Rend. Sem. Mat. Univ. Pol. Torino
Vol. 58, 2 (2000)
Geom., Cont. and Micros., II

R. Segev*

NOTES ON STRESSES FOR MANIFOLDS

Abstract. The geometric structure of stress theory on differentiable manifolds is considered. Mechanics is assumed to take place on an m-dimensional and no additional metric or parallelism structure is assumed. Two different approaches are described. The first is a generalisation of the traditional Cauchy approach where the resulting stresses are represented mathematically as vector valued $(m-1)$-forms. The second approach is variational and stresses are represented by densities valued in the dual of the first jet bundle. It is shown how a variational stress induces a Cauchy stress.

1. Introduction

This work describes some issues related to the formulation of stress theory on manifolds. In previous works (see [1, 2, 3, 4]), stress theory for the case where both body and space are modeled by differentiable manifolds rather the traditional Euclidean spaces was developed. In [1] a general weak formulation of stress theory was presented. On the basis of some general guidelines (see the motivation for the introduction of variational stresses below), stresses were presented as measures on the body manifold valued in the dual of a jet bundle. Such a stress measure represents a force using a representation theorem for the force functional. In that work, assuming that the stress measures may be represented by smooth densities, the additional geometric structure of a connection was used in order to allow the representation of a force by a body force field and a surface force field. In the sequel, we will refer to this approach as the variational approach. In the more recent works, [2, 3] stress theory was presented on manifolds without any additional geometric structure (e.g., a connection) from a point of view that is analogous to the classical Cauchy theory of stresses. In [2] the theory was presented for the case of scalar valued quantities and in [3] the theory was extended to forces. We will refer to this method as the generalized Cauchy approach. In [4], some aspects of the relation between the Cauchy approach and the variational approach were considered.

After a presentation of the generalized Cauchy approach in Section 2, Section 3 is concerned with the Cauchy postulates given in [3]. It is shown that the boundedness postulate in [3], that is a generalization of the balance of momentum in the traditional formulation, is not general enough. A revised version of the boundedness postulate is suggested and it is shown that the weaker assumption does not alter the proof of the generalized Cauchy theorem in the aforementioned paper.

Sections 4 and 5 review the variational approach and its relation to the generalized Cauchy approach presented in [4]. Section 6 extends this relation and shows how the representation of

*The research leading to this paper was partially supported by the Paul Ivanier Center for Robotics Research and Production Management at Ben-Gurion University.

forces by body forces and surface forces in the Cauchy approach is completely equivalent to the representation of forces by variational stress densities in the variational approach.

2. Cauchy's stress theory for manifolds

Let $\pi : W \to \mathcal{U}$ be a vector bundle over the m-dimensional orientable manifold $\mathcal{U}$. It is assumed that a particular orientation is chosen on $\mathcal{U}$. The vector bundle is interpreted as the bundle of generalized velocities over $\mathcal{U}$. The manifold $\mathcal{U}$ is interpreted as the universal body and the vector bundle is interpreted as the bundle of generalized velocities over $\mathcal{U}$. Cauchy's stress theory for manifolds, presented in [3], considers for each compact m-dimensional submanifold with boundary $\mathcal{R}$ of $\mathcal{U}$ linear functionals of the generalized velocity fields containing a volume term and a boundary term of the form

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w).$$

Here, using the notation $\bigwedge^p(T^*X)$ for the bundle of p-forms on a manifold X, w is a section of W, $\mathbf{b}_{\mathcal{R}}$, the *body force*, is a section of $L\big(W, \bigwedge^m(T^*\mathcal{R})\big)$ and $\mathbf{t}_{\mathcal{R}}$ the *boundary force* is a section of $L\big(W, \bigwedge^{m-1}(T^*\partial\mathcal{R})\big)$. The functional $F_{\mathcal{R}}$ is interpreted as the force, or power, functional and the value $F_{\mathcal{R}}(w)$ is classically interpreted as the power of the force for the generalized velocity field w.

Cauchy's postulates for the force system $\{F_{\mathcal{R}} = (\mathbf{b}_{\mathcal{R}}, \mathbf{t}_{\mathcal{R}})\}$ presented in [3] may be summarized as follows.

(*i*) For every $x \in \mathcal{U}$ and every body $\mathcal{R}$, $\mathbf{b}_{\mathcal{R}}(x) = \mathbf{b}(x)$, that is, the value of the body force at a point is independent of the body containing it. Accordingly, we will omit the subscript $\mathcal{R}$.

(*ii*) Let us consider the Grassmann bundle of hyperplanes $G_{m-1}(T\mathcal{U}) \to \mathcal{U}$ whose fiber $G_{m-1}(T_x\mathcal{U})$ at any point $x \in \mathcal{U}$ is the Grassmann manifold of hyperplanes, i.e., $(m-1)$-dimensional subspaces of the tangent space $T_x\mathcal{U}$. Let

$$L\Big(W, \bigwedge^{m-1} G_{m-1}(T\mathcal{U})^*\Big) \to G_{m-1}(T\mathcal{U})$$

be the vector bundle over $G_{m-1}(T\mathcal{U})$ whose fiber over a hyperplane $H \subset T_x\mathcal{U}$ is the vector space of linear mappings $L\big(W_x, \bigwedge^{m-1} H^*\big)$. Then, the dependence of $\mathbf{t}_{\mathcal{R}}$ on $\mathcal{R}$ is via a smooth section

$$\Sigma : G_{m-1}(T\mathcal{U}) \to L\Big(W, \bigwedge^{m-1} G_{m-1}(T\mathcal{U})^*\Big),$$

the *Cauchy section*, such that $\mathbf{t}_{\mathcal{R}} = \Sigma(H)$ where $H = T_x\partial\mathcal{R}$.

(*iii*) The Cauchy section Σ is continuous.

(*iv*) There is a section ζ of $L\big(W, \bigwedge^m(T^*\mathcal{U})\big)$ such that

$$\big|F_{\mathcal{R}}(w)\big| = \left|\int_{\mathcal{R}} \mathbf{b}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| \le \int_{\mathcal{R}} \zeta(w)$$

for every body $\mathcal{R}$.

Using the results of [2], it is shown in [3] that there is a unique section σ of $L\big(W, \bigwedge^{m-1}(T^*\mathcal{U})\big)$ called the *Cauchy stress* such that

$$\mathbf{t}_{\mathcal{R}}(w)(v_1, \ldots, v_{m-1}) = \sigma(w)(v_1, \ldots, v_{m-1}),$$

for any collection of $m-1$ vectors $(v_1, \ldots, v_{m-1}) \in T_x\partial\mathcal{R},\ x \in \partial\mathcal{R}$, where the dependence on x was omitted in order to simplify the notation. Using the notation $\iota\colon \partial\mathcal{R} \to \mathcal{U}$ for the natural inclusion mapping, so that $\iota^*\colon \bigwedge^{m-1}(T^*\mathcal{U}) \to \bigwedge^{m} -1\big(T^*\partial\mathcal{R}\big)$ is the restriction of forms, we may write $\mathbf{t}_{\mathcal{R}}(w) = \iota^*\big(\sigma(w)\big)$ which we will also write as $\mathbf{t}_{\mathcal{R}} = \iota^*(\sigma)$—the generalized Cauchy formula. We will refer to this result as the *generalized Cauchy theorem*.

Assume that (x^i, w^α) are local vector bundle coordinates in a neighborhood $\pi^{-1}(U) \subset W$, $U \subset \mathcal{U}$ with local basis elements $\{W^\alpha e_\alpha\}$ so a section of W is represented locally by $w^\alpha W^\alpha e_\alpha$. Then, denoting the dual base vectors by $\{W^\alpha e_\alpha\}$ a stress σ is represented locally by

$$\sigma_{\alpha 1\ldots\hat{k}\ldots m} W^\alpha e_\alpha \otimes dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^m,$$

where a "hat" indicates the omission of an item (an index or a factor). The value of $\sigma(w)$ is represented locally by

$$\sigma_{\alpha 1\ldots\hat{k}\ldots m} w^\alpha dx^1 \wedge \ldots \wedge \widehat{dx^k} \wedge \ldots \wedge dx^m.$$

3. The revised boundedness postulate

If we substitute the generalized Cauchy formula into the expression for $F_{\mathcal{R}}(w)$ we obtain

$$\begin{aligned} F_{\mathcal{R}}(w) &= \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w) \\ &= \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \iota^*\big(\sigma(w)\big) \\ &= \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \int_{\mathcal{R}} d\big(\sigma(w)\big), \end{aligned}$$

where Stokes' theorem was used in the last line. It is clear form the local expression for $\sigma(w)$ that the exterior derivative $d\sigma(w)$ depends on the derivative of w an not only on the local value of w. In other words, $F_{\mathcal{R}}(w)$ is a local linear functional on the first order jet $j^1(w)$.

Using the observation that $F_{\mathcal{R}}$ should be a local linear functional on the first jet of w, we replace the boundedness postulate (*iv*) by the following

Revised boundedness postulate

There is a section S of $L\big(J^1(W), \bigwedge^m(T^*\mathcal{U})\big)$ such that

$$\big|F_{\mathcal{R}}(w)\big| = \left|\int_{\mathcal{R}} \mathbf{b}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| \le \int_{\mathcal{R}} \left|S\big(j^1(w)\big)\right|,$$

where the absolute value of an m-form θ, $S\big(j^1(w)\big)$ in this case, is given as

$$|\theta(x)| = \begin{cases} \theta(x) & \text{if } \theta(x) \text{ is positively oriented,} \\ -\theta(x) & \text{if } \theta(x) \text{ is negatively oriented} \end{cases}$$

relatively to the orientation chosen on $\mathcal{U}$.

It is noted that the revised boundedness postulate may also be written as

$$\left|\int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| \leq \int_{\mathcal{R}} \left|S_0\big(j^1(w)\big)\right|,$$

for some section S_0 of $L\big(J^1(W), \bigwedge^m(T^*\mathcal{U})\big)$. This follows from

$$-\left|\int_{\mathcal{R}} \mathbf{b}(w)\right| + \left|\int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| \leq \left|\int_{\mathcal{R}} \mathbf{b}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| \leq \int_{\mathcal{R}} \left|S\big(j^1(w)\big)\right|$$

so

$$\begin{aligned}
\left|\int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)\right| &\leq \int_{\mathcal{R}} \left|S\big(j^1(w)\big)\right| + \left|\int_{\mathcal{R}} \mathbf{b}(w)\right| \\
&\leq \int_{\mathcal{R}} \left|S\big(j^1(w)\big)\right| + \int_{\mathcal{R}} \left|\mathbf{b}(w)\right| \\
&= \int_{\mathcal{R}} \Big(\left|S\big(j^1(w)\big)\right| + \left|\mathbf{b}(w)\right|\Big) \\
&\leq \int_{\mathcal{R}} \left|S_0\big(j^1(w)\big)\right|,
\end{aligned}$$

for some S_0.

For an arbitrary $x \in \mathcal{U}$ we want to show that

$$\mathbf{t}_{\mathcal{R}}(w) = \Sigma(T_x\partial\mathcal{R})(w) = \iota^*\big(\sigma(w)\big),$$

for a unique element of $L\big(W_x, \bigwedge^m -1(T_x\mathcal{U})\big)$, where in the equation above we omitted the dependence on x.

Just as in [3], the proof the generalized Cauchy theorem is based on the following points:

(a) The assertion is local and written in an invariant form and hence it may be proved in any vector bundle chart.

(b) Using a local basis $\{W^\alpha e_\alpha\}$ for the neighbohood where the vector bundle chart is used, any vector $w \in W_x$ may be expressed in the form $w = w^\alpha W^\alpha e_\alpha$, so $\mathbf{t}_{\mathcal{R}}(w) = w^\alpha \tau_{\mathcal{R}\alpha}$, where, $\tau_{\mathcal{R}\alpha} = \mathbf{t}_{\mathcal{R}}(W^\alpha e_\alpha)$.

(c) For the local vector field $W^\alpha e_\alpha$ in the chart neighborhood of x, the scalar valued extensive property given by the volume term $\beta_\alpha = \mathbf{b}(W^\alpha e_\alpha)$, the flux density term $\tau_{\mathcal{R}\alpha} = \mathbf{t}_{\mathcal{R}}(W^\alpha e_\alpha)$, and the source term $s_\alpha = \left|S\big(j^1(Wase_\alpha)\big)\right|$ satisfies the generalized Cauchy postulates for scalar valued quantities (see [2]). In particular, it is noted that if $S\big(j^1(w)\big)$ is represented locally by

$$S\big(j^1(w)\big)_{\alpha 1\ldots m} dx^1 \wedge \ldots \wedge dx^m = \big(S_{\alpha 1\ldots m} w^\alpha + S^i_{\alpha 1\ldots m} w^\alpha_{,i}\big) dx^1 \wedge \ldots \wedge dx^m$$

(the components dual to w^α and those dual to w^α_i differ in notation only by the number of indices), then, $s_\alpha = |S_{\alpha 1\ldots m}|$. Hence, by the Cauchy theorem for scalars [2], there is a unique collection of (dim W_x) $(m-1)$-forms σ_α such that $\tau_{\mathcal{R}\alpha} = \iota^*(\sigma_\alpha)$. These forms represent $\sigma(x) \in L\big(W_x, \bigwedge^{m-1} T_x\partial\mathcal{R}\big)$ in the given chart.

4. Variational stress densities

Let $\pi : W \to \mathcal{U}$ be a vector bundle as in the previous section. A *variational stress density* is a section of $L\big(J^1(W)_1 \bigwedge^m(T^*\mathcal{U})\big)$, where $J^1(W)$ is the first jet bundle associated with W.

For the vector bundle coordinate system (x^i, w^α), $i = 1, \ldots, m$, $\alpha = 1, \ldots, \dim(W_x)$, the jet of a section is represented locally by the functions $\{w^\alpha(x^i), w^\beta_{,j}(x^k)\}$, where a subscript following a comma indicates partial differentiation. A variational stress density will be represented locally by the functions $\{S_{\alpha 1\ldots m}, S^j_{\beta 1\ldots m}\}$ so that the single component of the m-form $S\big(j^1(w)\big)$ in this coordinate system is

$$S\big(j^1(w)\big)_{1\ldots m} = S_{\alpha 1\ldots m} w^\alpha + S^j_{\beta 1\ldots m} w^\beta_{,j}.$$

Note that the notation distinguishes between the components of S that are dual to the values of the section and those dual to the derivatives by the number of indices only. The next few paragraphs motivate the introduction of variational stress densities.

The rational behind the generalized variational formulation of stress theory is the framework for mechanical theories where a configuration manifold is constructed for the system under consideration, generalized velocities are defined as elements of the tangent bundle to the configuration manifold, and generalized forces are defined as elements of the cotangent bundle of the configuration space. For the mechanics of continuous bodies, a configuration is an embedding of the body $\mathcal{R}$ in space $\mathcal{M}$. The natural topology for the collection of embeddings is the C^1-topology for which the collection of embeddings is open in the collection of all C^1-mappings of the body into space. Using this topology, the tangent space to the configuration manifold at the configuration $\kappa : \mathcal{R} \to \mathcal{M}$ is $C^1\big(\kappa^*(T\mathcal{M})\big)$, the Banachable space of C^1-sections of the pullback $\kappa^*(T\mathcal{M})$. Thus forces in continuum mechanics are elements of $C^1\big(\kappa^*(T\mathcal{M})\big)^*$ – linear functionals on the space of differentiable vector fields equipped with the C^1-topology.

The basic representation theorem (see [1]) states that a force functional $F \in C^1\big(\kappa^*(T\mathcal{M})\big)^*$ may be represented by measures on $\mathcal{U}$ - the *variational stress measures* - valued in $J^1(\kappa^*(T\mathcal{M}))^*$, the dual of the first jet bundle $J^1(\kappa^*(T\mathcal{M})) \to \mathcal{U}$. Thus, the evaluation of a force $F_\mathcal{R}$ on the generalized velocity w is

$$F_\mathcal{R}(w) = \int_\mathcal{R} d\mu\big(j^1(w)\big),$$

where μ is the $J^1(\kappa^* T\mathcal{M})^*$-valued measure – a section Schwartz distribution.

Assuming that κ is defined on all the material universe $\mathcal{U}$, we use the notation W for $\kappa^*(T\mathcal{M})$. This vector bundle can be restricted to the individual bodies, and with some abuse of notation, we use the same notation for both the bundle and its restriction to the individual bodies.

Thus, in the smooth case, a variational stress measure is given in terms of a section S of the vector bundle of linear mappings $L\big(J^1(W), \bigwedge^{m-1}(T^*\mathcal{U})\big)$ so

$$F_\mathcal{R}(w) = \int_\mathcal{R} S(j^1(w)).$$

This expression makes sense as $S(j^1(w))$, is an $(m-1)$-form whose value at a point $x \in \mathcal{R}$ is $S(x)\big(j^1(w)(x)\big)$.

Since in the sequel we consider only the smooth case, we will use "variational stresses" to refer to the densities.

5. The Cauchy stress induced by a variational stress

In [4] we defined a canonical mapping

$$p_\sigma : L\big(J^1(W), \bigwedge^m (T^*\mathcal{U})\big) \to L\big(W, \bigwedge^{m-1} (T^*\mathcal{U})\big),$$

that assigns to a variational stress density S a Cauchy stress σ satisfying the following relation. At every $x \in \mathcal{U}$ (we suppress the evaluation at x in the notation)

$$\phi \wedge \sigma(w) = S(j_{\phi\otimes w}).$$

Here, $j_{\phi\otimes w}$ is roughly the jet at x of a section whose value is $0 \in W_x$ and its derivative is $\phi \otimes w$. More precisely, if $u : \mathcal{U} \to W$ is the section whose first jet at x is $j_{\phi\otimes w}$, then, u satisfies the following conditions: $u(x) = 0$; denoting the zero section of W by 0, $T_x u - T_x 0 \in L\big(T_x\mathcal{U}, T_{0(x)} W_x\big)$ induces the linear mapping $\phi\otimes w$ through the isomorphism of $T_{0(x)} W_x$ with W_x. The local representative of p_σ is as follows. If $\sigma = p_\sigma(S)$, then, using the local representatives of σ and S as in the previous sections,

$$\sigma_{\beta 1 \dots \hat{\imath} \dots m} = (-1)^{i-1} S^{+i}{}_{\beta 1 \dots m}, \quad \text{(no sum over } i\text{)}.$$

The mapping p_σ is clearly linear and surjective.

6. The divergence of a variational stress

Given a variational stress density S its generalized divergence Div S is the section of the bundle $L\big(W, \bigwedge^m (T^*\mathcal{U})\big)$ defined by

$$\mathrm{Div} S(w) = d\big(p_\sigma(S)(w)\big) - S\big(J^1(w)\big).$$

The local expression for Div $S(w)$ is

$$\big(S^i_{\alpha 1 \dots m, i} - S_{\alpha 1 \dots m}\big) w^\alpha dx^1 \wedge \dots \wedge dx^m,$$

which shows that Div S depends only on the values of w and not its derivative. With these definitions one obtains for the case where

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} S\big(j^1(w)\big)$$

that

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w)$$

where $\mathbf{t}_{\mathcal{R}}(w) = \iota^*_{\mathcal{R}}\big(\sigma(w)\big)$ and Div $S + \mathbf{b}_{\mathcal{R}} = 0$. We conclude that every variational stress induces a unique force system $\{(\mathbf{b}_{\mathcal{R}}, \mathbf{t}_{\mathcal{R}})\}$ through the Cauchy stress it induces and its divergence. Actually, we obtained a decomposition of $S\big(j^1(w)\big)$ into an exact differential and a term that is linear in the values of w. The converse is also true. If we have a force system that satisfies Cauchy's postulates, then, the induced Cauchy stress enables us to define a section S of $L\big(J^1(W), \bigwedge^{m-1}(T^*\mathcal{U})\big)$ by $S\big(j^1(w)\big) = \mathbf{b}(w) + d\sigma(w)$. Clearly, writing the local expression for S, it is linear in the jet of w. Hence,

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}(w) + \int_{\mathcal{R}} d\sigma(w) = \int_{\mathcal{R}} S\big(j^1(w)\big).$$

If for a given variational stress Div $S = 0$, then $S\big(j^1(w)\big) = d\sigma(w)$, for $\sigma = p_\sigma \circ S$.

References

[1] SEGEV R., *Forces and the existence of stresses in invariant continuum mechanics*, J. of Math. Phys. **27** (1986), 163–170.

[2] SEGEV R., *The geometry of Cauchy's fluxes*, Archive for Rat. Mech. and Anal. **154** (2000), 183–198.

[3] SEGEV R. AND RODNAY G., *Cauchy's theorem on manifolds*, Journal of Elasticity **56** (2000), 129–144.

[4] SEGEV R. AND RODNAY G., *Divergences of stresses and the principle of virtual work on manifolds*, Technische Mechanik **20** (2000), 129–136.

AMS Subject Classification: 73A05, 58A05.

Reuven SEGEV
Department of Mechanical Engineering
Ben-Gurion University
P. O. Box 653
Beer-Sheva 84105, ISRAEL
e-mail: `rsegev@bgumail.bgu.ac.il`