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ON NON-RIGID PROJECTIVE CURVES

Abstract. In this note, we consider the natural functorial rational map ϕ from the (restricted) Hilbert scheme $Hilb(d, g, r)$ to the moduli space $\mathcal{M}_g$, associating the nondegenerate projective model $p(C)$ of a smooth curve C to its isomorphism class $[C]$. We prove that ϕ is non constant in a neighbourhood of $p(C)$, for any $[C] \in U \subset \mathcal{M}_g$ (where $g \geq 1$ and U is a dense open subset of $\mathcal{M}_g$), provided $p(C)$ is a smooth point or a reducible singularity of $Hilb(d, g, r)_{red}$, the (restricted) Hilbert scheme with reduced structure.

1. Introduction

Let k be any algebraically closed field of characteristic zero and, as usual, let $\mathbb{P}^r := Proj(k[x_0, \ldots, x_r])$ be the associated projective space. Inside the Hilbert scheme $H(d, g, r)$, parametrizing closed subschemes of dimension 1, arithmetic genus g, degree d in $\mathbb{P}^r$, let us consider the so called *restricted* Hilbert scheme $Hilb(d, g, r)$, which is the subscheme of $H(d, g, r)$, consisting of those points $p(C)$, such that every irreducible component K of $H(d, g, r)$ containing $p(C)$ has smooth, non degenerate and irreducible general element (see Definition 1.31 of [12]).

The aim of the present note is to get some insight in the behaviour of the rational functorial map $\phi : Hilb(d, g, r) \to \mathcal{M}_g$, which associates to each point $p(C)$ in $Hilb(d, g, r)$ representing a smooth non degenerate irreducible curve C the corresponding isomorphism class $[C] \in \mathcal{M}_g$. In particular, we study in which cases the image of ϕ has positive dimension.

Any non degenerate smooth integral subscheme C of dimension 1 in $\mathbb{P}^r$ determines a point $p(C) \in Hilb(d, g, r)$. We give the following:

DEFINITION 1. *The projective curve $C \subset \mathbb{P}^r$ admits non-trivial first order deformations if the image of the map $D\phi : T_{p(C)}Hilb(d, g, r) \to T_{[C]}\mathcal{M}_g$ has positive dimension (or equivalently if $D\phi \neq 0$). In this case we say that the corresponding curve is non-rigid at the first order, for the given embedding.*

DEFINITION 2. *The projective curve $C \subset \mathbb{P}^r$ admits non-trivial deformations if there exists at least a curve $\gamma \subset Hilb(d, g, r)$, through $p(C)$, which is not contracted to a point via ϕ. Equivalently, if there exists an irreducible component of $Hilb(d, g, r)$*

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containing $p(C)$, such that its image in $\mathcal{M}_g$ through ϕ has positive dimension. In this case we say that the curve is non-rigid for the given embedding.

We can somehow get rid of the fixed embedding in some projective space taking into account *all* possible nondegenerate embeddings, as in the following:

DEFINITION 3. *The (abstract) smooth curve C is non-rigid at the first order, as a smooth non degenerate projective curve, if for* any *non degenerate projective embedding $j : C \hookrightarrow \mathbb{P}^r$, the corresponding map $D\phi : T_{p(C)}Hilb(d, g, r) \to T_{[C]}\mathcal{M}_g$ is non zero.*

Analogously, one has the following:

DEFINITION 4. *The (abstract) smooth curve C is non-rigid as a smooth non degenerate projective curve if, for any non degenerate projective embedding $j : C \hookrightarrow \mathbb{P}^r$, there exists an irreducible component of the associated $Hilb(d, g, r)$ containing $p(C)$, such that its image in $\mathcal{M}_g$ through ϕ has positive dimension.*

In this paper, we prove that there exists a dense open subset $U \subset \mathcal{M}_g$ ($g \geq 1$), such that any C, with $[C] \in U$, is non rigid at the first order as a smooth non degenerate projective curve in the sense of Definition 3; moreover, we prove that these curves are non-rigid (not only at the first order) under the additional assumption that $p(C)$ is a smooth point of $Hilb(d, g, r)_{red}$ (the restricted Hilbert scheme with reduced scheme structure) or at worst it is a reducible singularity of $Hilb(d, g, r)_{red}$ (see Definition 5 in section 3).

2. First order deformations

First of all we deal with the case of smooth projective curves of genus $g \geq 2$ in $\mathbb{P}^r$. We will prove that there exists a dense open subset $U^0_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U^0_{BN}$ and for *any* non degenerate smooth embedding of C in $\mathbb{P}^r$ the corresponding projective curve is non-rigid at the first order (in the sense of Definition 3).

From the fundamental exact sequence:

$$0 \to TC \to T\mathbb{P}^r_{|C} \to N_{C/\mathbb{P}^r} \to 0, \tag{1}$$

taking the associated long exact cohomology sequence, since $H^0(TC) = H^0(K_C^{-1}) = 0$ (genus $g \geq 2$), we get:

$$\begin{aligned} 0 \to H^0(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \stackrel{D\phi}{\to} H^1(TC) \to \\ \to H^1(T\mathbb{P}^r_{|C}) \to H^1(N_{C/\mathbb{P}^r}) \to 0. \end{aligned} \tag{2}$$

In sequence (2), as usual, we identify $H^0(N_{C/\mathbb{P}^r})$ with the tangent space $T_{p(C)}Hilb(d, g, r)$ to the Hilbert scheme at the point $p(C)$ representing C, and

$H^1(TC)$ with $T_{[C]}\mathcal{M}_g$ (see for example [11] and [12]). Thus the coboundary map $D\phi$ represents the differential of the map $\phi : Hilb(d, g, r) \to \mathcal{M}_g$ we are interested in. If $D\phi = 0$ (i.e. the corresponding curve is rigid also at the first order) the sequence above splits and in particular $h^0(T\mathbb{P}^r_{|C}) = h^0(N_{C/\mathbb{P}^r})$; thus imposing $h^0(T\mathbb{P}^r_{|C}) < h^0(N_{C/\mathbb{P}^r})$ and estimating the dimension of the cohomology groups, we get a relation involving d, g, r, which, if it is fulfilled implies that the corresponding curve is not rigid (at least at the first order). This is the meaning of the following:

PROPOSITION 1. *Let $C \subset \mathbb{P}^r$ a smooth non-degenerate curve of genus $g \geq 2$ and degree d. If $d > \frac{2}{r+1}[g(r-2)+3]$, or $\mathcal{O}_C(1)$ is non special (this holds if $d > 2g-2$), then $D\phi \neq 0$. Furthermore, if $C \subset \mathbb{P}^r$ is linearly normal, then $D\phi \neq 0$ provided that*

$$d > \frac{(r-2)g + r(r+1) + 3}{r+1}. \tag{3}$$

Proof. It is clear from the exactness of (2) that if $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, then $D\phi \neq 0$. On the other hand, $h^0(N_{C/\mathbb{P}^r}) = dim(T_{p(C)}Hilb(d, g, r)) \geq dim(Hilb(d, g, r))$ and $dim(Hilb(d, g, r)) \geq (r+1)d - (r-3)(g-1)$, where the last inequality always holds at points of $Hilb(d, g, r)$ parametrizing locally complete intersection curves (in particular smooth curves), see for example ([12]). Thus $h^0(N_{C/\mathbb{P}^r}) \geq (r+1)d - (r-3)(g-1)$. Now, applying Riemann-Roch to the vector bundle $T\mathbb{P}^r$ on C, we get $h^0(T\mathbb{P}^r_{|C}) = (r+1)d - r(g-1) + h^1(T\mathbb{P}^r_{|C})$. On the other hand, from the Euler sequence (twisted with $\mathcal{O}_C$):

$$0 \to \mathcal{O}_C \to (r+1)\mathcal{O}_C(1) \to T\mathbb{P}^r_{|C} \to 0, \tag{4}$$

we get immediately $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$ and by Riemann-Roch the latter is equal to $(r+1)(h^0(\mathcal{O}_C(1)) - d + g - 1)$. Now, if $\mathcal{O}_C(1)$ is non special (i.e. if $d > 2g-2$), then $h^1(T\mathbb{P}^r_{|C}) = 0$, so that, imposing $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, we get $3(g-1) > 0$, which is always satisfied (if $g \geq 2$). This means that a smooth curve of genus $g \geq 2$, which is embedded via a non special linear system, is always non-rigid at least at the first order.

If instead $\mathcal{O}_C(1)$ is special, by Clifford's theorem we have $h^0(\mathcal{O}_C(1)) \leq d/2 + 1$, so that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)(g - d/2)$. Imposing again $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, that is $(r+1)d - (r-3)(g-1) > (r+1)d - r(g-1) + (r+1)(g - d/2)$, we get the relation $d > \frac{2}{r+1}[g(r-2)+3]$.

Finally, if $C \subset \mathbb{P}^r$ is linearly normal and non degenerate, then $h^0(\mathcal{O}_C(1)) = r+1$. Substituting in $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1)) = (r+1)(h^0(\mathcal{O}_C(1)) - d + g - 1)$ and imposing the fundamental inequality $h^0(N_{C/\mathbb{P}^r}) > h^0(T\mathbb{P}^r_{|C})$, we get $d > \frac{(r-2)g+r(r+1)+3}{r+1}$.

□

Since the bound (3) is particularly good, but it holds only for linearly normal curves and since any curve can be obtained via a series of (generic) projections from a linearly normal curve, we are going to study what is the relation among first order deformations of a linearly normal curve and the first order deformations of its projections. This is the aim of the following:

PROPOSITION 2. *Let* $C \subset \mathbb{P}^r$ *a smooth curve of genus* $g \geq 2$ *which is non-rigid at the first order. Then any of its smooth projections* $C' := \pi_q(C) \subset \mathbb{P}^{r-1}$ *from a point* $q \in \mathbb{P}^r$ *is non-rigid at the first order (*q *is a point chosen out of the secant variety of* C*,* $Sec(C)$*).*

Proof. First of all, let us remark that the proposition states that in the following diagram:

$$\begin{array}{ccc} T_{p(C)}Hilb(d,g,r) & \overset{D\phi}{\rightarrow} & T_{[C]}\mathcal{M}_g \\ \downarrow & \nearrow\ D\phi` & \\ T_{p(C')}Hilb(d,g,r-1) & & \end{array}$$

if $Im(D\phi) \neq 0$, then $Im(D\phi`) \neq 0$. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & \rightarrow & ker(a) & \rightarrow & ker(b) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & TC & \rightarrow & T\mathbb{P}^r_{|C} & \rightarrow & N_{C/\mathbb{P}^r} & \rightarrow & 0 \\ & & \downarrow\cong & & \downarrow a & & \downarrow b & & \\ 0 & \rightarrow & TC' & \rightarrow & T\mathbb{P}^{r-1}_{|C'} & \rightarrow & N_{C'/\mathbb{P}^r} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & 0 & \rightarrow & 0 & & \end{array}$$

where the morphisms a and b are induced by the projection of C to C'. Clearly $TC \cong TC'$, because C and C' are isomorphic curves and moreover a and b are surjective by construction. Applying the snake lemma to the previous diagram, we see that $ker(a) \cong ker(b)$ and since a and b are surjective morphisms of vector bundles, it turns out that $ker(a) = ker(b) = \mathcal{L}$, where $\mathcal{L}$ is a line bundle on C. Restricting the attention to the last column of the previous diagram, it is clear from a geometric reasoning that the line bundle $\mathcal{L}$ can be identified with the ruling of the projective cone, with vertex q through which we project. Indeed, it is sufficient to look at the induced projection map b at a point $x \in C$: $b : N_{C/\mathbb{P}^r}, x \rightarrow N_{C'/\mathbb{P}^r}, \pi(x)$; the kernel is always the line on the cone with vertex q going trough x and this is never a subspace of TC, because $q \notin Sec(C)$. Clearly, we can identify the projective cone with vertex q through which we project, with the line bundle $\mathcal{L}$, since we can consider instead of just $\mathbb{P}^r$, the blowing-up $Bl_q(\mathbb{P}^r)$ in q in such a way to separate the ruling of the cone (this however does not affect our reasoning since we are dealing with line bundles over C and $q \notin C$).

Applying the cohomology functor to the previous commutative diagram and recalling that $h^0(TC) = 0$ since $g \geq 2$ we get the following diagram:

$$\begin{array}{ccccccccc}
 & & H^0(\mathcal{L}) & \cong & H^0(\mathcal{L}) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & H^0(N_{C/\mathbb{P}^r}) & \overset{D\phi}{\to} & H^1(TC) & \to & \dots \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & H^0(T\mathbb{P}^{r-1}_{|C'}) & \to & H^0(N_{C'/\mathbb{P}^{r-1}}) & \overset{D\phi'}{\to} & H^1(TC') & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & coker(\alpha) & \to & coker(\beta) & \to & 0 & \to & 0
\end{array}$$

Now, $Im(D\phi) \subset H^1(TC)$ and via the isomorphism γ it is mapped inside $H^1(TC')$. On the other hand, by commutativity of the square having as edges the maps $\beta, \gamma, D\phi$ and $D\phi'$ it is clear that $Im(D\phi) \subseteq Im(D\phi')$ so that if $D\phi \neq 0$, then a fortiori $D\phi' \neq 0$.

□

The following corollary gives two simple sufficient conditions for having $Im(D\phi) \cong Im(D\phi')$.

COROLLARY 1. *Let C, C', $D\phi$ and $D\phi'$ as in Proposition 2. Then if $\mathcal{O}_C(1)$ is non special or if $h^1(T\mathbb{P}^r_{|C}) = 0$, then $Im(D\phi) \cong Im(D\phi')$.*

Proof. Rewrite the previous diagram as:

$$\begin{array}{ccccccccc}
 & & H^0(\mathcal{L}) & \cong & H^0(\mathcal{L}) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & H^0(N_{C/\mathbb{P}^r}) & \to & Im(D\phi) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow & & \\
0 & \to & H^0(T\mathbb{P}^{r-1}_{|C'}) & \to & H^0(N_{C'/\mathbb{P}^{r-1}}) & \to & Im(D\phi') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & coker(\alpha) & \to & coker(\beta) & \to & coker(\alpha)/coker(\beta) & \to & 0
\end{array}$$

Observe that $coker(\alpha) \subseteq H^1(\mathcal{L})$ and the same is true for $coker(\beta)$. So if $H^1(\mathcal{L}) = 0$, then $Im(D\phi) = Im(D\phi')$. On the other hand, from the exact sequence $0 \to \mathcal{L} \to T\mathbb{P}^r_{|C} \to T\mathbb{P}^{r-1}_{|C'} \to 0$, taking Chern polynomials, we get that $\mathcal{L}$ is a line bundle of degree d (and one can identify $\mathcal{L}$ with $\mathcal{O}_C(1) \otimes \mathcal{L}'$ for some $\mathcal{L}' \in Pic^0(C)$). Thus, if $\mathcal{O}_C(1)$ is non special we conclude. If instead $h^1(T\mathbb{P}^r_{|C}) = 0$, then $coker(\alpha) = H^1(\mathcal{L})$ and $coker(\alpha) \subseteq coker(\beta) \subseteq H^1(\mathcal{L})$ so that $coker(\alpha) = coker(\beta)$ and we conclude again.

□

Now we deal with the much simpler case of curves of genus $g = 1$.

PROPOSITION 3. *For any smooth curve $[C] \in \mathcal{M}_1$ and for any non degenerate projective embedding of $C \hookrightarrow \mathbb{P}^r$, the corresponding projective curve is non-rigid at the first order.*

Proof. From the fundamental exact sequence:

$$0 \to T_C \to T\mathbb{P}^r_{|C} \to N_{C/\mathbb{P}^r} \to 0,$$

since $T_C \cong \mathcal{O}_C$ $(g = 1)$, we obtain the long exact cohomology sequence:

$$(5) \qquad 0 \to H^0(\mathcal{O}_C) \to H^0(T\mathbb{P}^r_{|C}) \to H^0(N_{C/\mathbb{P}^r}) \stackrel{D\phi}{\to} H^1(\mathcal{O}_C) \to H^1(T\mathbb{P}^r_{|C}) \dots$$

Twisting the Euler sequence with $\mathcal{O}_C$ and taking cohomology, we have that $h^1(T\mathbb{P}^r_{|C}) \leq (r+1)h^1(\mathcal{O}_C(1))$, but $\mathcal{O}_C(1)$ is always non special for a curve of genus $g = 1$ since $d > 2g - 2 = 0$. Thus $H^1(T\mathbb{P}^r_{|C}) = 0$ and being $h^1(\mathcal{O}_C) \neq 0$, from (5) we have that $D\phi \neq 0$ and it is even always surjective.

□

We conclude this section with the following theorem, which is the analogue of Proposition 3 for curves of genus $g \geq 2$ (in this case we do not work over all $\mathcal{M}_g$, but just on an open dense subset).

THEOREM 1. *For any $g \geq 2$, there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that for any $[C] \in U_{BN}$ and for any non degenerate projective embedding of $C \hookrightarrow \mathbb{P}^r$, the corresponding projective curve is non-rigid at the first order.*

Proof. According to theorem 1.8, page 216 of [1], there exists a dense open subset $U_{BN} \subset \mathcal{M}_g$ such that any $[C] \in U_{BN}$ can be embedded in $\mathbb{P}^r$ as a smooth non degenerate curve of degree d if and only if $\rho \geq 0$, where $\rho(d, g, r) := g - (r+1)(g - d + r)$ is the Brill-Noether number. Now we consider a curve $[C] \in U_{BN}$ and we embed it as a linerly normal curve $\bar{C}$ of degree d in some $\mathbb{P}^r$. Since $[C] \in U_{BN}$, we have that $\rho \geq 0$; on the other hand, $\bar{C}$ is linearly normal and the fundamental inequality (3) is satisfied since $\rho \geq 0$ (indeed, it is just a computation to see that (3) is equivalent to $\rho \geq -\epsilon$ for some $\epsilon > 0$). Thus, by Proposition 1 $\bar{C}$ is non-rigid at the first order, and moreover by Propositon 2 all of its smooth projections are non-rigid at the first order. To conclude, observe that any smooth non degenerate projective curve C such that $[C] \in U_{BN}$ can be obtained via a series of smooth projections from a linearly normal projective curve $\bar{C}$ with corresponding $\rho \geq 0$ (since for the curves in U_{BN} the Brill-Noether condition is necessary and sufficient).

□

3. Finite deformations

Our problem is now to extend the first order deformations studied in the previous section to finite deformations. By Theorem 1, we know that, for the curves C such that $[C] \in U_{BN}$ $(g \geq 2)$, the corresponding $Im(D\phi) \neq 0$ and an even stronger result holds for curves of genus $g = 1$. We need to prove that there exists a vector $v \in T_{p(C)}Hilb(d, g, r)$, corresponding to a smooth curve $\gamma \subset Hilb(d, g, r)$ through $p(C)$ such that the image of the curve via ϕ has positive dimension. To this aim, observe that if $[C] \in U_{BN}$ is not a smooth point of $\mathcal{M}_g$, then there are $w \in T_{[C]}U_{BN}$

which are obstructed deformations, that is which do not correspond to any curve in U_{BN} through $[C]$. We can easily get rid of this problem, just by restricting further the open subset U_{BN}. Indeed, for $g \geq 1$, there is a dense open subset $U^0 \subset \mathcal{M}_g$ such that any $[C] \in U^0$ is a smooth point (see for example [12]). Thus, for curves of genus $g \geq 2$ we consider the dense open subset $U^0_{BN} := U_{BN} \cap U^0$ and for any $w \in T_{[C]}U^0_{BN}$, the corresponding first order deformations are unobstructed, while for curves of genus $g = 1$ we just restrict to the smooth part of $\mathcal{M}_1$, that we denote as U^0_1.

We can draw a first conclusion of such an argument via the following:

PROPOSITION 4. *Let $[C] \in U^0_{BN}$ or $[C] \in U^0_1$ and let $C \hookrightarrow \mathbb{P}^r$ any projective embedding such that the corresponding point $p(C) \in Hilb(d, g, r)$ is a smooth point of the restricted Hilbert scheme. Then the projective curve $C \subset \mathbb{P}^r$ is non rigid.*

Proof. By Theorem 1 or Proposition 3, the associated map $D\phi \neq 0$, so that there exists a $w \in T_{p(C)}Hilb(d, g, r)$ such that $D\phi(w) \neq 0$. Since $p(C)$ is a smooth point of $Hilb(d, g, r)$, the tangent vector w corresponds to a smooth curve $\gamma \subset Hilb(d, g, r)$, through $p(C)$, such that $T_{p(C)}\gamma = w$. Now consider the image Z of this curve in U^0_{BN} via ϕ. Since $\mathcal{M}_g$ exists as a quasi-projective variety, in particular we can represent a neighbourhood of $[C] \in \mathcal{M}_g$, as $Spec(B)$, for some finitely generated k-algebra B. This implies that the map ϕ can be viewed locally around $p(C)$ as a morphism of affine schemes. Thus the image of the curve γ (which is a reduced scheme) via the morphism of affine schemes ϕ is the subscheme Z in $Spec(B)$. Then either Z is positive dimensional and in this case we are done, or it is a zero dimensional subscheme, supported at the point $[C]$; observe that this zero dimensional subscheme Z can not be the reduced point $[C]$, otherwise we would certainly have $D\phi(w) = 0$. So let us consider the case in which Z is a zero dimensional subscheme, supported at the point $[C]$, with non-reduced scheme structure: this case is clearly impossible since the image Z of a reduced subscheme (the curve γ) via the morphism of affine schemes ϕ can not be a non-reduced subscheme. Indeed, if it were the case, consider the rectriction of ϕ to γ: ϕ_γ, $Z_{red} = [C]$; then $\phi_\gamma^{-1}([C])$ is a reduced subscheme, which coincides with γ, since γ is reduced. But this would imply that $\phi(\gamma) = [C]$ and $D\phi(w) = 0$.

Thus, it turns out that Z has necessarily positive dimension and we conclude. □

The hypothesis of Proposition 4, according to which $p(C)$ is a smooth point of $Hilb(d, g, r)$ is extremely strong. Ideally, one would like to extend the result of Proposition 4 to *any* non degenerate projective embedding for curves $[C] \in U^0_{BN}$. Before giving a partial extension of Proposition 4 (Theorem 2), let us give the following:

DEFINITION 5. *A point $p(C) \in Hilb(d, g, r)_{red}$ is called a reducible singularity if it is in the intersection of two or more irreducible components of $Hilb(d, g, r)_{red}$, each of which is smooth in $p(C)$.*

THEOREM 2. *Let $[C] \in U^0_{BN}$ or $[C] \in U^0_1$ and let $C \hookrightarrow \mathbb{P}^r$ any projective embedding such that the corresponding point $p(C) \in Hilb(d, g, r)$ is a smooth point of*

$Hilb(d, g, r)_{red}$ (restricted Hilbert scheme with reduced structure) or such that $p(C)$ is a reducible singularity of $Hilb(d, g, r)_{red}$. Then the projective curve $C \subset \mathbb{P}^r$ is non rigid.

Proof. Let us consider the exact sequence:

$$0 \to H^0(T\mathbb{P}^r_{|C}) \to T_{p(C)}Hilb(d, g, r) \stackrel{D\phi}{\to} T_{[C]}\mathcal{M}_g \tag{6}$$

from which $ker(D\phi) \cong H^0(T\mathbb{P}^r_{|C})$. Take the reduced scheme $Hilb(d, g, r)_{red}$ and consider the induced morphism of schemes $r : Hilb(d, g, r)_{red} \to Hilb(d, g, r)$ (see for example [13], exercise 2.3, page 79). If $p(C)$ is a smooth point of $Hilb(d, g, r)_{red}$, then we have that $dim(Hilb(d, g, r)) = dim(T_{p(C)}Hilb(d, g, r)_{red})$. On the other hand, to prove that there are first order deformations we have just imposed $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d, g, r))$. Now, we want to prove that in the following diagram

$$\begin{array}{ccccccc} 0 & \to & H^0(T\mathbb{P}^r_{|C}) & \to & T_{p(C)}Hilb(d, g, r) & \stackrel{D\phi}{\to} & T_{[C]}\mathcal{M}_g \\ & & & & \uparrow Dr & \nearrow & \\ & & & & T_{p(C)}Hilb(g, d, r)_{red} & & \end{array}$$

the map Dr is injective, so that since $h^0(T\mathbb{P}^r_{|C}) < dim(Hilb(d, g, r)) = dim(T_{p(C)}Hilb(d, g, r)_{red})$, we can find a $w \in T_{p(C)}Hilb(d, g, r)_{red}$ whose image in $T_{[C]}\mathcal{M}_g$ is non zero and then we can argue as in the proof of Proposition 4. Setting $Hilb(d, g, r)_{red} = X_{red}$, $p(C) = x$ and $Hilb(d, g, r) = X$, we have to prove that given $r : X_{red} \to X$, the associated morphism on tangent spaces is injective $Dr : T_x X_{red} \to T_x X$. Since X_{red} is a scheme, we can always find an open affine subscheme U_{red} of X_{red} containing x such that $U_{red} = Spec(A_{red})$, where A_{red} is a finitely generated k-algebra without nilpotent elements and the closed point x corresponds to a maximal ideal m_x. Recall that, from the point of view of the functor of points, the closed point x corresponds to a morphism $\lambda : Spec(k) \to Spec(A_{red})$ (which is induced by $A_{red} \to A_{red,m_x} \to A_{red,m_x}/m_x A_{red,m_x} = k(x) = k$, where A_{red,m_x} is the localization of A_{red} at the maximal ideal m_x). Recall also that via the algebra map $k[\epsilon]/\epsilon^2 \to k$ and the corresponding inclusion of schemes $i : Spec(k) \to Spec(k[\epsilon]/\epsilon^2)$, $T_x X_{red}$ can be identified with $\{u \in Hom(Spec(k[\epsilon]/\epsilon^2, Spec(A_{red})))$ such that $u \circ i = \lambda\}$, (see for example [8]). Clearly, an analogous description holds for X and $T_x X$, (we denote the corresponding neighbourhood of x in X as $Spec(A)$). From the description of $T_x X_{red}$ just given, it turns out any $w \in T_x X_{red}$, $w \neq 0$, corresponds to a unique (non-zero) ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/(\epsilon)^2$, such that the following diagram is commutative:

$$\begin{array}{ccc} A_{red} & \stackrel{u^\natural}{\to} & k[\epsilon]/\epsilon^2 \\ & \lambda^\natural \searrow & \downarrow i^\natural \\ & & k(x) = k \end{array}$$

On the other hand, saying that $Dr(w) \neq 0$ is equivalent to say that we can lift the non zero ring homomorphism $u^\natural : A_{red} \to k[\epsilon]/\epsilon^2$ to a non zero ring homomorphism

$\tilde{u}^\natural : A \to k[\epsilon]/\epsilon^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \stackrel{\tilde{\lambda}^\natural}{\to} & k(x) = k \\ r^\natural \downarrow & \tilde{u}^\natural \searrow & \uparrow i^\natural \\ A_{red} & \stackrel{u^\natural}{\to} & k[\epsilon]/\epsilon^2 \\ & \lambda^\natural \searrow & \downarrow i^\natural \\ & & k(x) = k \end{array}$$

It is clear that we can always do such a lifting, since the homomorphisms $\tilde{u}^\natural$ and $\tilde{\lambda}^\natural$ are just given precomposing the corresponding homomorphisms from A_{red}, with $r^\natural$. Moreover, since $r^\natural$ is a non zero ring homorphism, it turns out that if $u^\natural \neq 0$, then also $\tilde{u}^\natural \neq 0$ and the previous diagram is commutative. This implies that $Dr(w) \neq 0$ and thus that $Dr : T_{p(C)}Hilb(d,g,r)_{red} \hookrightarrow T_{p(C)}Hilb(d,g,r)$ is injective. Reasoning as in the proof of Proposition 4, we can find a curve $\gamma \subset Hilb(d,g,r)_{red}$ through $p(C)$ in such a way that $D\phi \circ Dr(T_{p(C)}\gamma) \neq 0$. Thus the image of this curve via $\phi \circ r$ contains the point $[C]$ in U^0_{BN} and a tangent direction. On the other hand the image via $\phi \circ r$ of a reduced scheme can not be a non reduced point (always because we can represent a neighbourhood of $[C]$ in $\mathcal{M}_g$ as an affine scheme and consider $\phi \circ r$ locally as a morphism of affine schemes). Thus the image of γ through $\phi \circ r$ must have positive dimension and in this way we conclude if $p(C)$ is a smooth point of $Hilb(d,g,r)_{red}$.

Finally, if $p(C)$ is a reducible singularity of $Hilb(d,g,r)_{red}$, it will be sufficient to repeat the previous reasoning, substituting $T_{p(C)}Hilb(d,g,r)_{red}$, with $T_{p(C)}H$, where H is an irreducible component of $Hilb(d,g,r)_{red}$ through $p(C)$, smooth at $p(C)$ and of maximal dimension, so that $dim_{p(C)}H = dim_{p(C)}Hilb(d,g,r)_{red} = dim_{p(C)}Hilb(d,g,r)$. In the same way, one can find a smooth curve $\gamma \subset H$, through $p(C)$, such that its image in $\mathcal{M}_g$ is positive dimensional, arguing again as in the proof of Proposition 4 (the image of γ has to be a reduced scheme, hence necessarily positive dimensional, in order to have $D\phi \neq 0$).

□

REMARK 1. If $p(C)$ is a reducible singularity of $Hilb(d,g,r)_{red}$, for the Theorem 2 to work, it is not necessary that *all* irreducible components of $Hilb(d,g,r)_{red}$ through $p(C)$ are smooth in a neighbourhood of $p(C)$. Indeed, from the proof of Theorem 2, it is clear that it is sufficient that there exists an irreducible component of maximal dimension H of $Hilb(d,g,r)_{red}$, which is smooth at $p(C)$.

In the light of the previous theorem, let us discuss Mumford's famous example of a component of the restricted Hilbert scheme which is non reduced (see [15]). He considered smooth curves C on smooth cubic surfaces S in $\mathbb{P}^3$, belonging to the complete linear system $|4H + 2L|$, where H is the divisor class of a hyperplane section of S and L is the class of a line on S. It is immediate to see that the degree of such a curve is $d = 14$ and that its genus is $g = 24$. Therefore we are working with $Hilb(14, 24, 3)$. In [15], it is proved that the sublocus J_3 of $Hilb(14, 24, 3)$ cut out by curves C of this type, is dense in a component of the Hilbert scheme. Moreover, it turns out that this component is non reduced. Indeed, Mumford showed that the dimension of $Hilb(14, 24, 3)$ at the point $p(C)$ representing a curve C of the type just described, is 56, while the dimension of the tangent space to $Hilb(14, 24, 3)$ at $p(C)$ is 57. On the other hand, in [7] it is proved that for the points of type $p(C)$ an infinitesimal deformation (i.e. a deformation over $Spec(k[\epsilon]/\epsilon^2)$) is either obstructed at the second order (i.e. you can not lift the deformation to $Spec(k[\epsilon]/\epsilon^3)$), or at no order at all. This implies that the corresponding component of $Hilb(14, 24, 3)_{red}$ is smooth. Since for curves of this type, we have that $d > \frac{g+3}{2}$, by Proposition 1 we know that $D\phi \neq 0$. If $[C] \in \mathcal{M}_g$ is a smooth point, then by Theorem 2, being $Hilb(14, 24, 3)_{red}$ smooth at $p(C) \in J_3$, we have that the curve $C \hookrightarrow \mathbb{P}^3$ is non rigid for the given embedding.

For other interesting examples of singularities of Hilbert schemes of curves and related constructions, see [9], [14], [5] and [17].

In the light of Theorem 2, it would be extremely interesting to give an example of a smooth curve of genus $g \geq 2$, which is rigid for some embedding. Unfortunately, this is a difficult task; indeed, one of the main motivation for this paper was to prove that no such a curve exists. However, we did not succeed in proving this, and we prove a weaker statement (essentially Thoerem 2). This is strictly related to a question posed by Ellia: is there any component of the Hilbert scheme of curves of genus $g > 0$ in $\mathbb{P}^n$, which is the closure of the action of $Aut(\mathbb{P}^n)$? For this and related question see: [3], [4], [6] and [2].

4. Some special classes of curves in $\mathbb{P}^3$

In this section, we take into account some special classes of curves and prove that they are non-rigid at the first order or even non-rigid for the given embedding. As a first example, let us consider a projectively normal curve C in $\mathbb{P}^3$, which does not sit on a quadric or on a cubic. We prove that the curves of this class are non-rigid at the first order. Their ideal sheaf has a resolution of the type (with $a_j \geq 4$ and consequently $b_j \geq 5$):

$$0 \to \oplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^3}(-b_j) \to \oplus_{j=1}^{s+1} \mathcal{O}_{\mathbb{P}^3}(-a_j) \to \mathcal{I}_C \to 0,$$

from which, twisting with $T\mathbb{P}^3$, we get:

$$0 \to \oplus_{j=1}^{s} T\mathbb{P}^3(-b_j) \to \oplus_{j=1}^{s+1} T\mathbb{P}^3(-a_j) \to T_{\mathbb{P}^3} \otimes \mathcal{I}_C \to 0. \tag{7}$$

On the other hand, from the Euler sequence (suitably twisted) we have that $h^0(T\mathbb{P}^3(-k)) = 0$ and $h^1(T\mathbb{P}^3(-k)) = 0$ for $k \geq 4$. Thus, from (7) it follows that $h^0(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Moreover, $H^2(T\mathbb{P}^3(-b_j))$ is equal by Serre duality to $H^1(\Omega^1_{\mathbb{P}^3}(b_j - 4))^*$ and this is zero by Bott formulas (see for example [16]), since we assumed $b_j \geq 5$. Therefore, again from (7), it follows that $h^1(T\mathbb{P}^3 \otimes \mathcal{I}_C) = 0$. Finally, from the defining sequence of C, twisting by $T\mathbb{P}^3$, we get that $H^0(T\mathbb{P}^3) \cong H^0(T\mathbb{P}^3|_C)$. Now, $h^0(T\mathbb{P}^3) = 15$, so that $D\phi \neq 0$ as soon as $15 < 4d$ (recall that $h^0(N_{C/\mathbb{P}^3}) \geq 4d$), that this $D\phi \neq 0$ for $d \geq 4$. Now, recall the important fact that if C is a projectively normal curve, then $Hilb(\mathbb{P}^3)$ is smooth at the corresponding point $p(C)$ (see [10]) and this implies that the projectively normal curve is non rigid (Theorem 2) as soon as it does not sit on a quadric or a cubic surface.

Now we consider a projectively normal curve which sits on a smooth cubic surface S in $\mathbb{P}^3$ and prove that this curve is non-rigid at the first order and hence non-rigid always by Theorem 2 and by the result of [10]. From the exact sequence:

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_S|_C \to 0, \tag{8}$$

since $N_S|_C \cong \mathcal{O}_C(3)$ and $N_{C/S} \cong \omega_C \otimes \omega_S^{-1} \cong \omega_C(1) \cong \mathcal{O}_C(C)$, we get $\chi(N_{C/\mathbb{P}^3}) = \chi(\omega_C(1)) + \chi(\mathcal{O}_C(3))$. By Riemann-Roch $\chi(\mathcal{O}_C(3)) = 3d - g + 1$ and by Serre duality$h^1(\omega_C(1)) = h^0(\mathcal{O}_C(1)) = 0$, so that $\chi(\omega_C(1)) = C^2 + 1 - g$ and $\chi(N_{C/\mathbb{P}^3}) = 3d - g + 1 + h^0(\omega_C(1)) = 3d - 2g + 2 + C^2$. Again from the sequence (8), taking cohomology, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(3))$. On the other hand, from the exact sequence:

$$0 \to \mathcal{I}_C(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_C(3) \to 0,$$

assuming that C is projectively normal and that it sits on a unique cubic, we have $1 - 20 + h^1(\mathcal{O}_C(3)) + 3d - g + 1 = 0$, so that $h^1(N_{C/\mathbb{P}^3}) = 18 - 3d + g$. Thus $h^0(N_{C/\mathbb{P}^3}) = \chi(N_{C/\mathbb{P}^3}) + h^1(N_{C/\mathbb{P}^3}) = 20 - g + C^2$. As a remark, notice that since $h^0(N_{C/\mathbb{P}^3}) \geq 4d$, we obtain the inequality $4d \leq 20 - g + C^2$ for curves of this type. To give an estimate of $h^0(T\mathbb{P}^3_{|C})$, we use as before the Riemann-Roch Theorem and the Euler sequence, so that $h^0(T\mathbb{P}^3_{|C}) \leq 4d + 3(1-g) + 4h^1(\mathcal{O}_C(1))$. On the other hand, from the defining sequence of C twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$, assuming C projectively normal and nondegenerate, we get $h^1(\mathcal{O}_C(1)) = g - d + 3$, so that $h^0(T\mathbb{P}^3_{|C}) \leq g + 15$. Thus $D\phi \neq 0$ as soon as $g + 15 < 20 - g + C^2$. Using adjunction formula, i.e. $C.(C + K_S) = 2g - 2$, we can rewrite this as $C.K_S < 3$. Now, since S is a smooth cubic $K_S \equiv -H$ where H is an effective divisor representing a hyperplane section. Moreover any C is linearly equivalent to $al - \sum b_i e_i$ and $h \equiv 3l - \sum e_i$ (we identify S with $\mathbb{P}^2$ blown-up at 6 points in general position, i.e. no 3 on a line and no 6 on a conic), so that $D\phi \neq 0$ as soon as $3a - \sum b_i > 3$, but $3a - \sum b_i = d$, and so we get the condition $d \geq 4$.

Finally , as an example we consider the case of projectively normal curves on a smooth quadric Q, proving that these curves are non-rigid (indeed it is sufficient to assume that $h^1(\mathcal{I}_C(2)) = 0$). First of all, from the sequence:

$$0 \to N_{C/Q} \to N_{C/\mathbb{P}^3} \to N_Q|_C \to 0,$$

being $N_{C/Q} \cong \omega_C(2)$ and $N_Q|_C \equiv \mathcal{O}_C(2)$, we have that $h^1(N_{C/\mathbb{P}^3}) = h^1(\mathcal{O}_C(2))$; from the defining sequence $0 \to \mathcal{I}_C(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0$, since we assumed $h^1(\mathcal{I}_C(2)) = 0$, we have $1 - 10 + h^1(\mathcal{O}_C(2)) + 2d - g + 1 = 0$. Moreover, by Serre duality and Kodaira vanishing $h^1(\omega_C(2)) = h^1(N_{C/Q}) = 0$ so that $h^0(N_{C/\mathbb{P}^3}) = \chi(\omega_C(2)) + \chi(\mathcal{O}_C(2)) + h^1(N_{C/\mathbb{P}^3})$ and this is equal to $10 - g + C^2$. The previous estimate for $h^0(T\mathbb{P}^3_{|C})$ works also in this case (we just used the fact that C is linearly normal and non degenerate), so that $D\phi \neq 0$ as soon as $g + 15 < 10 - g + C^2$. By adjunction $2g - 2 = C.(C + K_Q)$, and by the fact that $K_Q \equiv -2H$, the inequality $g + 15 < 10 - g + C^2$ can be rewritten as $2C.H > 7$, so that, for $d \geq 4$, C is non rigid at the first order for the given embedding and so they are non-rigid (Theorem 2 and [10]).

Let us take into account the wider class of curves of maximal rank in $\mathbb{P}^3$. By definition a curve C is of maximal rank iff $h^0(\mathcal{I}_C(k))h^1(\mathcal{I}_C(k)) = 0$ for any $k \in \mathbb{Z}$. Since we have already dealt with projectively normal curves, from now on we assume that C is a smooth irreducible curve of maximal rank in $\mathbb{P}^3$, which is not projectively normal. As usual, let $s := min\{k/h^0(\mathcal{I}_C(k)) \neq 0\}$ be the postulation index of C. Observe that $h^1(\mathcal{I}_C(k)) = 0$ for any $k \geq s$, since C is of maximal rank. Thus, having set $c(C) := max\{k/h^1(\mathcal{I}_C(k)) \neq 0\}$, we have that $c(C) \leq s - 1$ ($c(C)$ is called the completeness index).

As a first case, let us consider $c = s - 2$ and assume $h^1(\mathcal{O}_C(s - 2)) = 0$ (which is certainly satisfied if $d(2 - s) + 2g - 2 < 0$ or equivalently $d > \frac{2g-2}{s-2}$, $s \geq 3$). Observe that in this case, C is s-regular, i.e. $h^i(\mathcal{I}_C(s - i)) = 0$ for any $i > 0$. Indeed, from the defining sequence of C, we have that $h^1(\mathcal{O}_C(k)) = h^2(\mathcal{I}_C(k))$ and since $h^1(\mathcal{O}_C(s - 2)) = 0$, we are done. Set $u := h^0(\mathcal{I}_C(s))$. Then, if

$$0 \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{3i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{2i}) \to \oplus\mathcal{O}_{\mathbb{P}^3}(-n_{1i}) \to \mathcal{I}_C \to 0$$

is the minimal free resolution of $\mathcal{I}_C$, setting $n_j^+ := max\{n_{ji}\}$ and $n_j^- := min\{n_{ji}\}$, it is easy to see that $n_3^+ = c + 4 = s + 2$. Moreover, we have $n_3^+ > n_2^+ > n_1^+$, $n_3^- > n_2^- > n_1^-$ and also $n_3^+ = s + 2 > n_2^+ \geq n_2^- > n_1^- = s$. From these we get $n_2^+ = n_2^- = s + 1$, that is $n_{2i} = s + 1$ for any i. Analogously, one gets $n_{3i} = s + 2$ for any i. Thus, in this case, the minimal free resolution is

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s - 2) \to x\mathcal{O}_{\mathbb{P}^3}(-s - 1) \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{9}$$

(resolution of the first kind), where $y = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s - 2))$. If we have a resolution of the first kind, we can split it as follows:

$$0 \to y\mathcal{O}_{\mathbb{P}^3}(-s - 2) \to x\mathcal{O}_{\mathbb{P}^3}(-s - 1) \to E \to 0, \tag{10}$$

$$0 \to E \to u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0 \tag{11}$$

where E is only a locally free sheaf (indeed, if it were free, then C would be projectively normal by (11)). Twisting (10) and (11) by $T\mathbb{P}^3$ and taking cohomology, we get:

$$0 \to uH^0(T\mathbb{P}^3(-s)) \to H^0(\mathcal{I}_C \otimes T\mathbb{P}^3) \to H^1(E \otimes T\mathbb{P}^3) \to \dots \tag{12}$$

$$\dots \to xH^1(T\mathbb{P}^3(-s - 1)) \to H^1(E \otimes T\mathbb{P}^3) \to yH^2(T\mathbb{P}^3(-s - 2)) \to \dots \tag{13}$$

On the other hand, in the sequence (13), $h^1(T\mathbb{P}^3(-s - 1)) = h^2(\Omega^1_{\mathbb{P}^3}(s - 3)) = 0$ by Serre duality and Bott formulas, while $h^2(T\mathbb{P}^3(-s - 2)) = h^1(\Omega^1_{\mathbb{P}^3}(s - 2)) = 0$, if $s \geq 3$. Thus, we get that if $s \geq 3$, then $H^1(E \otimes T\mathbb{P}^3) = 0$. Moreover, twisting the Euler sequence with $\mathcal{O}_{\mathbb{P}^3}(-s)$, we obtain that $h^0(T\mathbb{P}^3(-s)) = 0$ as soon as $s \geq 2$. Therefore, from the sequence (12), we have that $h^0(\mathcal{I}_C \otimes T\mathbb{P}^3) = 0$ as soon as $s \geq 3$.

Now, twisting the defining sequence of C by $T\mathbb{P}^3$ and taking cohomology, we get (assuming $s \geq 3$):

$$0 \to H^0(T\mathbb{P}^3) \to H^0(T\mathbb{P}^3_{|C}) \to H^1(\mathcal{I}_C \otimes T\mathbb{P}^3) \to 0. \tag{14}$$

We want to give an estimate to $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3)$. Continuing the long exact cohomology sequence (12), using again Serre duality and Bott formulas and assuming $s \geq 5$, we get that $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3) = h^2(E \otimes T\mathbb{P}^3)$. Moreover, going on with the sequence (13), applying Serre duality and Bott formulas ($s \geq 5$), we obtain $h^2(E \otimes T\mathbb{P}^3) \leq yh^3(T\mathbb{P}^3(-s-2)) = \frac{ys(s-1)(s-3)}{2}$. Hence $h^1(\mathcal{I}_C \otimes T\mathbb{P}^3) = \frac{ys(s-1)(s-3)}{2}$ and from (14) we get $h^0(T\mathbb{P}^3) \leq 15 + \frac{ys(s-1)(s-3)}{2}$, $s \geq 5$. Thus, if $4d > 15 + \frac{ys(s-1)(s-3)}{2}$, or equivalently $d \geq 4 + \frac{ys(s-1)(s-3)}{8}$, $s \geq 5$, then a curve C of maximal rank, with a resolution of the first kind and with $h^1(\mathcal{O}_C(s-2)) = 0$, is non rigid at the first order for the given embedding.

As a final example, let us consider a curve C of maximal rank, such that $h^0(\mathcal{I}_C(s)) \leq 2$ and $h^1(\mathcal{O}_C(s-3)) = h^1(\mathcal{O}_C(s-2)) = h^1(\mathcal{O}_C(s-1)) = h^1(\mathcal{O}_C(s)) = 0$ (this happens for example if $d > \frac{2g-2}{s}$ and assuming $s \geq 4$). In this case, we have $c(C) = s-1$. Indeed, if it were $c < s-2$, then C would be (s-1)-regular and this would contradict the fact that s is the postulation. Moreover, if it were $c = s-2$, then C would be s-regular and since $h^0(\mathcal{I}_C(s)) \leq 2$, C would be a complete intersection of type (s, s), and in particular it would be projectively normal.

Thus, $c(C) = s-1$ and from the given hypotheses, the fact that $h^2(\mathcal{I}_C(s-1)) = h^1(\mathcal{O}_C(s-1)) = 0$, and $h^1(\mathcal{I}_C(s)) = 0$ (since $c(C) = s-1$), it is easy to see that C is (s+1)-regular. This implies that the homogeneous ideal $I(C)$ is generated in degree less or equal to $s+1$. With notation as above, we have $n_3^+ = c+4 = s+3 > n_2^+ > n_1^+ = s+1$, where the last equality holds since $I(C)$ is generated in degree less or equal to $s+1$. From this, we get $n_2^+ = s+2$ and moreover $n_2^- > n_1^- = s$ so that $n_2^- \geq s+1$. On the other hand, we can say more, because the map $H^0(\mathcal{I}_C(s)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{I}_C(s+1))$ is injective; indeed, $h^0(\mathcal{I}_C(s)) \leq 2$ and from a relation of the form $H_1F_s = H_2F'_s$ between the two generators in degree s, we would have that $H_1|F'_s$ but this is clearly impossible. It turns out that we have no relations in degree (s+1) between the generators of $I(C)$. Thus $n_2^- > s+1$, $n_3^- > n_2^- \geq s+2$, so that $n_{3i} = s+3$ for any i and also $n_{2i} = s+2$ for any i.

Hence,in this case, the minimal free resolution of $\mathcal{I}_C$ is the following:

$$\begin{aligned} 0 \to v\mathcal{O}_{\mathbb{P}^3}(-s-3) \to x\mathcal{O}_{\mathbb{P}^3}(-s-2) \to \\ \to w\mathcal{O}_{\mathbb{P}^3}(-s-1) \oplus u\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_C \to 0, \end{aligned} \tag{15}$$

(resolution of the second kind), where $v = h^1(\mathcal{I}_C(c)) = h^1(\mathcal{I}_C(s-1))$. In this case, that is under the following hypotheses: $s \geq 5$, $h^1(\mathcal{O}_C(k)) = 0$ for $k = s, s-1, s-2, s-3$ (which is satisfied if for example $d > \frac{2g-2}{s}$, $s \geq 4$), $h^0(\mathcal{I}_C(s)) \leq 2$ ($c = s-1$), we start from the sequence (15) and we get that C is non rigid at the first order for the given embedding as soon as $d \geq 4 + \frac{vs(s-2)(s+1)}{8}$. We leave to the interested reader the details of this case, which is completely analogous to the previous one.

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