimates depending on a suitable low energy of the data and of the right-hand side.
- If the principal part is as in (1.1) but with coefficients which are B^∞ in x and LogLip in t, then the energy estimates depending on arbitrary high energy of the data and of the right-hand side.
- In all these energy estimates which exist for $t \in [0, T^*]$, where T^* is a suitable positive constant independent of the regularity of the data and right-hand side, the loss of derivatives depends on t.

It is clear, that these energy estimates are an important tool to prove (locally in t) well-posedness results.

Local condition

A second possibility to weaken the Lip-property with respect to t goes back to [6]. Under the assumptions

$$a \in C[0,T] \cap C^1(0,T],\ |ta'(t)| \le C \quad \text{for} \quad t \in (0,T], \tag{3}$$

the authors proved a C^∞ well-posedness result for $u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x)$ (even for more general Cauchy problems). They observed the effect of a finite loss of derivatives.

REMARK 3. Let us compare the local condition with the global one from the previous section. If $a = a(t) \in$ LogLip$[0,T]$, then the coefficient may have an irregular behaviour (in comparison with the Lip-property) on the whole interval $[0,T]$. In (3) the coefficient has an irregular behaviour only at $t = 0$. Away from $t = 0$ it belongs to C^1. Coefficients satisfying (3) don't fulfil the non-local condition

$$\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \le C\tau(|\ln\tau| + 1) \quad \text{for} \quad \tau \in (0, T/2]\,.$$

We will prove the next theorem by using the energy method and the following generalization of Gronwall's inequality to differential inequalities with singular coefficients. The method of proof differs from that of [6].

LEMMA 1 (LEMMA OF NERSESJAN [21]). *Let us consider the differential inequality*

$$y'(t) \le K(t)y(t) + f(t)$$

for $t \in (0,T)$, *where the functions* $K = K(t)$ *and* $f = f(t)$ *belong to* $C(0,T],\ T > 0$. *Under the assumptions*

- $\int_0^\delta K(\tau)d\tau = \infty\,,\quad \int_\delta^T K(\tau)d\tau < \infty\,,$
- $\lim_{\delta\to+0} \int_\delta^t \exp\left(\int_s^t K(\tau)d\tau\right) f(s)ds$ *exists,*
- $\lim_{\delta\to+0} y(\delta)\exp\left(\int_\delta^t K(\tau)d\tau\right) = 0$

for all $\delta \in (0,t)$ *and* $t \in (0,T]$, *every solution belonging to* $C[0,T] \cap C^1(0,T]$ *satisfies*

$$y(t) \le \int_0^t \exp\Big(\int_s^t K(\tau)d\tau\Big) f(s)ds\,.$$

THEOREM 7. *Let us consider the strictly hyperbolic Cauchy problem*

$$u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x)\ ,$$

where $a = a(t)$ satisfies with $\gamma \geq 0$ the conditions

$$a \in C[0,T] \cap C^1(0,T],\ |t^\gamma a'(t)| \leq C \quad \text{for} \quad t \in (0,T]\ . \tag{4}$$

Then this Cauchy problem is C^∞ well-posed iff $\gamma \in [0,1]$. If

- *$\gamma \in [0,1)$, then we have no loss of derivatives, that is, the energy inequalities $E_s(u)(t) \leq C_s\ E_s(u)(0)$ hold for $s \geq 0$;*
- *$\gamma = 1$, then we have a finite loss of derivatives, that is, the energy inequalities $E_{s-s_0}(u)(t) \leq C_s\ E_s(u)(0)$ hold for large s with a positive constant s_0.*

Proof. The proof will be divided into several steps.
Step 1. Cone of dependence
Let $u \in C^2([0,T], C^\infty(\mathbb{R}))$ be a solution of the Cauchy problem. If $\chi = \chi(x) \in C_0^\infty(\mathbb{R})$ and $\chi \equiv 1$ on $[x_0-\rho, x_0+\rho]$, then $v = \chi\, u \in C^2([0,T], \mathcal{A}')$ is a solution of

$$v_{tt} - a(t)v_{xx} = f(t,x)\ ,\quad v(0,x) = \tilde{\varphi}(x),\ v_t(0,x) = \tilde{\psi}(x)\ ,$$

where $\tilde{\varphi}, \tilde{\psi} \in C_0^\infty(\mathbb{R})$ and $f \in C([0,T], \mathcal{A}')$. Due to Theorem 5 we know that $v = v(t,x)$ is uniquely determined in $\{(t,x) \in [0,T] \times \mathbb{R}^n\colon |x-x_0| < \rho - \int_0^t \sqrt{|a(s)|}ds\}$. Hence, u is uniquely determined in this set, too. This implies $\varphi \equiv \psi \equiv 0$ on $[x_0-\rho, x_0+\rho]$ gives $u \equiv 0$ in this set. It remains to derive an energy inequality (see Section 3.1).
Step 2. The statement for $\gamma \in [0,1)$
If $\gamma \in [0,1)$, then

$$\int_0^{T-\tau} \left| \frac{a(t+\tau) - a(t)}{\tau} \right| dt \leq \int_0^{T-\tau} |a'(\theta(t,\tau))| dt \leq \int_0^{T-\tau} \frac{C}{t^\gamma}\ dt \leq C\ .$$

Thus the results from [5] are applicable.
Step 3. The statement for $\gamma > 1$
From the results of [6], we understand, that there is no C^∞ well-posedness for $\gamma > 1$. One can only prove well-posedness in suitable Gevrey spaces. Now let us consider the remaining case $\gamma = 1$.
Step 4. A family of auxiliary problems
We solve the next family of auxiliary problems:

$$\begin{aligned}
&u_{tt}^{(0)} = 0\ ,\quad u^{(0)}(0,x) = \varphi(x)\ ,\quad u_t^{(0)}(0,x) = \psi(x)\ ,\\
&u_{tt}^{(1)} = a(t)u_{xx}^{(0)}\ ,\quad u^{(1)}(0,x) = u_t^{(1)}(0,x) = 0\ ,\\
&u_{tt}^{(2)} = a(t)u_{xx}^{(1)}\ ,\quad u^{(2)}(0,x) = u_t^{(2)}(0,x) = 0\ ,\cdots,\\
&u_{tt}^{(r)} = a(t)u_{xx}^{(r-1)}\ ,\quad u^{(r)}(0,x) = u_t^{(r)}(0,x) = 0\ .
\end{aligned}$$

For the solution of our starting problem we choose the representation $u = \sum_{k=0}^{r} u^{(k)} + v$.
Then v solves the Cauchy problem $v_{tt} - a(t)v_{xx} = a(t)u^{(r)}_{xx},\ v(0,x) = v_t(0,x) = 0$.
Now let us determine the asymptotic behaviour of $u^{(r)}$ near $t = 0$. We have

$$|u^{(0)}(t,x)| \le |\varphi(x)| + t|\psi(x)|\ ,\ \ |u^{(1)}(t,x)| \le C\ t^2(|\varphi_{xx}(x)| + t|\psi_{xx}(x)|),$$
$$|u^{(2)}(t,x)| \le C\ t^4(|\partial_x^4\varphi| + t|\partial_x^4\psi|)$$

and so on.

LEMMA 2. *If* $\varphi \in H^{s+1}$, $\psi \in H^s$, *then* $u^{(k)} \in C^2([0,T], H^{s-2k})$ *and* $\|u^{(k)}\|_{C([0,t],H^{s-2k})} \le C_k t^{2k}$ *for* $k = 0, \cdots, r$ *and* $s \ge 2r + 2$.

Step 5. Application of Nersesjan's lemma
Now we are interested in deriving an energy inequality for a given solution $v = v(t,x)$ to the Cauchy problem

$$v_{tt} - a(t)v_{xx} = a(t)u^{(r)}_{xx}\ ,\ \ v(0,x) = v_t(0,x) = 0\ .$$

Defining the usual energy we obtain

$$\begin{aligned} E'(v)(t) &\le C_a|a'(t)|\ E(v)(t) + E(v)(t) + C\|u^{(r)}_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\le \frac{C_a}{t}\ E(v)(t) + E(v)(t) + C_a\|u^{(r)}_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\le \frac{C_a}{t}\ E(v)(t) + E(v)(t) + C_{a,r}t^{4r}\ . \end{aligned}$$

If $4r > C_a$ (C_a depends on $a = a(t)$ only), then Lemma 1 is applicable with $y(t) = E(v)(t),\ K(t) = \frac{C_a}{t}$ and $f(t) = C_{a,r}t^{4r}$. It follows that $E(v)(t) \le C_{a,r}t^{4r}$.

LEMMA 3. *If v is a solution of the above Cauchy problem which has an energy, then this energy fulfils* $E(v)(t) \le C_{a,r}t^{4r}$.

Step 6. Existence of a solution
To prove the existence we consider for $\varepsilon > 0$ the auxiliary Cauchy problems

$$v_{tt} - a(t+\varepsilon)v_{xx} = a(t)u^{(r)}_{xx} \in C([0,T], L^2(\mathbb{R}))\ ,$$

with homogeneous data. Then $a_\varepsilon = a_\varepsilon(t) = a(t+\varepsilon) \in C^1[0,T]$. For solutions $v_\varepsilon \in C^1([0,T], L^2(\mathbb{R}))$ which exist from strictly hyperbolic theory, the same energy inequality from the previous step holds. Usual convergence theorems prove the existence of a solution $v = v(t,x)$. The loss of derivatives is $s_0 = 2r + 2$. All statements of our theorem are proved.

□

A refined classification of oscillating behaviour

Let us suppose more regularity for a, let us say, $a \in L^\infty[0,T] \cap C^2(0,T]$. The higher regularity allows us to introduce a refined classification of oscillations.

DEFINITION 2. *Let us assume additionally the condition*

$$|a^{(k)}(t)| \leq C_k \left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^k \quad , \quad \text{for } k = 1, 2 \,. \tag{5}$$

We say, that the oscillating behaviour of a is

- *very slow if $\gamma = 0$,*
- *slow if $\gamma \in (0,1)$,*
- *fast if $\gamma = 1$,*
- *very fast if condition (3.3) is not satisfied for $\gamma = 1$.*

EXAMPLE 1. If $a = a(t) = 2 + \sin\left(\ln \frac{1}{t}\right)^\alpha$, then the oscillations produced by the sin term are very slow (slow, fast, very fast) if $\alpha \leq 1$ $(\alpha \in (1,2), \alpha = 2, \alpha > 2)$.

Now we are going to prove the next result yielding a connection between the type of oscillations and the loss of derivatives which appears. The proof uses ideas from the papers [7] and [14]. The main goal is the construction of WKB-solutions. We will sketch our approach, which is a universal one in the sense, that it can be used to study more general models from non-Lipschitz theory, weakly hyperbolic theory and the theory of $L_p - L_q$ decay estimates.

THEOREM 8. *Let us consider*

$$u_{tt} - a(t)u_{xx} = 0 \,, \quad u(0,x) = \varphi(x) \,, \quad u_t(0,x) = \psi(x) \,,$$

where $a = a(t)$ satisfies the condition (5), and the data φ , ψ belong to H^{s+1}, H^s respectively. Then the following energy inequality holds:

$$E(u)(t)\,|_{H^{s-s_0}} \leq C(T)E(u)(0)\,|_{H^s} \quad \textit{for all } t \in (0,T] \,, \tag{6}$$

where

- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there does not exist a positive constant s_0 satisfying (6) if $\gamma > 1$, that is, we have an infinite loss of derivatives.*

Proof. The proof will be divided into several steps. Without loss of generality we can suppose that T is small. After partial Fourier transformation we obtain

$$v_{tt} + a(t)\xi^2 v = 0\ ,\quad v(0,\xi) = \hat{\varphi}(\xi)\ ,\quad v_t(0,\xi) = \hat{\psi}(\xi)\ . \tag{7}$$

Step 1. Zones
We divide the phase space $\{(t,\xi) \in [0,T] \times \mathbb{R} : |\xi| \geq M\}$ into two zones by using the function $t = t_\xi$ which solves $t_\xi\langle\xi\rangle = N(\ln\langle\xi\rangle)^\gamma$. The constant N is determined later. Then the pseudo-differential zone $Z_{pd}(N)$, hyperbolic zone $Z_{hyp}(N)$, respectively, is defined by

$$Z_{pd}(N) = \{(t,\xi) : t \leq t_\xi\},\quad Z_{hyp}(N) = \{(t,\xi) : t \geq t_\xi\}\ .$$

Step 2. Symbols
To given real numbers $m_1, m_2 \geq 0,\ r \leq 2$, we define

$$S_r\{m_1,m_2\} = \{d = d(t,\xi) \in L^\infty([0,T]\times\mathbb{R}) :$$
$$|D_t^k D_\xi^\alpha d(t,\xi)| \leq C_{k,\alpha}\langle\xi\rangle^{m_1-|\alpha|}\left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^{m_2+k},\ k \leq r,\ (t,\xi) \in Z_{hyp}(N)\}.$$

These classes of symbols are only defined in $Z_{hyp}(N)$.
Properties:

- $S_{r+1}\{m_1,m_2\} \subset S_r\{m_1,m_2\}$;
- $S_r\{m_1-p,m_2\} \subset S_r\{m_1,m_2\}$ for all $p \geq 0$;
- $S_r\{m_1-p,m_2+p\} \subset S_r\{m_1,m_2\}$ for all $p \geq 0$, this follows from the definition of $Z_{hyp}(N)$;
- if $a \in S_r\{m_1,m_2\}$ and $b \in S_r\{k_1,k_2\}$, then $a\,b \in S_r\{m_1+k_1, m_2+k_2\}$;
- if $a \in S_r\{m_1,m_2\}$, then $D_t a \in S_{r-1}\{m_1, m_2+1\}$, and $D_\xi^\alpha a \in S_r\{m_1 - |\alpha|, m_2\}$.

Step 3. Considerations in $Z_{pd}(N)$
Setting $V = (\xi v, D_t v)^T$ the equation from (7) can be transformed to the system of first order

$$D_t V = \begin{pmatrix} 0 & \xi \\ a(t)\xi & 0 \end{pmatrix} V =: A(t,\xi)V\ . \tag{8}$$

We are interested in the fundamental solution $X = X(t,r,\xi)$ to (8) with $X(r,r,\xi) = I$ (identity matrix). Using the matrizant we can write X in an explicit way by

$$X(t,r,\xi) = I + \sum_{k=1}^{\infty} i^k \int_r^t A(t_1,\xi)\int_r^{t_1} A(t_2,\xi)\cdots\int_r^{t_{k-1}} A(t_k,\xi)\,dt_k\cdots dt_1\ .$$

The norm $\|A(t,\xi)\|$ can be estimated by $C\langle\xi\rangle$. Consequently

$$\int_0^{t_\xi} \|A(s,\xi)\|ds \le Ct_\xi\langle\xi\rangle = C_N(\ln\langle\xi\rangle)^\gamma .$$

The solution of the Cauchy problem to (8) with $V(0,\xi) = V_0(\xi)$ can be represented in the form $V(t,\xi) = X(t,0,\xi)V_0(\xi)$. Using

$$\|X(t,0,\xi)\| \le \exp(\int_0^t \|A(s,\xi)\|ds) \le \exp(C_N(\ln\langle\xi\rangle)^\gamma)$$

the next result follows.

LEMMA 4. *The solution to (8) with Cauchy condition $V(0,\xi) = V_0(\xi)$ satisfies in $Z_{pd}(N)$ the energy estimate*

$$|V(t,\xi)| \le \exp(C_N(\ln\langle\xi\rangle)^\gamma)|V_0(\xi)| .$$

REMARK 4. In $Z_{pd}(N)$ we are near to the line $t = 0$, where the derivative of the coefficient $a = a(t)$ has an irregular behaviour. It is not a good idea to use the hyperbolic energy $(\sqrt{a(t)}\xi\, v, D_t v)$ there because of the “bad” behaviour of $a' = a'(t)$. To avoid this fact we introduce the energy $(\xi\, v, D_t v)$.

Step 4. Two steps of diagonalization procedure

Substituting $V := (\sqrt{a(t)}\xi\, v, D_t v)^T$ (hyperbolic energy) brings the system of first order

$$D_t V - \begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} V - \frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = 0 . \tag{9}$$

The first matrix belongs to the symbol class $S_2\{1,0\}$, the second one belongs to $S_1\{0,1\}$. Setting $V_0 := MV,\ M = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, this system can be transformed to the first order system

$$D_t V_0 - M\begin{pmatrix} 0 & \sqrt{a(t)}\xi \\ \sqrt{a(t)}\xi & 0 \end{pmatrix} M^{-1}V_0 - M\frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1}V_0 = 0 ,$$

$$D_t V_0 - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_0 - \frac{D_t a}{4a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0 = 0 ,$$

where $\tau_{1/2} := \mp\sqrt{a(t)}\xi + \frac{1}{4}\,\frac{D_t a}{a}$. Thus we can write this system in the form $D_t V_0 - \mathcal{D}V_0 - R_0 V_0 = 0$, where

$$\mathcal{D} := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in S_1\{1,0\}\ ;\ R_0 = \frac{1}{4}\,\frac{D_t a}{a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S_1\{0,1\} .$$

This step of diagonalization is the *diagonalization of our starting system (9) modulo* $R_0 \in S_1\{0,1\}$.

Let us set

$$\mathcal{N}^{(1)} := -\frac{1}{4}\,\frac{D_t a}{a}\begin{pmatrix} 0 & \frac{1}{\tau_1-\tau_2} \\ \frac{1}{\tau_2-\tau_1} & 0 \end{pmatrix} = \frac{D_t a}{8a^{3/2}\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix $N_1 := I + \mathcal{N}^{(1)}$ is invertible in $Z_{hyp}(N)$ for sufficiently large N. This follows from the definition of $Z_{hyp}(N)$, from

$$\|N_1 - I\| = \|\mathcal{N}^{(1)}\| \le C_a \frac{1}{t|\xi|}\left(\ln\frac{1}{t}\right)^{\gamma} \le \frac{C_a}{N}\left(\frac{\ln\frac{1}{t}}{\ln\langle\xi\rangle}\right)^{\gamma} \le \frac{C_a}{N} \le \frac{1}{2},$$

if N is large, and from

$$\ln\langle\xi\rangle - \ln\frac{1}{t} \ge \ln N + \ln(\ln\langle\xi\rangle)^{\gamma}.$$

We observe that on the one hand $\mathcal{D}N_1 - N_1\mathcal{D} = R_0$ and on the other hand $(D_t - \mathcal{D} - R_0)N_1 = N_1(D_t - \mathcal{D} - R_1)$, where $R_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - R_0\mathcal{N}^{(1)})$. Taking account of $\mathcal{N}^{(1)} \in S_1\{-1,1\}$, $N_1 \in S_1\{0,0\}$ and $R_1 \in S_0\{-1,2\}$ the transformation $V_0 =: N_1 V_1$ gives the following first order system:

$$D_t V_1 - \mathcal{D}V_1 - R_1 V_1 = 0\,,\ \ \mathcal{D} \in S_1\{1,0\}\,,\ \ R_1 \in S_0\{-1,2\}\,.$$

The second step of diagonalization is the *diagonalization of our starting system (9) modulo* $R_1 \in S_0\{-1,2\}$.

Step 5. Representation of solution of the Cauchy problem

Now let us devote to the Cauchy problem

$$\begin{aligned} &D_t V_1 - \mathcal{D}V_1 - R_1 V_1 = 0\,, \\ &V_1(t_\xi,\xi) = V_{1,0}(\xi) := N_1^{-1}(t_\xi,\xi)M\,V(t_\xi,\xi). \end{aligned} \tag{10}$$

If we have a solution $V_1 = V_1(t,\xi)$ in $Z_{hyp}(N)$, then $V = V(t,\xi) = M^{-1}N_1(t,\xi)V_1(t,\xi)$ solves (9) with given $V(t_\xi,\xi)$ on $t = t_\xi$.

The matrix-valued function

$$E_2(t,r,\xi) := \begin{pmatrix} \exp\left(i\int_r^t\left(-\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)}\right)ds\right) & 0 \\ 0 & \exp\left(i\int_r^t\left(\sqrt{a(s)}\xi + \frac{D_s a(s)}{4a(s)}\right)ds\right) \end{pmatrix}$$

solves the Cauchy problem $(D_t - \mathcal{D})E(t,r,\xi) = 0$, $E(r,r,\xi) = I$. We define the matrix-valued function $H = H(t,r,\xi)$, $t, r \ge t_\xi$, by

$$H(t,r,\xi) := E_2(r,t,\xi)R_1(t,\xi)E_2(t,r,\xi)\,.$$

Using the fact that $\int\limits_r^t \frac{\partial_s a(s)}{4a(s)}\,ds = \ln\ a(s)^{1/4}\ |_r^t$ (this integral depends only on a, but is independent of the influence of a') the function H satisfies in $Z_{hyp}(N)$ the estimate

$$(11)\qquad \|H(t,r,\xi)\| \ \le\ \frac{C}{\langle\xi\rangle}\ \left(\frac{1}{t}\left(\ln\ \frac{1}{t}\right)^{\gamma}\right)^2\ .$$

Finally, we define the matrix-valued function $Q = Q(t,r,\xi)$ is defined by

$$Q(t,r,\xi) := \sum_{k=1}^{\infty} i^k \int\limits_r^t H(t_1,r,\xi)dt_1 \int\limits_r^{t_1} H(t_2,r,\xi)dt_2 \cdots \int\limits_r^{t_{k-1}} H(t_k,r,\xi)dt_k\ .$$

The reason for introducing the function Q is that

$$V_1 = V_1(t,\xi) := E_2(t,t_\xi,\xi)(I + Q(t,t_\xi,\xi))V_{1,0}(\xi)$$

represents a solution to (10).

Step 6. Basic estimate in $Z_{hyp}(N)$

Using (11) and the estimate $\int\limits_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds\ \le\ C_N(\ln\langle\xi\rangle)^{\gamma}$ we get from the representation for Q immediately

$$(12)\qquad \|Q(t,t_\xi,\xi)\| \le \exp\Big(\int\limits_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds\Big) \le \exp(C_N(\ln\langle\xi\rangle)^{\gamma})\ .$$

Summarizing the statements from the previous steps gives together with (12) the next result.

LEMMA 5. *The solution to (9) with Cauchy condition on $t = t_\xi$ satisfies in $Z_{hyp}(N)$ the energy estimate*

$$|V(t,\xi)| \le C\ \exp(C_N(\ln\langle\xi\rangle)^{\gamma})|V(t_\xi,\xi)|\ .$$

Step 7. Conclusions

From Lemmas 4 and 5 we conclude

LEMMA 6. *The solution $v = v(t,\xi)$ to*

$$v_{tt} + a(t)\xi^2 v = 0\ ,\quad v(0,\xi) = \hat\varphi(\xi)\ ,\quad v_t(0,\xi) = \hat\psi(\xi)$$

satisfies the a-priori estimate

$$\left|\begin{pmatrix} \xi\ v(t,\xi) \\ v_t(t,\xi)\end{pmatrix}\right| \le C\ \exp(C_N(\ln\langle\xi\rangle)^{\gamma})\left|\begin{pmatrix} \xi\ \hat\varphi(\xi) \\ \hat\psi(\xi)\end{pmatrix}\right|$$

for all $(t,\xi) \in [0,T]\times\mathbb{R}$.

The statement of Lemma 6 proves the statements of Theorem 8 for $\gamma \in [0, 1]$. The statement for $\gamma > 1$ follows from Theorem 9 (see next chapter) if we choose in this theorem $\omega(t) = \ln^q \frac{C(q)}{t}$ with $q \geq 2$.

□

REMARKS
1) From Theorem 5 and 8 we conclude the C^∞ well-posedness of the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0 \,, \quad u(0, x) = \varphi(x) \,, \quad u_t(0, x) = \psi(x),$$

under the assumptions $a \in L^\infty[0, T] \cap C^2(0, T]$ and (5) for $\gamma \in [0, 1]$.
2) Without any new problems all the results can be generalized to

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t)u_{x_k x_l} = 0 \,, \quad u(0, x) = \varphi(x), \; u_t(0, x) = \psi(x),$$

with corresponding assumptions for $a_{kl} = a_{kl}(t)$.
3) If we stop the diagonalization procedure after the first step, then we have to assume in Theorem 8 the condition (4). Consequently, we proposed another way to prove the results of Theorem 7. This approach was used in [6].

OPEN PROBLEM 3. In this section we have given a very effective classification of oscillations under the assumption $a \in L^\infty[0, T] \cap C^2(0, T]$. At the moment it does not seem to be clear what kind of oscillations we have if $a \in L^\infty[0, T] \cap C^1(0, T]$ satisfies $|a'(t)| \leq C \, \frac{1}{t} \left(\ln \frac{1}{t}\right)^\gamma$, $\gamma > 0$. If $\gamma = 0$, we have a finite loss of derivatives. What happens if $\gamma > 0$? To study this problem we have to use in a correct way the low regularity $C^1(0, T]$ (see next chapters).

OPEN PROBLEM 4. Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} + b(t)u_{xt} - a(t)u_{xx} = 0 \,, \quad u(0, x) = \varphi(x), \; u_t(0, x) = \psi(x) \,.$$

Does the existence of a mixed derivative of second order change the classification of oscillations from Definition 3.1? From the results of [1] we know that $a, b \in$ LogLip $[0, T]$ implies C^∞ well-posedness of the above Cauchy problem.

REMARK 5. *Mixing of different non-regular effects*
The survey article [11] gives results if we mix the different non-regular effects of Hölder regularity of $a = a(t)$ on $[0, T]$ and L_p integrability of a weighted derivative on $[0, T]$. Among all these results we mention only that one which guarantees C^∞ well-posedness of

$$u_{tt} - a(t)u_{xx} = 0 \,, \quad u(0, x) = \varphi(x) \,, \quad u_t(0, x) = \psi(x),$$

namely, $a = a(t)$ satisfies $t^q \partial_t a \in L^p(0, T)$ for $q + 1/p = 1$.

4. Hirosawa's counter-example

To end the proof of Theorem 8 we cite a result from [7] which explains that very fast oscillations have a deteriorating influence on C^∞ well-posedness.

THEOREM 9. *[see [7]] Let* $\omega : (0, 1/2] \to (0, \infty)$ *be a continuous, decreasing function satisfying* $\lim \omega(s) = \infty$ *for* $s \to +0$ *and* $\omega(s/2) \le c\, \omega(s)$ *for all* $s \in (0, 1/2]$. *Then there exists a function* $a \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})$ *with the following properties:*

- $1/2 \le a(t) \le 3/2$ *for all* $t \in \mathbb{R}$;
- *there exists a suitable positive* T_0 *and to each* p *a positive constant* C_p *such that*
$$|a^{(p)}(t)| \le C_p \omega(t) \Big(\frac{1}{t} \ \ln \ \frac{1}{t} \Big)^p \quad \text{for all} \ \ t \in (0, T_0) \ ;$$
- *there exist two functions* φ *and* ψ *from* $C^\infty(\mathbb{R})$ *such that the Cauchy problem* $u_{tt} - a(t) u_{xx} = 0, \quad u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x)$, *has no solution in* $C^0([0, r), \ \mathcal{D}'(\mathbb{R}))$ *for all* $r > 0$.

The coefficient $a = a(t)$ possesses the regularity $a \in C^\infty(\mathbb{R} \setminus \{0\})$. To attack the open problem 3 it is valuable to have a counter-example from [14] with lower regularity $a \in C^2(\mathbb{R} \setminus \{0\})$. To understand this counter-example let us devote to the Cauchy problem

$$(13) \qquad \begin{aligned} & u_{ss} - b\Big((\ln \frac{1}{s})^q \Big)^2 \bigtriangleup u = 0 \ , \quad (s, x) \in (0, 1] \times \mathbb{R}^n \ , \\ & u(1, x) = \varphi(x), \ u_s(1, x) = \psi(x), \qquad x \in \mathbb{R}^n \ . \end{aligned}$$

Then the results of [14] imply the next statement.

THEOREM 10. *Let us suppose that* $b = b(s)$ *is a positive, 1-periodic, non-constant function belonging to* C^2. *If* $q > 2$, *then there exist data* $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ *such that (13) has no solution in* $C^2([0, 1], \mathcal{D}'(\mathbb{R}^n))$.

Proof. We divide the proof into several steps.
Due to the cone of dependence property it is sufficient to prove H^∞ well-posedness. We will show that there exist positive real numbers $s_\xi = s(|\xi|)$ tending to 0 as $|\xi|$ tends to infinity and data $\varphi, \psi \in H^\infty(\mathbb{R}^n)$ such that with suitable positive constants C_1, C_2, and C_3,

$$|\xi| \ |\hat{u}(s_\xi, \xi)| + |\hat{u}_s(s_\xi, \xi)| \ge C_1 |\xi|^{\frac{1}{2}} \exp(C_2 (\ln C_3 |\xi|)^\gamma) \ .$$

Here $1 < \gamma < q - 1$. This estimate violates H^∞ well-posedness of the Cauchy problem (13). The assumption $b \in C^2$ guarantees that a unique solution $u \in C^2((0, T], H^\infty(\mathbb{R}^n))$ exists.

Step 1. Derivation of an auxiliary Cauchy problem
After partial Fourier transformation we get from (13)

$$v_{ss} + b\Big((\ln \frac{1}{s})^q\Big)^2 |\xi|^2 v = 0 \ , \quad (s,\xi) \in (0,1] \times \mathbb{R}^n \ ,$$
$$v(1,\xi) = \hat{\varphi}(\xi) \ , \ v_s(1,\xi) = \hat{\psi}(\xi) \ , \quad \xi \in \mathbb{R}^n \ ,$$

where $v(s,\xi) = \hat{u}(s,\xi)$. Let us define $w = w(t,\xi) := \tau(t)^{\frac{1}{2}} v(s(t),\xi)$, where $t = t(s) := (\ln \frac{1}{s})^q$, $\tau = \tau(t) := -\frac{dt}{ds}(s(t))$ and $s = s(t)$ denotes the inverse function to $t = t(s)$. Then w is a solution to the Cauchy problem

$$w_{tt} + b(t)^2 \lambda(t,\xi) w = 0 \ , \quad (t,\xi) \in [t(1),\infty) \times \mathbb{R}^n \ ,$$
$$w(t(1),\xi) = \tau(t(1))^{\frac{1}{2}} \hat{\varphi}(\xi) \ , \quad w_t(t(1),\xi) = \tau(t(1))^{-\frac{1}{2}} (\frac{1}{2}\tau_t(t(1))\hat{\varphi}(\xi) - \hat{\psi}(\xi)),$$

where $\lambda = \lambda(t,\xi) = \lambda_1(t,\xi) + \lambda_2(t)$, and

$$\lambda_1(t,\xi) = \frac{|\xi|^2}{\tau(t)^2} \ , \quad \lambda_2(t) = \frac{\theta(t)}{b(t)^2 \tau(t)^2} \ , \quad \theta = \tau'^2 - 2\tau''\tau \ .$$

Simple calculations show that $\tau(t) = q\, t^{\frac{q-1}{q}} \exp(t^{\frac{1}{q}})$ and $\theta(t) \approx -\exp(2t^{\frac{1}{q}})$. Hence, $\lim_{t\to\infty} \lambda_2(t) = 0$. Let λ_0 be a positive real number, and let us define $t_\xi = t_\xi(\lambda_0)$ by the definition $\lambda(t_\xi,\xi) = \lambda_0$. It follows from previous calculations that $\lim_{|\xi|\to\infty} t_\xi = \infty$.

Using the mean value theorem we can prove the following result.

LEMMA 7. *There exist positive constants C and δ such that*

$$|\lambda_1(t,\xi) - \lambda_1(t-d,\xi)| \le C\, d\, \frac{\tau'(t)}{\tau(t)} \lambda_1(t,\xi) \ , \quad |\lambda_2(t) - \lambda_2(t-d)| \le C\, \frac{\tau'(t)}{\tau(t)}$$

for any $0 \le d \le \delta\, \frac{\tau(t)}{\tau'(t)}$. In particular, we have

$$|\lambda(t_\xi,\xi) - \lambda(t_\xi - d,\xi)| \le Cd\, \frac{\tau'(t_\xi)}{\tau(t_\xi)} \lambda(t_\xi,\xi) \ , \quad 1 \le d \le \delta\, \frac{\tau(t_\xi)}{\tau'(t_\xi)} \ .$$

We have the hope that properties of solutions of $w_{tt} + b(t)^2\lambda(t,\xi)w = 0$ are not "far away" from properties of solutions of $w_{tt} + b(t)^2\lambda(t_\xi,\xi)w = 0$. For this reason let us study the ordinary differential equation $w_{tt} + \lambda_0 b(t)^2 w = 0$.

Step 2. Application of Floquet's theory
We are interested in the fundamental solution $X = X(t,t_0)$ as the solution to the Cauchy problem

$$\frac{d}{dt} X = \begin{pmatrix} 0 & -\lambda_0 b(t)^2 \\ 1 & 0 \end{pmatrix} X \ , \quad X(t_0,t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

It is clear that $X(t_0+1,t_0)$ is independent of $t_0 \in \mathbb{N}$.

LEMMA 8 (FLOQUET'S THEORY). *Let $b = b(t) \in C^2$, 1-periodic, positive and non-constant. Then there exists a positive real number λ_0 such that λ_0 belongs to an interval of instability for $w_{tt} + \lambda_0 b(t)^2 w = 0$, that is, $X(t_0+1, t_0)$ has eigenvalues μ_0 and μ_0^{-1} satisfying $|\mu_0| > 1$.*

Let us define for $t_\xi \in \mathbb{N}$ the matrix

$$X(t_\xi + 1, t_\xi) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} .$$

According to Lemma 8 the eigenvalues of this matrix are μ_0 and μ_0^{-1}. We suppose

$$|x_{11} - \mu_0| \geq \frac{1}{2} |\mu_0 - \mu_0^{-1}| . \tag{15}$$

Then we have $|x_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, too.

Step 3. A family of auxiliary problems

For every non-negative integer n we shall consider the equation

$$w_{tt} + \lambda(t_\xi - n + t, \xi) b(t_\xi + t)^2 w = 0. \tag{16}$$

It can be written as a first-order system which has the fundamental matrix $X_n = X_n(t, t_0)$ solving the Cauchy problem

$$\begin{aligned} & d_t X = A_n X \ , \ \ X(t_0, t_0) = I \\ & A_n = A_n(t, \xi) = \begin{pmatrix} 0 & -\lambda(t_\xi - n + t, \xi) b(t_\xi + t)^2 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

LEMMA 9. *There exist positive constants C and δ such that*

$$\max_{t_2, t_1 \in [0,1]} \|X_n(t_2, t_1)\| \ \leq \ e^{C\lambda_0}$$

for $0 \leq n \leq \delta \frac{\tau(t_\xi)}{\tau'(t_\xi)}$ and t_ξ large.

Proof. The fundamental matrix X_n has the following representation:

$$X_n(t_2, t_1) = I + \sum_{j=1}^{\infty} \int_{t_1}^{t_2} A_n(r_1, \xi) \int_{t_1}^{r_1} A_n(r_2, \xi) \cdots \int_{t_1}^{r_{j-1}} A_n(r_j, \xi) dr_j \cdots dr_1 .$$

By Lemma 7 we have

$$\begin{aligned} \max_{t_2, t_1 \in [0,1]} \|X_n(t_2, t_1)\| \ & \leq \exp(1 + b_1^2(\lambda_1(t_\xi - n, \xi) + \sup_{t(1) \leq t} |\lambda_2(t)|)) \\ & = \exp(1 + b_1^2(\lambda_1(t_\xi - n, \xi) - \lambda_1(t_\xi, \xi) + \lambda_0 - \lambda_2(t_\xi) + \sup_{t(1) \leq t} |\lambda_2(t)|)) \\ & \leq e^{C\lambda_0} \end{aligned}$$

for large t_ξ , $0 \le n \le \delta \, \frac{\tau(t_\xi)}{\tau'(t_\xi)}$, where $b_1 = \max\limits_{[0,1]} b(t)$.

□

LEMMA 10. *Let $\eta = \eta(t)$ be a function satisfying*

$$\lim_{t\to\infty} \eta(t) \, \frac{\tau'(t)}{\tau(t)} = 0 \, . \tag{18}$$

Then there exist constants C and δ such that $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\| \le C \, \lambda_0 \eta(t_\xi) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}$ for $0 \le n \le \delta \, \eta(t_\xi)$. Consequently, $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\| \le \varepsilon$ for any given $\varepsilon > 0$, sufficiently large $t_\xi \in \mathbb{N}$ and $0 \le n \le \delta \, \eta(t_\xi)$.

Proof. Using the representation of $X_n(1,0)$ and of $X(t_\xi + 1, t_\xi)$, then the application of Lemma 7 to $\|X_n(1,0) - X(t_\xi + 1, t_\xi)\|$ gives

$$\begin{aligned} \|X_n(1,0) - X(t_\xi + 1, t_\xi)\| &\le C \, \lambda_0 (n+1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)} \, \exp(C \, \lambda_0 (n+1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}) \\ &\le C \, \lambda_0 (\delta \, \eta(t_\xi) + 1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)} \, \exp(C \, \lambda_0 (\delta \, \eta(t_\xi) + 1) \, \frac{\tau'(t_\xi)}{\tau(t_\xi)}) \to 0 \end{aligned}$$

for $t_\xi \to \infty$ and $1 \le n \le \delta \, \eta(t_\xi)$.

□

Repeating the proofs of Lemmas 9 and 10 gives the following result.

LEMMA 11. *There exist positive constants C and δ such that*

$$\|X_{n+1}(1,0) - X_n(1,0)\| \le C \, \lambda_0 \, \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$$

for $1 \le n \le \delta \, \eta(t_\xi)$ and large ξ.

We will later choose $\eta = \eta(t) \sim t^\alpha$ with $\alpha \in \left(\frac{1}{2}\, , \, \frac{q-1}{q}\right)$. That the interval is non-empty follows from the assumptions of our theorem. If we denote $X_n(1,0) = \begin{pmatrix} x_{11}(n) & x_{12}(n) \\ x_{21}(n) & x_{22}(n) \end{pmatrix}$, then the statements of Lemmas 8 and 10 imply

- $|\mu_n - \mu_0| \le \varepsilon$, where μ_n and μ_n^{-1} are the eigenvalues of $X_n(1,0)$;
- $|\mu_n| \ge 1 + \varepsilon$ for $\varepsilon \le (|\mu_0| - 1)/2$;
- $|x_{11}(n) - \mu_n| \ge \frac{1}{4}|\mu_0 - \mu_0^{-1}|$, $|x_{22}(n) - \mu_n^{-1}| \ge \frac{1}{4}|\mu_0 - \mu_0^{-1}|$.

From Lemma 11 we conclude

- $|x_{ij}(n+1) - x_{ij}(n)| \le C \, \lambda_0 \, \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$. This implies

- $|\mu_{n+1} - \mu_n| \leq C\,\lambda_0\,\frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}$.

Step 4. An energy estimate from below

LEMMA 12. *Let n_0 satisfy $0 \leq n_0 \leq \delta\,\eta(t_\xi) \leq n_0 + 1$. Then there exist positive constants C_0 and C_1 such that the solution $w = w(t,\xi)$ to*

$$\begin{aligned} &w_{tt} + b(t)^2\lambda(t,\xi)w = 0,\\ &w(t_\xi - n_0 - 1, \xi) = 1\ , \quad w_t(t_\xi - n_0 - 1, \xi) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} \end{aligned}$$

satisfies

$$|w(t_\xi,\xi)| + |w_t(t_\xi,\xi)| \geq C_0\ \exp(C_1\,\eta(t_\xi)) \tag{19}$$

for large ξ and $\eta = \eta(t)$ fulfilling (18).

Proof. The function $w = w(t_\xi - n_0 + t, \xi)$ satisfies (16) with $n = n_0$. It follows that

$$\begin{aligned} \begin{pmatrix} \frac{d}{dt}\,w(t_\xi,\xi) \\ w(t_\xi,\xi) \end{pmatrix} &= X_1(1,0)X_2(1,0)\cdots \\ &\cdots X_{n_0-1}(1,0)X_{n_0}(1,0)\begin{pmatrix} \frac{d}{dt}\,w(t_\xi - n_0,\xi) \\ w(t_\xi - n_0,\xi) \end{pmatrix} . \end{aligned}$$

The matrix

$$B_n = \begin{pmatrix} \frac{x_{12}(n)}{\mu_n - x_{11}(n)} & 1 \\ 1 & \frac{x_{21}(n)}{\mu_n^{-1} - x_{22}(n)} \end{pmatrix}$$

is a diagonalizer for $X_n(1,0)$, that is, $X_n(1,0)B_n = B_n$ diag (μ_n, μ_n^{-1}). Since det $X_n(1,0) = 1$ and trace of $X_n(1,0)$ is $\mu_n + \mu_n^{-1}$ we get det $B_n = \frac{\mu_n - \mu_n^{-1}}{\mu_n^{-1} - x_{22}(n)}$. Using the properties of μ_n from the previous step we conclude $|\det\ B_n| \geq C > 0$ for all $0 < n \leq \delta\,\eta(t_\xi)$. Moreover, by Lemma 9 we have $|x_{ij}(n)| \leq C$, $\|B_n\| + \|B_n^{-1}\| \leq C$ for all $0 < n \leq \delta\,\eta(t_\xi)$. All constants C are independent of n. These estimates lead to

$$\|B_{n-1}^{-1}B_n - I\| = \|B_{n-1}^{-1}(B_n - B_{n-1})\| \leq C\,\lambda_0\,\frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)} \tag{20}$$

for large t_ξ. If we denote $G_n := B_{n-1}^{-1} B_n - I$, then we can write

$$
\begin{aligned}
& X_1(1,0)X_2(1,0)\cdots X_{n_0-1}(1,0)X_{n_0}(1,0) \\
= \;& B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} B_1^{-1}B_2 \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \\
& B_2^{-1}B_3\cdots B_{n_0-1}^{-1}B_{n_0} \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} B_{n_0}^{-1} \\
= \;& B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I+G_2) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \\
& (I+G_3)\cdots(I+G_{n_0}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} B_{n_0}^{-1}.
\end{aligned}
$$

We shall show that the $(1,1)$ element y_{11} of the matrix

$$
\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I+G_2) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} (I+G_3)\cdots
\cdots(I+G_{n_0}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix}
$$

can be estimated with suitable positive constants C_0 and C_1 by $C_0 \exp(C_1 \eta(t_\xi))$. It is evident from (20) that

$$
|y_{11} - \prod_{n=1}^{n_0} \mu_n| \;\le\; C \prod_{n=1}^{n_0} |\mu_n| \; \sum_{n=1}^{n_0} \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)}
$$

for large t_ξ. We have

$$
\begin{aligned}
\sum_{n=1}^{n_0} \frac{\tau'(t_\xi - n)}{\tau(t_\xi - n)} &\le \int_0^{\delta\,\eta(t_\xi)} \frac{\tau'(t_\xi - t - 1)}{\tau(t_\xi - t - 1)}\, dt \;\le\; \ln \frac{\tau(t_\xi - 1)}{\tau(t_\xi - \delta\,\eta(t_\xi) - 1)} \\
&\le \ln\left(1 - \delta\,\eta(t_\xi) \quad \frac{\tau'(t_\xi - 1)}{\tau(t_\xi - 1)}\right)^{-1} \to 0 \quad \text{as} \quad t_\xi \to \infty\,.
\end{aligned}
$$

Hence, we can find a positive real ν such that

$$
|y_{11}| \;\ge\; (1-\nu)\prod_{n=1}^{n_0}|\mu_n| \;\ge\; (1-\nu)(\mu_0 - \varepsilon)^{n_0} \;\ge\; (1-\nu)(\mu_0-\varepsilon)^{\delta\,\eta(t_\xi)-1}.
$$

The vector of data on $t = t_\xi - n_0$ is an eigenvector of B_{n_0}. Thus the estimate for y_{11} holds for the vector $(d_t w(t_\xi,\xi), w(t_\xi,\xi))^T$ too. This proves the energy estimate from below of the lemma.

□

Step 5. Conclusion

After choosing $s_\xi = s(t_\xi) = \exp(-t_\xi^{1/q})$ for large t_ξ and taking account of $w_t(t,\xi) = \frac{1}{2}\ \tau_t(t)\tau(t)^{-\frac{1}{2}}\ v(s(t),\xi) + \tau(t)^{\frac{1}{2}}\ v_s(s(t),\xi)$ we obtain

$$\begin{aligned} & |w(t(s),\xi)| + |w_t(t(s),\xi)| \\ \leq\ & \tau(t(s))^{\frac{1}{2}}\left(1+\ \frac{\tau_t(t(s))}{2\tau(t(s))}\right)|v(s,\xi)| + \tau(t(s))^{-\frac{1}{2}}|v_s(s,\xi)| \\ \leq\ & 2\tau(t(s))^{\frac{1}{2}}|v(s,\xi)| + \tau(t(s))^{-\frac{1}{2}}|v_s(s,\xi)| \end{aligned}$$

for large ξ. Finally, we use $\tau(t(s)) \sim |\xi|$. This follows from the definition $\lambda(t_\xi,\xi) = \lambda_0$ and $\lim\limits_{t_\xi\to\infty} \lambda_2(t_\xi) = 0$. Thus we have shown

$$|\xi|\ |\hat{u}(s_\xi,\xi)|\ +\ |\hat{u}_s(s_\xi,\xi)|\ \geq\ C_1|\xi|^{\frac{1}{2}}\ \exp(C_2\ \eta(t_\xi))\ .$$

The function $\eta(t) = t^\alpha$ satisfies (18) if $\alpha < \frac{q-1}{q}$. The function t_ξ behaves as $(\ln|\xi|)^q$.

Together these relations give

$$\begin{aligned} |\xi|\ |\hat{u}(s_\xi,\xi)|\ +\ |\hat{u}_s(s_\xi,\xi)|\ &\geq\ C_1|\xi|^{\frac{1}{2}}\ \exp(C_2\ (\ln|\xi|)^{q\alpha}) \\ &\geq C_1|\xi|^{\frac{1}{2}}\ \exp(C_2\ (\ln|\xi|)^{\gamma})\quad,\quad \text{where}\quad \gamma\in(1,q-1)\ . \end{aligned}$$

From this inequality we conclude the statement of Theorem 10.

□

REMARK 6. The idea to apply Floquet's theory to construct a counter-example goes back to [25] to study C^∞ well-posedness for weakly hyperbolic equations. This idea was employed in connection to $L_p - L_q$ decay estimates for solutions of wave equations with time-dependent coefficients in [24]. The merit of [14] is the application of Floquet's theory to strictly hyperbolic Cauchy problems with non-Lipschitz coefficients. We underline that the assumed regularity $b \in C^2$ comes from statements of Floquet's theory itself. An attempt to consider non-Lipschitz theory, weakly hyperbolic theory and theory of $L_p - L_q$ decay estimates for solutions of wave equations with a time-dependent coefficient is presented in [23].

5. How to weaken $\mathbf{C^2}$ regularity to keep the classification of oscillations

There arises after the results of [6] and [7] the question whether there is something between the conditions

(21) • $a \in L^\infty[0,T]\cap C^1(0,T]\ ,\ |t^\gamma a'(t)| \leq C \quad \text{for}\ \ t\in(0,T]\ ;$

(22) • $a \in L^\infty[0,T]\cap C^2(0,T]\ ,\ |a^{(k)}(t)| \leq C_k\left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^k$ for $\ t\in(0,T],\ k=1,2.$

The paper [15] is devoted to the model Cauchy problem

$$(23) \qquad u_{tt} - a(t,x) \triangle u = 0 , \quad u(T,x) = \varphi(x), \quad u_t(T,x) = \psi(x) ,$$

where $a = a(t,x) \in L^\infty([0,T], B^\infty(\mathbb{R}^n))$ and $a_0 \le a(t,x)$ with a positive constant a_0.

DEFINITION 3. *Definition of* admissible space of coefficients. *Let T be a positive small constant, and let $\gamma \in [0,1]$ and $\beta \in [1,2]$ be real numbers. We define the weighted spaces of Hölder differentiable functions $\Lambda_\gamma^\beta = \Lambda_\gamma^\beta((0,T])$ in the following way:*

$$\Lambda_\gamma^\beta((0,T]) = \{a = a(t,x) \in L^\infty([0,T], B^k(\mathbb{R}^n)) : \sup_{t\in(0,T]} \|a(t)\|_{B^k(\mathbb{R}^n)}$$

$$+ \sup_{t\in(0,T]} \frac{\|\partial_t a(t)\|_{B^k(\mathbb{R}^n)}}{t^{-1}(\ln t^{-1})^\gamma} + \sup_{t\in(0,T]} \frac{\|\partial_t a\|_{M^{\beta-1}([t,T],B^k(\mathbb{R}^n))}}{(t^{-1}(\ln t^{-1})^\gamma)^\beta} \quad \textit{for all} \ \ k \ge 0\} ,$$

where $\|F\|_{M^{\beta-1}(I)}$ with a closed interval I is defined by

$$\|F\|_{M^{\beta-1}(I)} = \sup_{s_1,s_2\in I, s_1\neq s_2} \frac{|F(s_1) - F(s_2)|}{|s_1 - s_2|^{\beta-1}} .$$

- If a satisfies (21) with $\gamma = 1$, then $a \in \Lambda_0^1$.
- If a satisfies (22) with $\gamma \in [0,1]$, then $a \in \Lambda_\gamma^2$.

DEFINITION 4. Space of solutions. *Let σ and γ be non-negative real numbers. We define the exponential-logarithmic scale $H_{\gamma,\sigma}$ by the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying*

$$\|f\|_{H_{\gamma,\sigma}} := \left(\int_{\mathbb{R}^n} |\exp(\sigma(\ln\langle\xi\rangle)^\gamma)\, \hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty .$$

In particular, we denote $H_\gamma = \bigcup_{\sigma>0} H_{\gamma,\sigma}$.

THEOREM 11. *Let $\gamma \in [0,1]$ and $\beta \in (1,2]$. If $a \in \Lambda_\gamma^\beta((0,T])$, then the Cauchy problem (23) is well-posed in H_γ on $[0,T]$, that is, there exist positive constants $C_{\gamma,\beta}$, σ and σ' with $\sigma \le \sigma'$ such that*

$$\|(\nabla u(t), u_t(t))\|_{H_{\gamma,\sigma}} \le C_{\gamma,\beta} \|(\nabla\varphi, \psi)\|_{H_{\gamma,\sigma'}} \ \ \textit{for all} \ t \in [0,T].$$

REMARK 7. In the Cauchy problem (23) we prescribe data φ and ψ on the hyperplane $t = T$. It is clear from Theorem 4, that a unique solution of the backward Cauchy problem (23) exists for $t \in (0,T]$. The statement of Theorem 11 tells us that in the case of very slow, slow or fast oscillations ($\gamma \in [0,1]$), the solution possesses a continuous extension to $t = 0$.

OPEN PROBLEM 5. Try to prove the next statement:
If $a = a(t,x) \in \Lambda_\gamma^\beta((0,T])$ with $\gamma > 1$ and $\beta \in (1,2)$, then these oscillations are very fast oscillations!

The energy inequality from Theorem 11 yields the same connection between the type of oscillations and the loss of derivatives as Theorem 8.

THEOREM 12. *Let us consider the Cauchy problem (23), where $a \in \Lambda_\gamma^\beta((0,T])$ with $\gamma \in [0,1]$ and $\beta \in (1,2]$. The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:*

$$E(u)(t)\Big|_{H^{s-s_0}} \le C(T)E(u)(0)\Big|_{H^s} \quad \textit{for all } \; t \in [0,T]\,,$$

where

- *$s_0 = 0$ if $\gamma = 0$ (very slow oscillations),*
- *s_0 is an arbitrary small positive constant if $\gamma \in (0,1)$ (slow oscillations),*
- *s_0 is a positive constant if $\gamma = 1$ (fast oscillations).*

Proof of Theorem 11. The proof follows that for Theorem 8. But now the coefficient depends on spatial variables, too. Our main goal is to present modifications to the proof of Theorem 8.

To Step 2. Symbols
To given real numbers $m_1, m_2 \ge 0$, we define $S\{m_1,m_2\}$ and T^{m_1} as follows:

$$\begin{aligned} S\{m_1,m_2\} &= \{a = a(t,x,\xi) \in L^\infty_{loc}((0,T), C^\infty(\mathbb{R}^{2n})) : \\ &\quad |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \le C_{\tau,\eta}\langle\xi\rangle^{m_1-|\eta|}\left(\frac{1}{t}\Big(\ln\frac{1}{t}\Big)^\gamma\right)^{m_2} \;\text{ in }\; Z_{hyp}(N)\}; \\ T^{m_1} &= \{a = a(t,x,\xi) \in L^\infty((0,T), C^\infty(\mathbb{R}^{2n})) : \\ &\quad |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \le C_{\tau,\eta}\langle\xi\rangle^{m_1-|\eta|} \;\text{ in }\; Z_{pd}(N)\}. \end{aligned}$$

Regularization
Our goal is to carry out the first two steps of the diagonalization procedure because only two steps allow us to understand a refined classification of oscillations. But the coefficient $a = a(t,x)$ doesn't belong to C^2 with respect to t. For this reason we introduce a regularization a_ρ of a. Let $\chi = \chi(s) \in B^\infty(\mathbb{R})$ be an even non-negative function having its support on $(-1,1)$. Let this function satisfy $\int \chi(s)ds = 1$. Moreover, let the function $\mu = \mu(r) \in B^\infty[0,\infty)$ satisfy $0 \le \mu(r) \le 1$, $\mu(r) = 1$ for $r \ge 2$ and $\mu(r) = 0$ for $r \le 1$. We define the pseudo-differential operator $a_\rho = a_\rho(t,x,D_x)$ with the symbol

$$a_\rho(t,x,\xi) = \mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\underbrace{b_\rho(t,x,\xi)}_{Z_{hyp}(N)} + \left(1-\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\underbrace{a_0}_{Z_{pd}(N)}\,,$$

where

$$b_\rho(t,x,\xi) = \langle\xi\rangle \underbrace{\int_{\mathbb{R}} a(s,x)\chi((t-s)\langle\xi\rangle)\,ds}_{\text{regularization of } a} .$$

LEMMA 13. *The regularization a_ρ has the following properties:*

- $a_\rho(t,x,\xi) \geq a_0$*;*
- $a_\rho(t,x,\xi) \in S^0_{1,0}$*;*
- $\partial_t a_\rho(t,x,\xi) \in S\{0,1\} \cap T^{-\infty}$*;*
- $\partial_t^2 a_\rho(t,x,\xi) \in S\{-\beta+2,\beta\} \cap T^{-\infty}$*;*
- $a(t,x) - a_\rho(t,x,\xi) \in S\{-\beta,\beta\} \cap T^0$.

To Step 4. Two steps of diagonalization procedure
We start with $u_{tt} - a(t,x) \triangle u = 0$. The vector-valued function $U = (\sqrt{a_\rho}\langle D_x\rangle u, D_t u)^T$ is a solution of the first order system

$$\begin{aligned}
&(D_t - A_0 - B_0 - R_0)U = 0,\\
&A_0 := \begin{pmatrix} 0 & \sqrt{a_\rho}\langle D_x\rangle \\ \sqrt{a_\rho}\langle D_x\rangle & 0 \end{pmatrix},\\
&B_0 := \begin{pmatrix} \mathrm{Op}\left[\frac{D_t a_\rho}{2a_\rho}\right] & 0 \\ (a-a_\rho)\langle D_x\rangle \sqrt{a_\rho}^{\sharp} & 0 \end{pmatrix},
\end{aligned}$$

where $R_0 \in S^0$ uniformly for all $t \in [0,T]$, that is, $R_0 = R_0(t,x,\xi) \in L^\infty([0,T], S^0)$.
First step of diagonalization, diagonalization modulo $L^\infty([0,T],\ S\{0,1\} \cap T^1)$.
Using the same diagonalizer in the form of a constant matrix we obtain from the above system

$$\begin{aligned}
&(D_t - A_1 - B_1 - R_1)U_1 = 0 ,\\
&A_1 := \sqrt{a_\rho}\langle D_x\rangle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
&B_1 \in L^\infty([0,T], S\{0,1\} \cap T^1),\\
&R_1 \in L^\infty([0,T], S^0).
\end{aligned}$$

REMARK 8. We can split B_1 into two parts

$$\begin{aligned}
B_{10} &:= \mathrm{Op}\left[\frac{D_t a_\rho}{4a_\rho}\right] \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\
B_{11} &:= \frac{1}{2}\,(a-a_\rho)\langle D_x\rangle \sqrt{a_\rho}^{\sharp} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\end{aligned}$$

The second part B_{11} belongs to $S\{-\beta+1,\beta\}\cap T^1$ for all $t\in[0,T]$. If $\beta>1$, then this class is better than $S\{0,1\}\cap T^1$. We need $\beta>1$ later, to understand that the influence of B_{11} is not essential. This is the reason we exclude in Theorem 11 the value $\beta=1$.

Second step of diagonalization, diagonalization modulo $L^\infty([0,T],\ S\{-\beta+1,\beta\}\cap T^1)+L^\infty([0,T],S^0)$.

We define the diagonalizer $M_2 = M_2(t,x,D_x) := \begin{pmatrix} I & -p \\ p & I \end{pmatrix}$, where $p = p(t,x,\xi) = \frac{D_t a_\rho}{8a_\rho\sqrt{a_\rho}\langle\xi\rangle}$. Then a suitable transformation $U_2 := M_2 U_1$ changes the above system to

$$\begin{aligned}&(D_t - A_1 - A_2 - B_2 - R_2)U_2 = 0\ ,\\ &A_2 := \mathrm{Op}\left[\frac{D_t a_\rho}{4a_\rho}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\\ &B_2 \in L^\infty([0,T],S\{-\beta+1,\beta\}\cap T^1),\\ &R_2 \in L^\infty([0,T],S^0).\end{aligned}$$

Transformation by an elliptic pseudo-differential operator.

We define $M_3 = M_3(t,x,\xi) := \exp\left(-\int_t^T \frac{D_s a_\rho}{4a_\rho}\,ds\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The transformation $U_2 := M_3 U_3$ gives from the last system $(D_t - A_1 - B_3 - R_3)U_3 = 0$, where $B_3,\ R_3$ belong to the same symbol classes as $B_2,\ R_2$, respectively.

REMARK 9. The last step corresponds to the fact from the proof of Theorem 8, that $\int_r^t \frac{\partial_s a(s)}{4a(s)}\,ds$ depends only on a.

Application of sharp Gårding's inequality for matrix-valued operators.

We generalize an idea from [2] to our model problem.

GOAL. Let us find a pseudo-differential operator $\theta=\theta(t,D_x)$ in such a way that after transformation $V(t,x) := e^{-\int_t^T \theta(s,D_x)ds}U_3(t,x)$ the operator equation $(D_t - A_1 - B_3 - R_3)U_3 = 0$ is transformed to $(\partial_t - P_0 - P_1)V = 0$, where we can show that for the solution V of the Cauchy problem an energy estimate without loss of derivatives holds.

A simple computation leads to

$$\begin{aligned} P_0 + P_1 &= i(A_1 + B_3 + R_3) + \theta(t,D_x)I \\ &+ i\left[e^{-\int_t^T \theta(s,D_x)ds}, A + B + R\right] e^{\int_t^T \theta(s,D_x)ds}. \end{aligned}$$

The matrix-valued operator A_1 brings no loss of derivatives, here we feel the strict hyperbolicity. Taking account of the symbol classes for B_3, R_3 and our strategy due to Gårding's inequality that $\theta=\theta(t,\xi)$ should majorize $i(B_3(t,x,\xi)+R_3(t,x,\xi))$ the symbol of θ should consist at least of two parts:

- a positive constant K, due to $R_3 \in L^\infty([0,T], S^0)$;

- $K\,\theta_0(t,\xi) := K\,\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\frac{1}{\langle\xi\rangle^{\beta-1}}\left(\frac{1}{t}\left(\ln\frac{1}{t}\right)^\gamma\right)^\beta$
$+K\left(1-\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\langle\xi\rangle$, due to $B_3 \in L^\infty([0,T], S\{-\beta+1,\beta\}\cap T^1)$.

It turns out that the symbol of the commutator doesn't belong to one of these symbol classes. For this reason we introduce a third part

- $K\,\theta_1(t,\xi) := K\,\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\left(\ln\frac{1}{t}\right)^\gamma + K\left(1-\mu\left(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\right)\right)\left(\ln\frac{1}{t_\xi}\right)^\gamma.$

Defining

- $P_0 = i(A_1+B_3+R_3) + K(1+\theta_0(t,D_x))I,$

- $P_1 = K\,\theta_1(t,D_x)I + i\left[e^{-\int_t^T \theta(s,D_x)ds}, A_1+B_3+R_3\right]e^{\int_t^T \theta(s,D_x)ds}$

one can show

$$\det\left(\frac{P_0+P_0^*}{2}\right)(t,x,\xi) \geq \theta_0(t,\xi) \in L^\infty([0,T], S^1_{1,0})\ ,$$
$$\det\left(\frac{P_1+P_1^*}{2}\right)(t,x,\xi) \geq \theta_1(t,\xi) \in L^\infty([0,T], S^\varepsilon_{\varepsilon,0})\ .$$

We use the sharp Gårding's inequality with (see [19]) with

- $c_0 = 0\ ,\quad m = 1\ ,\quad \rho = 1\ ,\quad \delta = 0\quad$ for P_0 ,

- $c_0 = 0\ ,\quad m = \varepsilon\ ,\quad \rho = \varepsilon\ ,\quad \delta = 0\quad$ for P_1 ,

thus $\mathrm{Re}(P_k u, u) \geq -C_k\ \|u\|^2_{L_2}$ for $k = 1, 2$. These are the main inequalities for proving the energy estimate

$$\|V(t,\cdot)\|^2_{L_2} \leq e^{CT}\|V(T,\cdot)\|^2_{L_2} \qquad \text{for } t\in[0,T]\ .$$

It remains to estimate $\int_0^T \theta(s,\xi)ds$. This is more or less an exercise. A careful calculation brings $\int_0^T \theta(s,\xi)ds \leq C\ (\ln\langle\xi\rangle)^\gamma$. The statements of Theorem 11 are proved.

□

6. Construction of parametrix

In this section we come back to our general Cauchy problem (1) taking account of the classification of oscillations supposed in Definition 2 and (5). We assume

$$a_{kl} \in C([0,T],\mathcal{B}^\infty(\mathbb{R}^n)) \cap C^\infty((0,T],\mathcal{B}^\infty(\mathbb{R}^n)). \tag{24}$$

The non-Lipschitz behaviour of coefficients is characterized by

$$|D_t^k D_x^\beta a_{kl}(t,x)| \le C_{k,\beta}\left(\frac{1}{t}\left(\ln \frac{1}{t}\right)^\gamma\right)^k \tag{25}$$

for all k, β and $(t,x) \in (0,T] \times \mathbb{R}^n$, where T is sufficiently small and $\gamma \ge 0$. The transformation $U = (\langle D_x\rangle u, D_t u)^T$ transfers our starting Cauchy problem (1) to a Cauchy problem for $D_t U - AU = F$, where $A = A(t,x,D_x)$ is a matrix-valued pseudo-differential operator. The goal of this section is the construction of parametrix to $D_t - A$.

DEFINITION 5. *An operator $E = E(t,s)$, $0 \le s \le t \le T_0$, is said to be a parametrix to the operator $D_t - A$ if $D_t E - AE \in L^\infty([0,T_0]^2, \Psi^{-\infty}(\mathbb{R}^n))$. Here $\Psi^{-\infty}$ denotes the space of pseudo-differential operators with symbols from $S^{-\infty}$ (see [19]).*

We will prove that *E is a matrix Fourier integral operator*. The considerations of this section are based on [17], where the case $\gamma = 1$ was studied, and on [23]. We will sketch this construction of the parametrix and show how the different loss of derivatives appears. It is more or less standard to get from the parametrix to the existence of C^1 solutions in t of (1) with values in Sobolev spaces.

Step 1. Tools

With the function $t = t_\xi$ from the proof of Theorem 8 we define for $\langle\xi\rangle \ge M$ the *pseudo-differential zone* $Z_{pd}(N)$, *hyperbolic zone* $Z_{hyp}(N)$, respectively, by

$$Z_{pd}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \le t_\xi\}\,, \tag{26}$$

$$Z_{hyp}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \ge t_\xi\}. \tag{27}$$

Moreover, we divide $Z_{hyp}(N)$ into the so-called *oscillations subzone* $Z_{osc}(N)$ and the *regular subzone* $Z_{reg}(N)$. These subzones are defined by

$$Z_{osc}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t_\xi \le t \le \tilde{t}_\xi\}\,, \tag{28}$$

$$Z_{reg}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ \tilde{t}_\xi \le t\}\,, \tag{29}$$

where $t = \tilde{t}_\xi$ solves

$$\tilde{t}_\xi \langle\xi\rangle = 2N(\ln\langle\xi\rangle)^{2\gamma}. \tag{30}$$

In each of these zones we define its own class of symbols.

DEFINITION 6. *By T_{2N} we denote the class of all amplitudes $a = a(t,x,\xi) \in L^\infty([0,T], C^\infty(\mathbb{R}^{2n}))$ satisfying for $(t,x,\xi) \in Z_{pd}(2N)$ and all α, β the estimates*

$$\operatorname{ess\,sup}_{(t,x)\in[0,t_\xi]\times\mathbb{R}^n} |\partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \le C_{\beta\alpha} \langle\xi\rangle^{1-|\alpha|}. \tag{31}$$

By $S^m_{\rho,\delta}(\mathbb{R}^n)$ we will denote the usual symbol spaces (see [19]).
To describe the behaviour in oscillations subzone $Z_{osc}(N)$ we need the following class of symbols.

DEFINITION 7. *By $S_N\{m_1, m_2\},\ m_2 \ge 0$, we denote the class of all amplitudes $a = a(t,x,\xi) \in C^\infty((0,T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \le C_{k\beta\alpha} \langle\xi\rangle^{m_1-|\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2+k} \tag{32}$$

for all k, α, β and $(t,x,\xi) \in Z_{hyp}(N)$.

Finally, we use symbols describing the behaviour of the solution in the regular part $Z_{reg}(N)$ of $Z_{hyp}(N)$.

DEFINITION 8. *By $S_N^\star\{m_1, m_2\},\ m_2 \ge 0$, we denote the class of all amplitudes $a = a(t,x,\xi) \in C^\infty((0,T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \le C_{k\beta\alpha} \langle\xi\rangle^{m_1-|\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2+k} \tag{33}$$

for all k, α, β and $(t,x,\xi) \in Z_{reg}(N)$.

To all these symbol classes one can define corresponding pseudo-differential operators. To get a calculus for these symbol classes it is useful to know that under assumptions on the behaviour of the symbols in $Z_{pd}(N)$ we have relations to classical parameter-dependent symbol classes.

LEMMA 14. *Assume that the symbol $a \in S_N\{m_1, m_2\}$ is constant in $Z_{pd}(N)$. Then*

$$a \in L^\infty([0,T],\ S_{1,0}^{\max(0,m_1+m_2)}(\mathbb{R}^n)),\ \partial_t^k a \in L^\infty([0,T],\ S_{1,0}^{m_1+m_2+k}(\mathbb{R}^n)) \tag{34}$$

for all $k \ge 1$.

The statements (34) allow us to apply the standard rules of classical symbolic calculus. One can show

a hierarchy of symbol classes $S_N\{m_{1,k}, m_2\}$ for $m_{1,k} \to -\infty$.

LEMMA 15. *Assume that the symbols $a_k \in S_N\{m_{1,k}, m_2\},\ k \ge 0$, vanish in $Z_{pd}(N)$ and that $m_{1,k} \to -\infty$ as $k \to \infty$. Then there is a symbol $a \in S_N\{m_{1,0}, m_2\}$ with support in $Z_{hyp}(N)$ such that*

$$a - \sum_{l=0}^{k-1} a_l \in S_N\{m_{1,k}, m_2\} \qquad \text{for all } k \ge 1.$$

The symbol is uniquely determined modulo $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

a hierarchy of symbol classes $S_N\{m_1-k, m_2+k\}$ *for* $k \geq 0$.

LEMMA 16. *Assume that the symbols* $a_k \in S_N\{m_1-k, m_2+k\}$, $k \geq 0$, *vanish in* $Z_{pd}(N)$. *Then there is a symbol* $a \in S_N\{m_1, m_2\}$ *with support in* $Z_{hyp}(N)$ *such that*

$$a - \sum_{l=0}^{k-1} a_l \in S_N\{m_1-k, m_2+k\} \quad \text{for all} \quad k \geq 1 \, .$$

The symbol is uniquely determined modulo $\bigcap_{l \geq 0} S_N\{m_1 - l, m_2 + l\}$.

Asymptotic representations of symbols vanishing in $Z_{pd}(N)$ *by using these hierarchies.*

A composition formula of pseudo-differential operators whose symbols are constant in $Z_{pd}(N)$.

LEMMA 17. *Let A and B be pseudo-differential operators with symbols* $a := \sigma(A)$ *and* $b := \sigma(B)$ *from* $S_N\{m_1, m_2\}$ *and* $S_N\{k_1, k_2\}$, *where we use the representations*

$$A(t,x,D_x)u = \tfrac{1}{(2\pi)^n} \text{Os-} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\xi} a(t,x,\xi) u(x+y) d\xi dy \, ;$$
$$B(t,x,D_x)u = \tfrac{1}{(2\pi)^n} \text{Os-} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy\xi} b(t,x,\xi) u(x+y) d\xi dy .$$

Let us suppose that both symbols a and b are constant in $Z_{pd}(N)$. *Then the operator* $A \circ B$ *has a symbol* $c = c(x,t,\xi)$ *which belongs to* $S_N\{m_1+k_1, m_2+k_2\}$ *and satisfies*

$$c(t,x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha a(t,x,\xi) \partial_x^\alpha b(t,x,\xi)$$

modulo a regularizing symbol from $C^\infty([0,T], S^{-\infty})$.

The existence of parametrix to elliptic matrix pseudo-differential operators belonging to $S_N\{0,0\}$ *and which are constant in* $Z_{pd}(N)$.

LEMMA 18. *Assume that the symbol* $a := \sigma(A)$ *of the matrix pseudo-differential operator A belongs to* $S_N\{0,0\}$ *and is a constant matrix in* $Z_{pd}(N)$. *If A is elliptic, this means* $|\det a(t,x,\xi)| \geq C > 0$ *for all* $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}$, *then there exists a parametrix* $A^\sharp$, *where* $a^\sharp := \sigma(A^\sharp) \in S_N\{0,0\}$ *is a constant matrix in* $Z_{pd}(N)$, *too.*

Proof. We set $a_0^\sharp(t,x,\xi) := a(t,x,\xi)^{-1}$. The symbol $a_0^\sharp$ belongs to $S_N\{0,0\}$. Using Lemma 14 we can recursively define symbols $a_k^\sharp$ by

$$\sum_{|\alpha|=1}^{k} \tfrac{1}{\alpha!} \Big(D_\xi^\alpha a(t,x,\xi) \Big) \Big(\partial_x^\alpha a_{k-|\alpha|}^\sharp(t,x,\xi) \Big) =: -a(t,x,\xi) a_k^\sharp(t,x,\xi).$$

It is clear that $a_k^\sharp(t,x,\xi) \equiv 0$ in $Z_{pd}(N)$ and $a_k^\sharp \in S_N\{-k,0\}$.
The application of Lemma 15 gives a symbol $a_R^\sharp \in S_N\{0,0\}$ and a right parametrix $A_R^\sharp$ with symbol $\sigma(A_R^\sharp) =: a_R^\sharp$ and

$$a_R^\sharp - \sum_{l=0}^{k-1} a_l^\sharp \in S_N\{-k,0\},$$
$$a_R^\sharp(t,x,\xi) = a_0^\sharp(t,x,\xi) \text{ in } Z_{pd}(N),$$
$$AA_R^\sharp - I \in C^\infty([0,T],\Psi^{-\infty}),$$

where I denotes the identity operator. In the same way we can show the existence of a left parametrix $A_L^\sharp$ with $A_L^\sharp A - I \in C^\infty([0,T],\Psi^{-\infty})$. As usual one can show that $A_L^\sharp$ and $A_R^\sharp$ coincide modulo $C^\infty([0,T],\Psi^{-\infty})$. This gives the existence of a parametrix with symbol belonging to $S_N\{0,0\}$. It is uniquely determined modulo $C^\infty([0,T],\Psi^{-\infty})$.

□

Step 2. Diagonalization procedure
We have to carry out perfect diagonalization. The main problem is to understand what the perfect diagonalization procedure means. Here we follow the following strategy:

- The first step of perfect diafonalization we carry out in all zones.
- The second step of perfect diagonalization we only carry out in $Z_{hyp}(N)$.
- The perfect diagonalization we only carry out in $Z_{reg}(N)$.

Perfect diagonalization means diagonalization modulo $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{r\geq 0} S^\star_{2N}\{-r, r+1\} \Big\}$.

Let us explain these steps more in detail. We start with

$$Lu := D_t^2 u - \sum_{k,l=1}^{n} a_{kl}(t,x) D^2_{x_k x_l} u = g,$$
$$u(0,x) = \varphi(x),$$
$$D_t u(0,x) = -i\psi(x),$$

where $g := -f$ from (1). The transformation $U = (U_1,U_2)^T = \Big(\langle D_x\rangle u, D_t u\Big)^T$ transfers this Cauchy problem to

$$D_t U - AU = G\ ,\ U(0,x) = \begin{pmatrix} \langle D_x\rangle \varphi(x) \\ -i\psi(x) \end{pmatrix}, \tag{35}$$

where

$$A := \begin{pmatrix} 0 & \langle D_x\rangle \\ \sum\limits_{k,l=1}^{n} a_{kl}(t,x) D^2_{x_k x_l} \langle D_x\rangle^{-1} & 0 \end{pmatrix}\ ,\ G := \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

LEMMA 19. *Symbol $\sigma(A)$ belongs to $T_{2N} \cap S_N\{1,0\}$.*

Now we care for the main step of diagonalization, this means, for the step which transforms A to a *diagonal matrix pseudo-differential operator* modulo an operator with symbol from $T_{2N} \cap S_N\{0,1\}$. Therefore we define the pseudo-differential operators of first order $\Phi_k = \Phi_k(t,x,D_x),\ k = 1,2$, having symbols

$$\varphi_k(t,x,\xi) = d_k\langle\xi\rangle\ \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big) + \tau_k(t,x,\xi)\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big)\,. \tag{36}$$

Here $d_2 = -d_1$ is a positive constant and

$$\tau_k(t,x,\xi) = (-1)^k\sqrt{a(t,x,\xi)}\ ,\ a(t,x,\xi) := \sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l. \tag{37}$$

The function $\chi = \chi(s)$ is from $C_0^\infty(\mathbb{R}),\ \chi(s) \equiv 1$ for $|s| \le 1,\ \chi(s) \equiv 0$ for $|s| \ge 2$ and $0 \le \chi(s) \le 1$.

LEMMA 20. *a) The non-vanishing symbols $\varphi_k = \varphi_k(t,x,\xi),\ k = 1,2$, belong to $T_{2N} \cap S_N\{1,0\}$.*

b) The special choice of $d_k,\ k = 1,2$, yields $\varphi_2 - \varphi_1 = 2\varphi_2$.

To start the diagonalization procedure we define the matrix pseudo-differential operator ($h(D_x) = \langle D_x\rangle$)

$$M(t,x,D_x) = \begin{pmatrix} I & I \\ \Phi_1(t,x,D_x)h^{-1}(D_x) & \Phi_2(t,x,D_x)h^{-1}(D_x) \end{pmatrix}.$$

Due to Lemma 18 we have the existence of $M^\sharp$. This follows from the analysis of

$$\sigma(M) = \begin{pmatrix} 1 & 1 \\ \frac{\varphi_1(t,x,\xi)}{h(\xi)} & \frac{\varphi_2(t,x,\xi)}{h(\xi)} \end{pmatrix},$$

that by (36) and (37) the symbol $\sigma(M)$ is a constant matrix in $Z_{pd}(N)$, $\det\sigma(M) = \frac{2\varphi_2(t,x,\xi)}{\langle\xi\rangle} \ge C > 0$ for $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n}$. Hence, M is elliptic with a symbol belonging to $S_N\{0,0\}$. The parametrix $M^\sharp$ belongs to $S_N\{0,0\}$, too. We will later apply Duhamel's principle to find a representation of the solution to (35). Therefore we devote to find a fundamental solution to (35), this is a solution $E = E(t,s)$ satisfying

$$D_tE - AE = 0\ ,\ E(s,s) = I. \tag{38}$$

Setting $E_0 = M^\sharp E$ leads to

$$\begin{aligned} D_tE_0 &= M^\sharp D_tE + D_tM^\sharp E = M^\sharp AE + D_tM^\sharp E \\ &= M^\sharp AME_0 + D_tM^\sharp ME_0 + R_\infty E, \end{aligned}$$

where $R_\infty \in C^\infty([0,T],\Psi^{-\infty})$. The symbols $\sigma(M^\sharp)$, $\sigma(M)$ are constant in $Z_{pd}(N)$. Consequently,

$$\sigma(M^\sharp AM) = \sigma(M^\sharp)\sigma(A)\sigma(M) + f_0(t,x,\xi) + r_\infty(t,x,\xi),$$

where

$$f_0(t,x,\xi) = \begin{cases} 0 \quad \text{in} \quad Z_{pd}(N) \\ \in T_{2N} \cap S_N\{0,0\}\,, \end{cases} \tag{39}$$

and $r_\infty \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$. Straightforward calculations yield

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{cases} d(t,x,\xi) \quad \text{in} \quad Z_{hyp}(2N) \\ \in T_{2N}, \end{cases} \tag{40}$$

where

$$d(t,x,\xi) = \begin{pmatrix} \tau_1(t,x,\xi) & 0 \\ 0 & \tau_2(t,x,\xi) \end{pmatrix}$$

and

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{pmatrix} \frac{\tau_1^2+\varphi_1^2}{2\varphi_1} & \frac{\varphi_2^2-\tau_2^2}{2\varphi_2} \\ \frac{\varphi_1^2-\tau_1^2}{2\varphi_1} & \frac{\tau_2^2+\varphi_2^2}{2\varphi_2} \end{pmatrix}(t,x,\xi) \quad \text{in} \quad Z_{pd}(2N).$$

Consequently, the following identity holds in $Z_{pd}(2N)$:

$$\sigma(M^\sharp)\sigma(A)\sigma(M) = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \sigma(Q),$$

where the symbol $\sigma(Q) \in T_{2N} \cap S_N\{1,0\}$ and $\sigma(Q) \equiv 0$ in $Z_{hyp}(2N)$. Finally, let us devote to $D_t M^\sharp M = -M^\sharp D_t M + R_\infty$. We have

$$\sigma(M^\sharp D_t M) = \sigma(M^\sharp)\sigma(D_t M) + f_0(t,x,\xi) + r_\infty(t,x,\xi)\ ,$$

where

$$f_0(t,x,\xi) = \begin{cases} 0 \quad \text{in} \quad Z_{pd}(N) \\ \in T_{2N} \cap S_N\{-1,1\}, \end{cases} \tag{41}$$

and $r_\infty \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$. Using

$$\begin{aligned} \sigma(M^\sharp)\sigma(D_t M) &= \left(\tfrac{\varphi_2-\varphi_1}{h} \right)^{-1} \begin{pmatrix} \frac{\varphi_2}{h} & -1 \\ -\frac{\varphi_1}{h} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D_t\frac{\varphi_1}{h} & D_t\frac{\varphi_2}{h} \end{pmatrix} \\ &= \left(\tfrac{\varphi_2-\varphi_1}{h} \right)^{-1} \begin{pmatrix} -D_t\frac{\varphi_1}{h} & -D_t\frac{\varphi_2}{h} \\ D_t\frac{\varphi_1}{h} & D_t\frac{\varphi_2}{h} \end{pmatrix} \end{aligned}$$

and (38) to (41) we arrive at the next result. In the formulation of this result we use due to the influence of Q some symbols in $Z_{hyp}(2N)$ and take into consideration that symbols from $S_N\{1,0\}$ supported in the transition zone $Z_{pd}(2N) \setminus Z_{pd}(N)$ belong to T_{2N}.

LEMMA 21. *The fundamental solution* $E = E(t,s)$ *solving (38) can be represented in the form* $E(t,s) = M(t)E_0(t,s)M^\sharp(s)$, *where* M *is an elliptic operator with symbol* $\sigma(M) \in S_N\{0,0\}$ *and* $E_0 = E_0(t,s)$ *solves*

$$D_tE_0 - \mathcal{D}E_0 + P_1E_0 + P_2E_0 + QE_0 + R_\infty E = 0. \tag{42}$$

The matrix pseudo-differential operators $\mathcal{D}$, P_1, P_2, Q, R_∞ *possess the following properties:*

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$,
$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2}D_t\frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2}D_t\frac{\varphi_2}{h} \end{pmatrix};$$
- P_1: *diagonal,* $\sigma(P_1) \in T_{2N} \cap S_N\{0,0\}$, $\sigma(P_1) \equiv 0$ *in* $Z_{pd}(N)$;
- P_2: *antidiagonal,* $\sigma(P_2) \in T_{2N} \cap S_N\{0,1\}$, $\sigma(P_2) \equiv 0$ *in* $Z_{pd}(N)$;
- Q: $\sigma(Q) \in T_{2N}$, $\sigma(Q) \equiv 0$ *in* $Z_{hyp}(2N)$;
- R_∞: $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

This finishes the first step of perfect diagonalization, this step yields a diagonalization modulo $T_{2N} \cap S_{2N}\{0,1\}$.

In the next step of perfect diagonalization our goal consists in the *diagonalization of the antidiagonal matrix operator* P_2 with symbol $\sigma(P_2)$ modulo $S_{2N}\{-1,2\}$. In the hierarchy of symbols described in Lemma 16 the corresponding pseudo-differential operator has a better smoothing property than pseudo-differential operators with symbols from $S_{2N}\{0,1\}$. We restrict ourselves to

$$D_tE_0 - \mathcal{D}E_0 + P_1E_0 + P_2E_0 + QE_0 = 0. \tag{43}$$

LEMMA 22. *There exist an elliptic pseudo-differential operator* N_1 *with* $\sigma(N_1) \in S_N\{0,0\}$,
$\sigma(N_1) \equiv I$ *in* $Z_{pd}(N)$, *and pseudo-differential operators* F_1 *of diagonal structure and* P_3 *with* $\sigma(F_1) \in T_{2N} \cap S_N\{0,0\}$, $\sigma(F_1) \equiv 0$ *in* $Z_{pd}(N)$, *and* $\sigma(P_3) \in T_{2N} \cap S_{2N}\{-1,2\}$ *such that*

$$(D_t - \mathcal{D} + P_1 + P_2 + Q)N_1 = N_1(D_t - \mathcal{D} + F_1 + P_3) \tag{44}$$

holds modulo an regularizing operator R_∞ *with symbol* $\sigma(R_\infty)$ *belonging to* $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

Proof. We localize our considerations to $Z_{hyp}(N)$ by using the pseudo-differential operator $I - \chi(t, D_x)$ with symbol $I\Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big)$. We define F_1 with the symbol $\sigma(F_1)(t,x,\xi) = \Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big)\sigma(P_1)(t,x,\xi)$, which belongs to $T_{2N} \cap S_N\{0,0\}$.

Moreover, we introduce

$$n_1^{(1)}(t,x,\xi) := \begin{pmatrix} 0 & \frac{p_{12}}{\varphi_1-\varphi_2} \\ \frac{p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^{\gamma}}\Big)\Big) \in T_{2N}\cap S_N\{-1,1\},$$

where

$$\sigma(P_2) = \begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix} \in T_{2N}\cap S_N\{0,1\}.$$

Setting $N_1 = I + N_1^{(1)}$, $\sigma(N_1^{(1)}) = n_1^{(1)}$, we are able to conclude that the symbol $\sigma(B^{(1)})$ of

$$\begin{aligned} B^{(1)} &:= (D_t - \mathcal{D} + P_1 + P_2 + Q)(I + N_1^{(1)}) - (I + N_1^{(1)})(D_t - \mathcal{D} + F_1) \\ &= P_1 + P_2 + Q - [\mathcal{D}, N_1^{(1)}] - F_1 + D_t N_1^{(1)} + (P_1 + P_2 + Q)N_1^{(1)} \\ &\quad - N_1^{(1)} F_1 \end{aligned}$$

belongs to $T_{2N}\cap S_{2N}\{-1,2\}$. This follows from

- $\sigma(D_t N_1^{(1)}) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma(D_t N_1^{(1)}) \equiv 0$ in $Z_{pd}(N)$;
- $\sigma((P_1+P_2)N_1^{(1)} - N_1^{(1)}F_1) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma((P_1+P_2)N_1^{(1)} - N_1^{(1)}F_1) \equiv 0$ in $Z_{pd}(N)$;
- $\sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}]) \in T_{2N}\cap S_N\{-1,2\}$, $\sigma((1-\chi)P_2 - [D, N_1^{(1)}]) \equiv 0$ in $Z_{pd}(N)$.

The last relation is a conclusion from

$$\begin{aligned} \sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}]) &= \begin{pmatrix} 0 & (1-\chi)p_{12} \\ (1-\chi)p_{21} & 0 \end{pmatrix} \\ &- \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{(1-\chi)p_{12}}{\varphi_1-\varphi_2} \\ \frac{(1-\chi)p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \frac{(1-\chi)p_{12}}{\varphi_1-\varphi_2} \\ \frac{(1-\chi)p_{21}}{\varphi_2-\varphi_1} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2} D_t \frac{\varphi_2}{h} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mod\ S_N\{-1,2\}. \end{aligned}$$

The symbol $\sigma((1-\chi)P_2 - [\mathcal{D}, N_1^{(1)}])$ vanishes in $Z_{pd}(N)$ because of $\sigma(P_2) = \sigma(N_1^{(1)}) \equiv 0$ and belongs to T_{2N}. The remainder $R_1 := (P_1 + P_2)\chi + QN_1$ belongs to T_{2N} and vanishes in $Z_{hyp}(2N)$. Summarizing these observations we see that $B^{(1)} = \tilde{B}^{(1)} + R_1$, where $\sigma(\tilde{B}^{(1)}) \in T_{2N}\cap S_N\{-1,2\}$, $\equiv 0$ in $Z_{pd}(N)$ and $\sigma(R_1) \in T_{2N}$, $\equiv 0$ in $Z_{hyp}(2N)$. Now let us show that a sufficiently large N in (30) guarantees that N_1 is an elliptic pseudo-differential operator with symbol belonging to $S_N\{0,0\}$. Due to our construction $\sigma(N_1) \equiv I$ in $Z_{pd}(N)$. We know that

$$|n_1^{(1)}(x,t,\xi)| \le \frac{C}{\langle\xi\rangle}\frac{1}{t}(\ln\frac{1}{t})^\gamma \le \frac{C_1}{N} \ \text{ in } \ Z_{hyp}(N).$$

Consequently, a large N yields $|\sigma(N_1)| \ge 1/2$ in $[0,T]\times\mathbb{R}^{2n}$. Using $\sigma(N_1) = I$ in $Z_{pd}(N)$ gives together with Lemma 18 the existence of $N_1^\sharp$ with $\sigma(N_1^\sharp) \in S_N\{0,0\}$. It is clear that the symbol of

$$P_3 := N_1^\sharp B^{(1)} = N_1^\sharp(\tilde{B}^{(1)} + R_1) \tag{45}$$

belongs to $T_{2N}\cap S_{2N}\{-1,2\}$ modulo a regularizing operator R_∞ with symbol $\sigma(R_\infty)$ belonging to $C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

□

This finishes the second step of perfect diagonalization, this step yields a diagonalization modulo $T_{2N}\cap S_{2N}\{-1,2\}$.

Summarizing we have proved the next result.

LEMMA 23. *The fundamental solution $E_0 = E_0(t,s)$ solving (43) can be represented in the form $E_0(t,s) = N_1(t)E_1(t,s)N_1^\sharp(s)$, where $N_1^\sharp$ and N_1 are elliptic pseudo-differential operators with symbols $\sigma(N_1^\sharp),\ \sigma(N_1) \in S_N\{0,0\}$, both symbols are constant in $Z_{pd}(N)$. The matrix operator $E_1 = E_1(t,s)$ solves*

$$D_tE_1 - \mathcal{D}E_1 + F_1E_1 + P_3E_1 + R_\infty E_1 = 0,$$

where the matrix pseudo-differential operators $\mathcal{D},\ F_1,\ P_3,\ R_\infty$ possess the following properties:

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N}\cap S_N\{1,0\}$, $\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle} & 0 \\ 0 & \varphi_2 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle}\end{pmatrix}$;
- F_1: *diagonal,* $\sigma(F_1) \in T_{2N}\cap S_N\{0,0\}$, $\sigma(F_1) \equiv 0$ in $Z_{pd}(N)$;
- P_3: $\sigma(P_3) \in T_{2N}\cap S_{2N}\{-1,2\}$;
- R_∞: $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

Now let us sketch the perfect diagonalization.

REMARK 10. Let us explain our philosophy to carry out further steps of perfect diagonalization. We will localize further steps of diagonalization to $Z_{reg}(N)$. In this part of $Z_{hyp}(N)$ we get the improvement of smoothness of the remainder P_{p+2}. This improvement of smoothness can be understood after calculating for $\gamma \in (0,1]$

$$\begin{aligned}&\int_{\tilde{t}_\xi}^t \Big|\sigma(P_{p+2})(\tau,x,\xi)\Big|d\tau \\ &\le \int_{\tilde{t}_\xi}^t \frac{C_p}{\langle\xi\rangle^p}\Big(\frac{1}{\tau}\Big(\ln\frac{1}{\tau}\Big)^\gamma\Big)^{p+1}d\tau \le \frac{C_p(\ln\langle\xi\rangle)^{\gamma(p+1)}}{(\langle\xi\rangle\tilde{t}_\xi)^p} = \frac{C_p}{(2N)^p}(\ln\langle\xi\rangle)^{\gamma(p+1)-2\gamma p},\end{aligned}$$

where $\tilde{t}_\xi$ is defined as in formula (30). In the oscillations subzone we use for the construction of parametrix a behaviour of the symbol of remainder like $S_{2N}\{0,0\}+S_{2N}\{-1,2\}$. It turns out that the perfect diagonalization means diagonalization modulo operators with symbols from $T_{2N}\cap(S_{2N}\{0,0\}+S_{2N}\{-1,2\})\cap\Big\{\bigcap\limits_{p\ge 0}S^{\star}_{2N}\{-p,p+1\}\Big\}$.

LEMMA 24. *There exist a matrix elliptic operator N_2 with $\sigma(N_2)\in S_N\{0,0\}$, $\sigma(N_2)\equiv I$ in $Z_{pd}(N)\cup Z_{osc}(N)$, a diagonal matrix pseudo-differential operator F_2 with $\sigma(F_2)\in(S^{\star}_N\{0,0\}+S^{\star}_N\{-1,2\})$, $\sigma(F_2)\equiv 0$ in $Z_{pd}(N)\cup Z_{osc}(N)$, and a matrix pseudo-differential operator P_∞ with $\sigma(P_\infty)(t,x,\xi)\in T_{2N}\cap(S_{2N}\{0,0\}+S_{2N}\{-1,2\})\cap\Big\{\bigcap\limits_{p\ge 0}S^{\star}_{2N}\{-p,p+1\}\Big\}$ such that*

$$(D_t-\mathcal{D}+F_1+P_3)N_2=N_2(D_t-\mathcal{D}+F_2+P_\infty). \tag{46}$$

This identity holds modulo a regularizing operator R_∞ with symbol $\sigma(R_\infty)$ belonging to $C^\infty([0,T],S^{-\infty}(\mathbb{R}^n))$.

Proof. We choose the representation $N_2\sim I+\sum\limits_{r\ge 1}N_2^{(r)}$ and $F_2\sim\sum\limits_{r\ge 0}F_2^{(r)}$. Our goal is to show the relation

$$(D_t-\mathcal{D}+F_1+P_3)(I+\sum_{r\ge 1}N_2^{(r)})\sim(I+\sum_{r\ge 1}N_2^{(r)})(D_t-\mathcal{D}+\sum_{r\ge 0}F_2^{(r)}+P_\infty).$$

For further constructions we use $P_3=P_{3,1}+P_{3,2}$, where $P_{3,1}$ denotes the *diagonal part* of P_3 and $P_{3,2}$ denotes the *antidiagonal part*.
We localize our considerations to $Z_{reg}(N)$ by using the pseudo-differential operator $I-\chi I$ with symbol $I\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big)$. We define $F_2^{(0)}$ with the symbol $\sigma(F_2^{(0)})(t,x,\xi)=\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big)\sigma(F_1+P_{3,1})(t,x,\xi)$, which belongs to $S^{\star}_N\{0,0\}+S^{\star}_N\{-1,2\}$. Moreover, we introduce

$$n_2^{(1)}(t,x,\xi):=\begin{pmatrix}0 & \frac{p_{13}}{\varphi_1-\varphi_2}\\ \frac{p_{31}}{\varphi_2-\varphi_1} & 0\end{pmatrix}\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)\Big)\in S^{\star}_N\{-2,2\},$$

where

$$\sigma(P_{3,2})=\begin{pmatrix}0 & p_{13}\\ p_{31} & 0\end{pmatrix}\in T_{2N}\cap S_{2N}\{-1,2\}.$$

Setting $N_2=I+N_2^{(1)}$, $\sigma(N_2^{(1)})=n_2^{(1)}$, we get similar as in the proof of Lemma 22 that the symbol $\sigma(B^{(1)})$ of

$$B^{(1)}:\ =\ (D_t-\mathcal{D}+F_1+P_3)(I+N_2^{(1)})-(I+N_2^{(1)})(D_t-\mathcal{D}+F_2^{(0)})$$

belongs to $T_{2N}\cap(S_{2N}\{0,0\}+S_{2N}\{-1,2\})\cap S^{\star}_{2N}\{-2,3\}$. Moreover, we can show that $B^{(1)}=\tilde{B}^{(1)}+R_1$, where $\sigma(\tilde{B}^{(1)})\in S^{\star}_N\{-2,3\}$, $\equiv 0$ in $Z_{pd}(N)\cup Z_{osc}(N)$ and

$\sigma(R_1) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\})$, $\equiv 0$ in $Z_{reg}(2N)$. Now we are able to start an induction procedure. Let us suppose that $\tilde{B}^{(r)}$ is already constructed and its symbol $\sigma(\tilde{B}^{(r)}) \in S_N^\star\{-(r+1), r+2\}$, $\equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. Then $F_2^{(r)} := \tilde{B}_1^{(r)}$ has the same properties, where $\tilde{B}_1^{(r)}$, $\tilde{B}_2^{(r)}$ denote the diagonal part and antidiagonal part of $\tilde{B}^{(r)}$, respectively. We introduce

$$n_2^{(r+1)}(t,x,\xi) := \begin{pmatrix} 0 & \frac{p_{1(r+3)}}{\varphi_1-\varphi_2} \\ \frac{p_{(r+3)1}}{\varphi_2-\varphi_1} & 0 \end{pmatrix}$$

as the symbol of $N^{(r+1)}$, where

$$\sigma(\tilde{B}_2^{(r)}) = \begin{pmatrix} 0 & p_{1(r+3)} \\ p_{(r+3)1} & 0 \end{pmatrix} \in S_N^\star\{-(r+1), r+2\}, \equiv 0 \text{ in } Z_{pd}(N) \cup Z_{osc}(N).$$

Then we have to check the operator

$$B^{(r+1)} := (D_t - \mathcal{D} + F_1 + P_3)(I + \sum_{l=1}^{r+1} N_2^{(l)}) - (I + \sum_{l=1}^{r+1} N_2^{(l)})(D_t - \mathcal{D} + \sum_{l=0}^{r} F_2^{(l)})$$

and can show that $B^{(r+1)} = \tilde{B}^{(r+1)} + R_1$, where $\sigma(\tilde{B}^{(r+1)}) \in S_N^\star\{-(r+2), r+3\}$, $\equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$ and R_1 is as above. By Lemma 16 we find a symbol $n_2 = n_2(t,x,\xi) \sim I + \sum_{r\ge 1} \sigma(N_2^{(r)})(t,x,\xi)$, $n_2 \in S_N^\star\{0,0\}$ modulo $\bigcap_{r\ge 0} S_N^\star\{-r,r\}$, and $n_2 \equiv I$ in $Z_{pd}(N) \cup Z_{osc}(N)$, and a symbol $f_2 = f_2(t,x,\xi) \sim \sum_{r\ge 0} \sigma(F_2^{(r)})(t,x,\xi)$, $f_2 \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$ modulo $\bigcap_{r\ge 0} S_N^\star\{-r,r+1\}$, $f_2 \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. Then the above operator identity holds with $\sigma(N_2) := n_2$ and $\sigma(F_2) := f_2$, where P_∞ can be represented in the form $P_\infty = F_\infty + R$, where $\sigma(R) = \sigma(F_1 + P_3)\chi\Big(\frac{t\langle\xi\rangle}{2N(\ln\langle\xi\rangle)^{2\gamma}}\Big)$.
The first pseudo-differential operator F_∞ has a symbol $\sigma(F_\infty)$ from $\Big\{\bigcap_{r\ge 0} S_N^\star\{-r, r+1\}\Big\}$, $\sigma(F_\infty) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$. The second pseudo-differential operator R has a symbol $\sigma(R)$ belonging to $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\})$. Moreover, $\sigma(R)$ vanishes in $Z_{reg}(2N)$.

□

Thus we finished our perfect diagonalization modulo $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{\bigcap_{r\ge 0} S_{2N}^\star\{-r, r+1\}\Big\}$.

To complete the perfect diagonalization it remains to understand that a parametrix $N_2^\sharp$ to N_2 exists. From the construction we know that $\sigma(N_2 - I) \in S_N^\star\{-1,1\}$ and vanishes in $Z_{pd}(N) \cup Z_{osc}(N)$. A suitable large constant N in the definition of zones guarantees that N_2 is elliptic and its symbol is equal to I in $Z_{pd}(N) \cup Z_{osc}(N)$. Hence, the statement of Lemma 18 gives the existence of $N_2^\sharp$ with symbol from $S_N^\star\{0,0\}$ and equal to I in $Z_{pd}(N)\cup Z_{osc}(N)$. Thus we can formulate the next result.

LEMMA 25. *The fundamental solution $E_0 = E_0(t,s)$ solving (43) can be represented in the form $E_0(t,s) = N_1(t)N_2(t)E_1(t,s)N_2^\sharp(s)N_1^\sharp(s)$, where $N_1^\sharp$, N_1 and $N_2^\sharp$, N_2 are elliptic operators with symbols $\sigma(N_1^\sharp), \sigma(N_1) \in S_N\{0,0\}$, both symbols are $\equiv I$ in $Z_{pd}(N)$ and $\sigma(N_2^\sharp)$, $\sigma(N_2) \in S_N^\star\{0,0\}$, both symbols are $\equiv I$ in $Z_{pd}(N)\cup Z_{osc}(N)$. The matrix operator $E_1 = E_1(t,s)$ solves*

$$D_tE_1 - \mathcal{D}E_1 + F_2E_1 + P_\infty E_1 + R_\infty E_1 = 0,$$

where the matrix pseudo-differential operators $\mathcal{D},\ F_2,\ P_\infty,\ R_\infty$ possess the following properties:

- $\mathcal{D}:\ \sigma(\mathcal{D}) \in T_{2N}\cap S_N\{1,0\}$,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{h}{2\varphi_2}D_t\frac{\varphi_2}{h} & 0 \\ 0 & \varphi_2 + \frac{h}{2\varphi_2}D_t\frac{\varphi_2}{h} \end{pmatrix};$$

- F_2*: diagonal,* $\sigma(F_2) \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N)\cup Z_{osc}(N)$*;*
- P_∞*:* $\sigma(P_\infty) \in T_{2N}\cap(S_{2N}\{0,0\}+S_{2N}\{-1,2\})\cap\Big\{\bigcap_{p\geq 0} S_{2N}^\star\{-p,p+1\}\Big\}$*;*
- R_∞*:* $\sigma(R_\infty) \in C^\infty([0,T], S^{-\infty}(\mathbb{R}^n))$.

All the statements together yield the following result.

LEMMA 26. *The determination of a parametrix to the matrix pseudo-differential operator $D_t - A$ can be reduced, after transformations by elliptic matrix pseudo-differential operators (corresponding to perfect diagonalization), to the determination of a parametrix to the matrix pseudo-differential operator $D_t - \mathcal{D} + F_2 + P_\infty$, where the matrix pseudo-differential operators $\mathcal{D},\ F_2,\ P_\infty$, possess the following properties:*

- $\mathcal{D}:\quad \sigma(\mathcal{D}) \in T_{2N}\cap S_N\{1,0\}$,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle} & 0 \\ 0 & \varphi_2 + \frac{\langle\xi\rangle}{2\varphi_2}D_t\frac{\varphi_2}{\langle\xi\rangle} \end{pmatrix};$$

- F_2*: diagonal,* $\sigma(F_2) \in (S_N^\star\{0,0\} + S_N^\star\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N)\cup Z_{osc}(N)$*;*
- P_∞*:* $\sigma(P_\infty) \in T_{2N}\cap(S_{2N}\{0,0\}+S_{2N}\{-1,2\})\cap\Big\{\bigcap_{p\geq 0} S_{2N}^\star\{-p,p+1\}\Big\}$.

Here we use

$$\varphi_k(t,x,\xi) = d_k\langle\xi\rangle\ \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big) + \tau_k(t,x,\xi)\Big(1-\chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big),$$

where $d_2 = -d_1$ *is a positive constant and*

$$\tau_k(t,x,\xi) = (-1)^k\sqrt{a(t,x,\xi)}\ ,\ a(t,x,\xi) := \sum_{k,l=1}^{n} a_{k,l}(t,x)\xi_k\xi_l.$$

The function $\chi = \chi(s)$ *is from* $C_0^\infty(\mathbb{R})$, $\chi(s)\equiv 1$ *for* $|s|\le 1$, $\chi(s)\equiv 0$ *for* $|s|\ge 2$ *and* $0\le\chi(s)\le 1$.

Step 3. Construction of parametrix

We need four steps for the construction of the parametrix.

Transformation by an elliptic pseudo-differential operator.
Let K be the diagonal elliptic pseudo-differential operator with symbol

$$\sigma(K) = \begin{pmatrix} \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} & 0 \\ 0 & \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} \end{pmatrix}.$$

This symbol is constant in $Z_{pd}(N)$, $\sigma(K)\in S_N\{0,0\}$. Then the following operator-valued identity holds modulo a regularizing operator:

$$(D_t - \mathcal{D} + F_2)K = K(D_t - \mathcal{D}_1 + F_3), \tag{47}$$

where

$$\sigma(\mathcal{D}_1) := \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix},\ \sigma(F_3)\equiv 0 \ \text{ in } \ Z_{pd}(N),$$
$$\sigma(F_3)\in T_{2N}\cap(S_N\{0,0\} + S_N^\star\{-1,2\}).$$

REMARK 11. This transformation corresponds to the special structure of our starting operator and explains that we have no contribution to the loss of derivatives from $\mathcal{D}$. This we already observed in Section 3 during the proof of Theorem 8. In the representation of V_1 from (10) there appears $E_2 = E_2(t,t_\xi,\xi)$. Although in E_2 there appears the term $\frac{1}{2}\frac{D_s a}{a}$ which belongs to $S_1\{0,1\}$ (see Definition 7), this term has no contribution to the loss of derivatives.

Parametrix to $D_t - \mathcal{D}_1$.

LEMMA 27. *The parametrix* $E_2(t,s) =$ diag$(E_2^-(t,s), E_2^+(t,s))$ *to* $D_t - \mathcal{D}_1$ *is a diagonal Fourier integral operator with*

$$E_2^\mp(t,s)w(x) = \int_{\mathbb{R}^n} e^{i\phi^\mp(t,s,x,\xi)} e_2^\mp(t,s,x,\xi)\hat{w}(\xi)d\xi\ ,$$
$$\phi^\mp(s,s,x,\xi) = x\cdot\xi\ ,\ e_2^\mp(s,s,x,\xi) = 1\ .$$

The phase functions $\phi^\mp$ *satisfy*

- $\phi^{\mp}(t,s,x,\xi) = x\cdot\xi + d_k\langle\xi\rangle(t-s),\ k=1$ *for* ϕ^-, $k=2$ *for* ϕ^+ *if* $0 \le s,t \le t_\xi$;
- $|\partial_\xi^\alpha \partial_x^\beta(\phi^{\mp}(t,s,x,\xi) - x\cdot\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}\max(s,t)$ *if* $\max(s,t) \ge t_\xi$.

The amplitude functions $e_2^{\mp}$ *satisfy*

- $e_2^{\mp}(t,s,x,\xi) = 1$ *if* $0 \le s,t \le t_\xi$;
- $e_2^{\mp} \in C([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.

To prove this result we follow the following steps:
Study of the Hamiltonian flow generated by $\varphi_1 = \varphi_1(t,x,\xi)$ *and* $\varphi_2 = \varphi_2(t,x,\xi)$.
Construction of phase functions
Let us denote by $\lambda = \lambda(t,x,\xi)$ one of the functions $\varphi_k = \varphi_k(t,x,\xi),\ k=1,2$. The *Hamiltonian flow* $(q,p) = (q,p)(t,s,y,\eta) =: H_{s,t}(y,\eta)$ is the solution to

$$\frac{dq}{dt} = \nabla_\xi\lambda(t,q,p),\quad q(s,s,y,\eta) = y\ ;\ \frac{dp}{dt} = -\nabla_x\lambda(t,q,p),\quad p(s,s,y,\eta) = \eta.$$

Using $\sigma(\lambda) \in T_{2N} \cap S_N\{1,0\}$ we know that the growth of λ with respect to q and p is at most linear. Thus the solution (q,p) exists globally in time, $t \in [0,T]$, for all (y,η). For the following considerations we need suitable estimates for $q = q(t,s,y,\eta)$ and $p = p(t,s,y,\eta)$. Following the approach of [12] and [26] one can prove the next results.

LEMMA 28. *There exists a (in general small) positive constant* T_0 *such that*

$$\tfrac{q(t,s)-y}{t-s},\ \partial_t q(t,s),\ \partial_s q(t,s) \in L^\infty([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta));$$
$$\tfrac{p(t,s)-\xi}{t-s},\ \partial_t p(t,s),\ \partial_s p(t,s) \in L^\infty([0,T_0]^2, S^1_{1,0}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)).$$

LEMMA 29. *If* T_0 *is small, then the inverse function* $y = y(t,s,x,\eta)$ *to* $x = q(t,s,y,\eta)$ *exists and satisfies*

$$\tfrac{y(t,s)-x}{t-s},\ \partial_t y(t,s),\ \partial_s y(t,s) \in L^\infty([0,T_0]^2,\ S^0_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\eta)).$$

Construction of phase functions $\phi^{\mp}$ *solving the eikonal equations.*

Let us construct the phase function $\phi = \phi(t,s,x,\xi)$ solving the *eikonal equation* $\partial_t\phi(t,s,x,\xi) - \lambda(t,x,\nabla_x\phi(t,s,x,\xi)) = 0$, $\phi(s,s,x,\xi) = x\cdot\xi$.

LEMMA 30. *The phase function* $\phi = \phi(t,s,x,\xi)$ *is defined as follows:*
$\phi(t,s,x,\xi) := v(t,s,y(t,s,x,\xi),\xi)$, *where*

$$v(t,s,y,\xi) = y\cdot\xi - \int_s^t \Big(p\cdot\nabla_\xi\lambda - \lambda\Big)\Big(\tau, q(\tau,s,y,\xi), p(\tau,s,y,\xi)\Big)d\tau.$$

Construction of amplitudes $e_2^{\mp}$ by solving the transport equations and by using the asymptotic representation theorem.

Following our representation

$$E_2^{\mp}(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^{\mp}(t,s,x,\xi)} e_2^{\mp}(t,s,x,\xi)\hat{w}(\xi)d\xi$$

with $\phi^{\mp}(s,s,x,\xi) = x\cdot\xi$, $e_2^{\mp}(s,s,x,\xi) = 1$, as usual, the asymptotic representation

$$e_2^{\mp}(t,s,x,\xi) \sim \sum_{j=0}^{\infty} e_{2,j}^{\mp}(t,s,x,\xi) \quad \text{modulo} \;\; C([0,T_0]^2, S^{-\infty}(\mathbb{R}^n)),$$
$$e_{2,0}^{\mp}(s,s,x,\xi) = 1\;,\; e_{2,j}^{\mp}(s,s,x,\xi) = 0 \quad \text{for} \;\; j\geq 1,$$

allows us to derive so-called *transport equations*.
We have to study the action of $D_t - \varphi_1(t,x,D_x)$, $D_t - \varphi_2(t,x,D_x)$ respectively on E_2^- , E_2^+. We consider $(D_t - \varphi_1)E_2^-$ and suppose that all assumptions are satisfied for the action of the pseudo-differential operator $D_t - \varphi_1(t,x,D_x)$ on the Fourier integral operator E_2^-. On the one hand we get formally

$$D_t E_2^-(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^-(t,s,x,\xi)}\Big(\partial_t\phi^- \sum_{j=0}^{\infty} e_{2,j}^- + \tfrac{1}{i}\partial_t \sum_{j=0}^{\infty} e_{2,j}^-\Big)(t,s,x,\xi)\hat{w}(\xi)d\xi;$$

on the other hand we use formally

$$\begin{aligned}
&\varphi_1(t,x,D_x)E_2^-(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^-(t,s,x,\xi)}\Big[\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\\
&\sum_{j=0}^{\infty} e_{2,j}^-(t,s,x,\xi) + \nabla_\xi\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\cdot\tfrac{1}{i}\sum_{j=0}^{\infty}\nabla_x e_{2,j}^-(t,s,x,\xi)\\
&-\tfrac{i}{2}\sum_{k,l=1}^{n}\partial^2_{\xi_k\xi_l}\varphi_1\Big(t,x,\nabla_x\phi^-(t,s,x,\xi)\Big)\Big(\partial^2_{x_kx_l}\phi^-\sum_{j=0}^{\infty} e_{2,j}^-\Big)(t,s,x,\xi)\\
&+r_2(t,s,x,\xi)\Big]\hat{w}(\xi)d\xi,
\end{aligned}$$

where

$$\begin{aligned}
r_2(t,s,x,\xi) \quad \sim \quad & \sum_{|\alpha|=2}^{\infty}\tfrac{1}{\alpha!}D_y^{\alpha}\Big(\partial_\xi^{\alpha}\varphi_1\Big(t,x,\int_0^1\nabla_x\phi^-(t,s,y+r(x-y),\xi)dr\Big)\\
&\sum_{j=0}^{\infty} e_{2,j}^-(t,s,y,\xi)\Big)_{y=x}.
\end{aligned}$$

Supposing that all series converge uniformly and using that ϕ^- solves the eikonal equation with $\lambda = \varphi_1$ we arrive at the *transport equations* to determine $e_{2,j}^{\mp}$ for $j\geq 0$. Finally we arrive at the statements of Lemma 27.

Parametrix to $D_t - \mathcal{D}_1 + F_3$.

LEMMA 31. *The parametrix $E_4 = E_4(t,s)$ to the operator $D_t - \mathcal{D}_1 + F_3$ can be written as $E_4(t,s) = E_2(t,s)Q_4(t,s)$, where $E_2 = E_2(t,s)$ is the diagonal Fourier integral operator from Lemma 27 and $Q_4 = Q_4(t,s)$ is a diagonal pseudo-differential operator with symbol belonging to $W^{1,\infty}([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.*

To prove this result we follow the following steps:

Application of Egorov's theorem, that is, conjugation of F_3 by Fourier integral operators, here we use the diagonal structure.

We will construct the parametrix to $D_t - \mathcal{D}_1 + F_3$. Using $E_2 = E_2(t,s)$ from the previous step we choose the representation

$$E_4(t,s) = E_2(t,s)Q_4(t,s), \quad Q_4(s,s) \sim I.$$

This implies the Cauchy problem

$$D_t Q_4 + E_2(s,t)F_3(t)E_2(t,s)Q_4 \sim 0\ , \quad Q_4(s,s) \sim I.$$

According to Egorov's theorem [26] (here we can use the diagonal structure of $D_t - \mathcal{D}_1 + F_3$) the matrix operator $R_4(t,s) := E_2(s,t)F_3(t)E_2(t,s)$ is a pseudo-differential operator whose symbol is $r_4 = r_4(t,s,x,\xi) = f_3(t, H_{s,t}(x,\xi)),\ f_3 := \sigma(F_3)$, modulo a symbol from $S_N\{-1,0\} + S^\star_N\{-2,2\}$, where $H_{s,t}(x,\xi)$ denotes the Hamiltonian flow starting at (x,ξ) and generated by the symbols $\varphi_k = \varphi_k(t,x,\xi),\ k = 1,2$.

For $t \in [0,T_0]$ with a sufficiently small T_0 we understand to which zone the Hamiltonian flow belongs to.

We can write $f_3(t,x,\xi) = f_{3,0}(t,x,\xi) + f_{3,1}(t,x,\xi)$, where $f_{3,0} \in S_N\{0,0\}$, $f_{3,1} \in S^\star_N\{-1,2\}$,
$f_{3,0} \equiv 0$ in $Z_{pd}(N)$, $f_{3,1} \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$.

LEMMA 32. *Let us denote by $\lambda = \lambda(t,x,\xi)$ one of the functions $\varphi_k = \varphi_k(t,x,\xi),\ k = 1,2$. The Hamiltonian flow $(q,p) = (q,p)(t,s,y,\eta) =: H_{s,t}(y,\eta)$ is the solution to*

$$\frac{dq}{dt} = \nabla_\xi \lambda(t,q,p), \quad q(s,s,y,\eta) = y\ ;\ \frac{dp}{dt} = -\nabla_x \lambda(t,q,p), \quad p(s,s,y,\eta) = \eta.$$

Then the symbols $f_{3,0}$ and $f_{3,1}$ satisfy

$$\left|\partial_x^\beta \partial_\xi^\alpha f_{3,0}(t, H_{s,t}(x,\xi))\right| \le C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|},$$

$$\left|\partial_x^\beta \partial_\xi^\alpha f_{3,1}(t, H_{s,t}(x,\xi))\right| \le C_{\alpha,\beta}\Big(\tfrac{1}{t}(\ln\tfrac{1}{t})^\gamma\Big)^2\langle\xi\rangle^{-1-|\alpha|}$$

for all $(t,x,\xi) \in [0,T_0] \times \mathbb{R}^{2N}$.

The statement of this lemma makes it clear that the following representation is reasonable for $Q_4 = Q_4(t,s)$:

$$Q_4(t,s)w(x) = \int_{\mathbb{R}^n} e^{i\,x\cdot\xi} q_4(t,s,x,\xi)\hat{w}(\xi)d\xi, \quad q_4(s,s,x,\xi) = I.$$

We determine the matrix amplitude q_4 by equivalence to a series, that is $q_4(t,s,x,\xi) \sim \sum_{j=0}^{\infty} q_{4,j}(t,s,x,\xi)$. After determination of $q_{4,j} = q_{4,j}(t,s,x,\xi)$ for $j \geq 0$ we obtain the statement of Lemma 31.

Parametrix to $D_t - \mathcal{D} + F_2$.

LEMMA 33. *The parametrix* $E_3 = E_3(t,s)$ *to the operator* $D_t - \mathcal{D} + F_2$ *can be written as* $E_3(t,s) = K(t)E_2(t,s)Q_4(t,s)K^{\sharp}(s)$, *where* K *and its parametrix* $K^{\sharp}$ *having symbols from* $L^{\infty}([0,T_0], S^0_{1,0}(\mathbb{R}^n)) \cap C^{\infty}((0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$ *are the elliptic pseudo-differential operators from the above transformation.*

REMARK 12. From Lemma 33 we conclude that the parametrix to $D_t - \mathcal{D} + F_2$ gives no loss of derivatives of the solution to (1). In the next point we will see that this loss comes from P_∞.

Parametrix to $D_t - \mathcal{D} + F_2 + P_\infty$.

LEMMA 34. *The parametrix* $E_1 = E_1(t,s)$ *to the operator* $D_t - \mathcal{D} + F_2 + P_\infty$ *can be written as* $E_1(t,s) = E_3(t,s)Q_1(t,s)$, *where* $Q_1 = Q_1(t,s)$ *is a matrix pseudo-differential operator with symbol from* $L^{\infty}([0,T_0]^2, S^{K_0}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)) \cap W^{1,\infty}([0,T_0]^2, S^{K_0+1+\varepsilon}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n))$ *for every small* $\varepsilon > 0$. *Here the constant* K_0 *describes the loss of derivatives coming from the pseudo-differential zone* $Z_{pd}(2N)$ *and the oscillations subzone* $Z_{osc}(2N)$.

To prove this result we use the next observations and ideas:

- Egorov's theorem is not applicable because P_∞ has no diagonal structure.
- We have to use the properties of P_∞ after perfect diagonalization.
- The next result is a base to get a relation between the type of oscillations and the loss of derivatives.

LEMMA 35. *The Fourier integral operator* $P_\infty(t)E_3^{\mp}(t,s)$ *is a pseudo-differential operator with the representation*

$$P_\infty(t)E_3^{\mp}(t,s)w(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\tilde{r}^{\mp}(t,s,x,\xi)\hat{w}(\xi)d\xi,$$

where the symbol satisfies the estimates

$$\left|\partial_x^\beta \partial_\xi^\alpha \tilde{r}^{\mp}(t,s,x,\xi)\right| \leq \begin{cases} C_{\alpha\beta\varepsilon p}\left(\frac{1}{t}(\ln\frac{1}{t})^\gamma\right)^{p+1}\langle\xi\rangle^{-p+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{reg}(2N), \\ C_{\alpha\beta\varepsilon}\left(1+\left(\frac{1}{t}(\ln\frac{1}{t})^\gamma\right)^2\langle\xi\rangle^{-1}\right)\langle\xi\rangle^{\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{osc}(2N), \\ C_{\alpha\beta\varepsilon}\langle\xi\rangle^{1+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{pd}(2N), \end{cases}$$

for every $p \geq 0$, *small* $\varepsilon > 0$ *and all* $s \in [0,t]$.

Step 4. Conclusion

Using Lemma 34 and the backward transformation (from the steps of perfect diagonalization) we obtain the parametrix for $D_t - A$. The backward transformation doesn't bring an additional loss of derivatives. Therefore we can conclude the following result.

THEOREM 13. *Let us consider*

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = 0, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

where the coefficients satisfy the conditions (24) and (25). The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:

$$E(u)|_{H^{s-s_0}}(t) \le C(T) E(u)|_{H^s}(0) \textit{ for all } t \in (0,T], \tag{48}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there doesn't exist a positive constant* s_0 *satisfying (48) if* $\gamma > 1$, *that is, we have an infinite loss of derivatives.*

It seems to be remarkable that we can prove the same relation between types of oscillations and loss of derivatives as in Theorem 8.

7. Concluding remarks

Let us mention further results which are obtained for model problems with non-Lipschitz behaviour and more problems which could be of interest.

REMARK 13. *Lower regularity with respect to x.* The results and the approach from [15] motivate the study of the question of how to weaken the regularity with respect to x (compare with [9]). From this paper we understand to which class the remainder should belong after diagonalization. Thus pseudo-differential operators with symbols of finite smoothness or maybe paradifferential operators should be used.

REMARK 14. *Quasi-linear models.* Quasi-linear models with behaviour of suitable derivatives as $O(\frac{1}{t})$ were studied in [3] and [18]. Here the log-effect from (5) could not be observed.

REMARK 15. *Applications to Kirchhoff type equations.* A nice application of non-Lipschitz theory with behaviour $a'(t) = O((T-t)^{-1})$ for $t \to T-0$ to Kirchhoff equations was described in [16]. The assumed regularity of data could be weakened in [13] by proving that these very slow oscillations (in the language of Definition 2) produce no loss of derivatives (see Theorem 8).

REMARK 16. *p-Evolution equations.* The paper [1] is devoted to the Cauchy problem for p-evolution equations with LogLip coefficients. The paper [4] is devoted among other things to p-evolution equations of higher order with non-Lipschitz coefficients. Concerning our starting model this means p-evolution equations of second order with respect to t with coefficients behaving like $|ta'(t)| \leq C$ on $(0, T]$. An interesting question is to find p-evolution models with log-effect from (5).

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