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A NOTE ON THE HILBERT SCHEME OF CURVES OF DEGREE d AND GENUS $\binom{d-3}{2} - 1$

Abstract. This note is inspired by a lecture given during the school "Liason theory and related topics" and contains a summary of the results in [15] about the connectedness of the Hilbert scheme of curves of degree d and genus $\binom{d-3}{2} - 1$. The only novelty is the list of degrees for which smooth and irreducible curves appear.

This short note was inspired by a talk I gave at the Politecnico of Torino during the School *"Liaison theory and related topics"*. The question of the connectedness of the Hilbert schemes $H_{d,g}$ of locally Cohen–Macaulay curves $\mathcal{C} \subset \mathbb{P}^3$ of degree d and arithmetic genus g arose naturally after Hartshorne proved in his PhD thesis that the Hilbert scheme of all one dimensional schemes with fixed Hilbert polynomial is connected. The result is somewhat too general since, even to connect one smooth curve to another, it involves curves with embedded or isolated points. On the other hand, if the question is addressed under the more restrictive hypothesis of smooth curves, then the Hilbert scheme need not be connected: a counterexample can be found for $(d, g) = (9, 10)$. In the recent years, after the developing of liaison theory, it has become clear that, even though one can be interested in the classification of smooth curves, the natural class to look at is the class of locally Cohen–Macaulay curves, i.e. the class of schemes of equidimension 1 with all their local rings Cohen–Macaulay. In other words, they are 1 dimensional schemes with no embedded or isolated points. The answer to the question in case of locally Cohen–Macaulay curves is known, so far, only for low degrees or high genera. The scheme $H_{d,g}$ is non empty when $d \geq 1$ and $g = \binom{d-1}{2}$ (that corresponds to the case of plane curves), or $d > 1$ and $g \leq \binom{d-2}{2}$. After the paper [9], it is well known that $H_{d,g}$ contains an irreducible component consisting of extremal curves (i.e. curves having the largest possible Rao function). This is the only component for $d \geq 5$ and $(d-3)(d-4)/2 + 1 < g \leq (d-2)(d-3)/2$ while in the cases $d \geq 5$, $g = (d-3)(d-4)/2 + 1$ and $d \geq 4$, $g = (d-3)(d-4)/2$ the Hilbert scheme is not irreducible, but it is connected (see [1], [12]). The connectedness is trivial for $d \leq 2$ since the scheme is irreducible, see [5], while it has been proved for $d = 3$, $d = 4$ and any genus in [11], [13] respectively. Note that for $d = 3, 4$ there is a large number of irreducible components: they are approximatively $\frac{1}{3}|g|$ for $d = 3$ and $\frac{1}{24}g^2$ for $d = 4$. The paper [4] has given a new light to the problem, in fact Hartshorne provides some methods to connect particular classes of curves to the irreducible component of extremal curves, while in the paper [14] Perrin has proved that all the curves whose Rao module is Koszul can be connected to the components of extremal curves. This note deals with the first unknown case for high genus, i.e. $\tilde{g} = (d-3)(d-4)/2 - 1$ and its purpose is to give an overview of the results in the forthcoming [15]. Since it contains only a brief state of the art, for a more complete treatment of the topic the reader is referred to [4], [5].

In [15] we have studied the connectedness of the Hilbert scheme $H_{d,\tilde{g}}$ of locally Cohen–Macaulay curves in $\mathbb{P}^3 = \mathbb{P}^3_k$, where k is an algebraically closed field of characteristic zero. A way one can follow to prove the connectedness of $H_{d,\tilde{g}}$, is to first identify its irreducible components for every d and then to connect them to extremal curves using [4] and its continuation [18]. Following this idea, we used the so called *spectrum* of a curve (see [16], [17]) to find all the possible Rao functions and then all the possible Rao modules occurring for curves in $H_{d,\tilde{g}}$. For $d \geq 9$ it is possible to show that there are only four possible modules (see [15], Theorem 3.3) and that each of them characterizes an irreducible family of curves. Those families turn out to be the components of $H_{d,\tilde{g}}$ and their general member is described in the following:

THEOREM 1. *The Hilbert scheme $H_{d,\tilde{g}}$ of curves of degree $d \geq 9$ and genus $\tilde{g}$ has four irreducible components:*

1. *The family H_1 of extremal curves, whose dimension is $\frac{d(d+5)}{2} - 1$.*
2. *The closure H_2 of the family of subextremal curves whose general member is the disjoint union of two plane curves of degrees $d - 2$ and 2. The dimension of H_2 is $\frac{d(d-1)}{2} + 10$.*
3. *The closure H_3 of the family of curves whose general member is obtained by a biliaison of height 1 on a surface of degree $d - 2$ from a double line of genus -2 and corresponds to the union of a plane curve $\mathcal{C}_{d-2}$ of degree $d - 2$ with a double line of genus -2 intersecting $\mathcal{C}_{d-2}$ in a zero–dimensional subscheme of length 2. The dimension of H_3 is $\frac{d(d-1)}{2} + 9$.*
4. *The closure H_4 of the family of curves whose general member is the union of a plane curve $\mathcal{C}_{d-2}$ of degree $d - 2$ with two skew lines, one of them intersecting transversally $\mathcal{C}_{d-2}$ in one point. The dimension of H_4 is $\frac{d(d-1)}{2} + 9$.*

For curves of degree $d \leq 8$ we have that the Hilbert scheme $H_{d,g}$ with $d = 2$, $g \leq 0$ is irreducible hence connected, while the case $d = 3$ and the case $d = 4$ were studied for all the possible values of the genus in [10] and [13] respectively. Finally, $H_{5,0}$ was dealt by Liebling in his PhD thesis [7]. Then we only have to consider $(d, g) \in \{(6, 2), (7, 5), (8, 9)\}$. In these cases, we have proved that the Rao modules of the type occurring for $d \geq 9$ are still possible but the spectrum allows more possibilities that were determined using the notion of *triangle* introduced by Liebling in [7]. Each Rao module is associated to a family of curves that is not necessarily a component of the Hilbert scheme $H_{d,\tilde{g}}$ as it appears clear by looking at their dimension (see [15], Theorem 4.3 and 4.5). The components of the Hilbert scheme are listed in the following

THEOREM 2. *The Hilbert schemes $H_{6,2}$, $H_{7,5}$, $H_{8,9}$ have five components: the four components listed in Theorem 1, moreover*

1. *$H_{6,2}$ contains the closure H_5 of the family of curves in the biliaison class of the disjoint union of a line and a conic.*
2. *$H_{7,5}$ contains the closure of the family H_6 of ACM curves.*
3. *$H_{8,9}$ contains the closure of the family H_7 of ACM curves.*

Now we can state our main result (see [15], Theorem 4.8) whose proof rests on the fact that all the curves in the families listed in the previous Theorems 0.1 and 0.2 can be connected by flat families to extremal curves:

THEOREM 3. *The Hilbert scheme $H_{d,\tilde{g}}$ is connected for $d \geq 3$.*

To complete the description of $H_{d,\tilde{g}}$ given in [15] we specify where smooth and irreducible curves can be found. In what follows, R is the ring $k[X, Y, Z, T]$ and M denotes the Rao module.

PROPOSITION 1. *The Hilbert scheme $H_{d,\tilde{g}}$ contains smooth and irreducible curves if and only if*

1. $d = 5$ *and* M *is dual to a module of the type* $M = R/(X, Y, Z^2, ZT, T^2)$
2. $d = 6$ *and* $M = R/(X, Y, Z, T^2)(-1)$
3. $d = 7$ *and* $M = 0$
4. $d = 8$ *and* $M = 0$.

Proof. By the results of Gruson and Peskine [2] there exist smooth irreducible (non degenerate) curves if and only if either $0 \leq \tilde{g} \leq d(d-3)/6 + 1$ or $d = a + b$, $\tilde{g} = (a-1)(b-1)$ with $a, b > 0$. This implies that either $d = 5, 6, 7$ or $d = 8$, $a = b = 4$. Looking at the possible Rao modules (see [7] for the complete list occurring in the case $d = 5$) the only Rao modules with cohomology compatible with smooth curves are the ones listed.

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References

[1] AIT-AMRANE S., *Sur le schéma de Hilbert des courbes gauches de degré d et genre* $g = (d-3)(d-4)/2$, Ann. Inst. Fourier **50** (6) (2000), 1671–1708.

[2] GRUSON L. AND PESKINE C., *Genre des courbes de l'espace projectif*, Ann. Scient. Ec. Norm. Sup. **15** (1982), 401–418.

[3] HARTSHORNE R., *Connectedness of the Hilbert scheme*, Publ. Math. IHES **29** (1966), 5–48.

[4] HARTSHORNE R., *On the connectedness of the Hilbert scheme of curves in* $\mathbf{P}^3$, Comm. in Algebra **28** (2000), 6059–6077.

[5] HARTSHORNE R., *Questions on the connectedness of the Hilbert scheme of curves in* $\mathbf{P}^3$, preprint (2001).

[6] HARTSHORNE R. AND SCHLESINGER E. *Curves in the double plane*, Comm. in Algebra **28** (2000), 5655–5676.

[7] LIEBLING R., *Classification of space curves using initial ideals*, Berkeley Ph. D. Thesis, 1996.

[8] MARTIN-DESCHAMPS M. AND PERRIN D., *Sur la classification des courbes gauches I*, Astérisque **184–185**, Soc. Math. France 1990.

[9] MARTIN-DESCHAMPS M. AND PERRIN D., *Le schéma de Hilbert de courbes gauches localement de Cohen–Macaulay n'est presque jamais réduit*, Ann. Sci. Ec. Norm. Sup. **29** (1996), 757–785.

[10] NOLLET S., *Subextremal curves*, Man. Math. **94** (1997), 303–317.

[11] NOLLET S., *The Hilbert schemes of degree three curves*, Ann. Sci. Ec. Norm. Sup. **30** (1997), 367–384.

[12] NOLLET S., *A remark on connectedness in Hilbert schemes*, Comm. Alg. **28** (2000), 5745–5747.

[13] NOLLET S. AND SCHLESINGER E., *The Hilbert schemes of degree four curves*, preprint (2001), math.AG/0112167.

[14] PERRIN D., *Un pas vers la connexité du schéma de Hilbert: les courbes de Koszul sont dans la composante des extrémales*, Coll. Math. **52** (3) (2001), 295–319.

[15] SABADINI I., *On the Hilbert scheme of curves of degree d and genus* $\binom{d-3}{2} - 1$, Le Matematiche **LV** (2000), 517–531.

[16] SCHLESINGER E., *The spectrum of projective curves*, Ph. D. Thesis, Berkeley 1996.

[17] SCHLESINGER E., *On the spectrum of certain subschemes of* P^n, J. Pure Appl. Alg. **136** (1999), 267–283.

[18] SCHLESINGER E., *Footnote to a paper by Hartshorne*, Comm. in Algebra **28** (2000), 6079–6083.

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