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ON PERTURBATIONS OF MINIMUM PROBLEMS WITH UNBOUNDED CONTROLS

Abstract.

A typical optimal control problem among those considered in this work includes dynamics of the form $f(x,c) = g_0(x) + \tilde{g}_0(x)|c|^\alpha$ (here x and c represent the state and the control, respectively) and a Lagrangian of the form $l(x,c) = l_0(x) + \tilde{l}_0(x)|c|^\beta$, with $\alpha \leq \beta$, and c belonging to a closed, unbounded subset of $\mathbb{R}^m$. We perturb this problem by considering dynamics and Lagrangians $f_n(x,c) = g_n(x) + \tilde{g}_n(x)|c|^{\alpha_n}$, and $l_n(x,c) = l_{0_n}(x) + \tilde{l}_{0_n}(x)|c|^\beta$ respectively, with $\alpha_n \leq \beta$, and f_n and l_n approaching f and l. We show that the value functions of the perturbed problems converge, uniformly on compact sets, to the value function of the original problem. For this purpose we exploit some comparison results for Bellman equations with fast gradient-dependence which have been recently established in a companion paper. Of course the fast growth in the gradient of the involved Hamiltonians is connected with the presence of unbounded controls. As an easy consequence of the convergence result, an optimal control for the original problem turns out to be nearly optimal for the perturbed problems. This is true in particular, for very general perturbations of the LQ problem, including cases where the perturbed problem is *not* coercive, that is, $\alpha_n = \beta (= 2)$.

1. Introduction

Let us consider a Boltz optimal control problem,

$$(P) \qquad \begin{array}{c} \text{minimize } \int_{\bar{t}}^{T} l(t,x,c)\,dt + g(x(T)) \\ \dot{x} = f(t,x,c) \quad x(\bar{t}) = \bar{x} \\ (\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^k \,, \end{array}$$

where $c = c(t)$ is a control which takes values in $\mathbb{R}^m$. Let us also consider a sequence of *perturbations* of this problem,

$$(P_n) \qquad \begin{array}{c} \text{minimize } \int_{\bar{t}}^{T} l_n(t,x,c)dt + g_n(x(T)) \\ \dot{x} = f_n(t,x,c) \quad x(\bar{t}) = \bar{x} \end{array}$$

where the triples (f_n, l_n, g_n) converge to (f, l, g), in a sense to be made precise.

In the present note we address the following question:

Q_1. *Assume that for every initial data $(\bar{t},\bar{x})$ an optimal control $c_{(\bar{t},\bar{x})} : [\bar{t},T] \to \mathbb{R}^m$ is known. Are these controls nearly optimal for the problem (P_n)?*

(Here *nearly optimal* means that the value of the cost functional of (P_n) when the control $c_{(\bar{t},\bar{x})}$ is implemented differs from the optimal value by an error which approaches zero when n tends to ∞).

An analogous question can be posed when an optimal feedback control $c = c(t, x)$ of problem (P) is known:

Q_2. *Is the feedback control $c(t, x)$ nearly optimal for the problem (P_n)?*

The practical usefulness of studying such a theoretical problem is evident: it may happen that the construction of an optimal control for problem (P) is relatively easy, while the same task for the perturbed problem (P_n) might result hopeless. In this case, one could be tempted to implement the (P) optimal control for problem (P_n) as well. And positive answers to questions like $\mathbf{Q}_1$ and $\mathbf{Q}_2$ would guarantee that these strategies would be safe. (For a general account on perturbation theory see e.g. [3]).

Since we are interested in the case when the controls c are unbounded, questions concerning the growth in c of f and l turn out to be quite relevant. The crucial hypotheses (see $\mathbf{A}_1$-$\mathbf{A}_5$ in Section 2) here assumed on the dynamics f and the Lagrangian l are as follows: there exist α, β, both greater than or equal to 1, such that if $Q \subset \mathbb{R}^k$ is a compact subset and $x, y \in Q$, then

$$|f(t, x, c) - f(t, y, c)| \leq L(1 + |c|^\alpha)|x - y| \tag{1}$$

$$|l(t, x, c)| \geq l_0|c|^\beta - C \tag{2}$$

for all $c \in \mathbb{R}^m$, where L depends only on Q. The same kind of hypotheses are assumed on the perturbed pairs f_n, l_n, with the same growth-exponent β for the Lagrangians l_n, while the growth-exponents α_n of the f_n are allowed to depend on n. Moreover, *weak coercivity* relations, namely $\alpha_n \leq \beta$, $\alpha \leq \beta$, are assumed. Let us observe that when $\alpha < \beta$ (*strict coercivity*) the optimal trajectories turn out to be (absolutely) continuous, while, if $\alpha = \beta$, an optimal path may contain *jumps* (in a non trivial sense which *cannot* be resumed by a distributional approach, see e.g. [7, 8]).

Answers to questions $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are given in Theorems 6, 7 below, respectively. The main theoretical tool on which these results rely consists in a so-called stability theorem (see Theorem 1) for a class of Hamilton-Jacobi-Bellman equations with fast gradient-dependence. In order to prove the stability theorem we exploit some uniqueness and regularity results for this class of equations that have been recently established in a companion paper [8] (see also [1] and [6]). Let us notice that questions like $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be approached with more standard uniqueness results as soon as the controls c are bounded.

Similar questions were addressed in a paper by M. Bardi and F. Da Lio [1], where the authors assumed the following stronger hypothesis on f:

$$|f(x, c) - f(y, c)| \leq L|x - y| \tag{3}$$

(actually a monotonicity hypothesis, weaker than (3) is assumed in [1]; however this is irrelevant at this stage, while the main point in assuming (3) consists in the fact that it is Lipschitz in x *uniformly with respect to c*). Observe that hypothesis (3) still allows for fields growing as $|c|^\alpha$ in the variable c. Yet, while a field of the form $f(x, c) \doteq g_0(x) + g_1(x)|c|^\alpha$ agrees with hypothesis (1), it does not satisfy hypothesis (3) unless $g_1(x)$ is constant. Furthermore, in [1] the exponent α is required to be strictly less than β (strict coercivity).

The relevance of weakening hypothesis (3) (and the position $\alpha < \beta$) is perhaps better understood by means of an application to a perturbation question for the linear quadratic problem. In this case one has: $\alpha = 1, \beta = 2$, $f(x, c) = Ax + Bc$, $l(x, c) = x^*Dx + x^*Ec + c^*Fc$, $g(x) = x^*Sx$. Here the coercivity hypothesis reduces to the fact that F is positive definite. As it is well known, (see e.g. [4]) under suitable hypotheses on A, B, D, E and F, this problem admits a smooth optimal feedback, which can be actually computed by solving the corresponding Riccati equation. It is obvious that a crucial point in questions $\mathbf{Q}_1$ and $\mathbf{Q}_2$ consists in the specification of *which* class of perturbation problems (P_n) has to be considered. Of course, since in practical situations the nature of the perturbation is only partially known, the larger this class is the better. In [1] a positive answer to $\mathbf{Q}_1$ is provided when the perturbed fields are of the form

$$f_n(x) = Ax + Bx + \epsilon(n)\varphi(x, c)$$

with φ verifying (3) and $\epsilon(n)$ infinitesimal. So, for instance, a perturbed dynamics like

$$f_n = Ax + Bx + \frac{1}{n}xc$$

is *not* allowed. On the contrary, hypothesis (1) assumed in the present paper is not in contrast with this (and much more general) kind of perturbation. A further improvement is represented by the fact that the f_n's growth exponents α_n are allowed to be different from the f's growth exponent α (=1, in this case), and moreover, they can be less than or *equal* to β (which in this example is equal to 2). So, for example, perturbed dynamics like

$$f_n = Ax + Bc + \epsilon(n)(g(x)c + h(x)|c|^2)$$

may be well considered. In this case, the possibility of implementing a (P)-optimal control c in the perturbed problem (P_n) may be of particular interest. Indeed, the problems (P_n) are quite irregular, in that the lack of a sufficient degree of coercivity may give rise to optimal trajectories with *jumps* (see Remark 2).

The general approach of the present paper, which is partially inspired by [1], relies on proving the convergence of the value functions of the problems (P_n) to the value function of (P) via a PDE argument. However, the enlarged generality of the considered problems makes the exploitation of very recent results on Hamilton-Jacobi-Bellman equations with fast gradient-dependence crucial (see [8]). In particular, by allowing mixed type boundary conditions, these results cover the weak coercivity case ($\alpha = \beta$). Moreover they do not require an assumption of local Lipschitz continuity of the solution of the associated dynamic programming equation. Actually, as a consequence of the fact that we allow value functions which are not equicontinuous, the Ascoli-Arzelà argument exploited in the stability theorem of [1] does not work here. In order to overcome this difficulty we join ordinary convergence arguments originally due to G. Barles and B. Perthame [2] with the reparameterization techniques introduced in [8].

2. A convergence result

For every $\bar{t} \in [0, T]$, let $\mathcal{C}(\bar{t})$ denote the set of Borel-measurable maps which belong to $L^\beta([\bar{t}, T], \mathbb{R}^m)$. $\mathcal{C}(\bar{t})$ is called the set of controls starting at $\bar{t}$. Let us point out that the choice of the whole $\mathbb{R}^m$ as the set where the controls take values is made just for the sake of simplicity. Indeed, in view of the Appendix in [8] it is straightforward to generalize the results presented here to situations where the controls can take values in a (possibly unbounded) closed subset of $\mathbb{R}^m$. For every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k$ and every $c \in \mathcal{C}(\bar{t})$, by the assumptions $\mathbf{A}_1$-$\mathbf{A}_5$ listed below, there exists a unique solution of the Cauchy problem

$$(E) \qquad \begin{cases} \dot{x} = f(t, x, c) \text{ for } t \in [\bar{t}, T] \\ x(\bar{t}) = \bar{x}\,, \end{cases}$$

(where the dot means differentiation with respect to t). We will denote this solution by $x_{(\bar{t},\bar{x})}[c](\cdot)$ (or by $x[c](\cdot)$ if the initial data are meant by the context). For every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k$ let us consider the optimal control problem

$$(P) \qquad \underset{c\in\mathcal{C}(\bar{t})}{\text{minimize}}\ J(\bar{t}, \bar{x}, c)$$

where

$$J(\bar{t}, \bar{x}, c) \doteq \int_{\bar{t}}^{T} l(t, x[c](t), c(t))\, dt + g(x[c](T))\,,$$

and let us define the *value function* $V : [0, T[\times \mathbb{R}^k \to \mathbb{R}$, by setting

$$V(\bar{t}, \bar{x}) \doteq \inf_{c \in \mathcal{C}(\bar{t})} J(\bar{t}, \bar{x}, c)\,.$$

We consider also a sequence of *perturbed* problems

$$(P_n) \qquad \underset{c \in \mathcal{C}(\bar{t})}{\text{minimize}}\; J_n(\bar{t}, \bar{x}, c)$$

where

$$J_n(\bar{t}, \bar{x}, c) \doteq \int_{\bar{t}}^{T} l_n(t, x_n[c](t), c(t))\, dt + g_n(x_n[c](T))\,,$$

where $x_n[c]$ (or $x^n_{(\bar{t},\bar{x})}[c]$ if one wishes to specify the initial data), denotes the solution – existing unique by hypotheses $\mathbf{A}_1$-$\mathbf{A}_5$ below – of

$$(E_n) \qquad \begin{cases} \dot{x} = f_n(t, x, c) \text{ for } t \in [\bar{t}, T] \\ x(\bar{t}) = \bar{x} \end{cases}$$

Let us define the value function V_n of (P_n) by setting

$$V_n(\bar{t}, \bar{x}) \doteq \inf_{c \in \mathcal{C}(\bar{t})} J_n(\bar{t}, \bar{x}, c)\,.$$

We assume that there exist numbers α, α_n, β satisfying $1 \le \alpha \le \beta$, $1 \le \alpha_n \le \beta$, such that the following hypotheses hold true:

$\mathbf{A}_1$ *the maps f and f_n are continuous on $[0, T] \times \mathbb{R}^k \times \mathbb{R}^m$ and, for every compact subset $Q \subset \mathbb{R}^k$, there exists a positive constant L and a modulus ρ_f verifying*

$$\begin{aligned} |f(t_1, x_1, c) - f(t_2, x_2, c)| &\le (1 + |c|^{\alpha})(L|x_1 - x_2| + \rho_f(|t_1 - t_2|)\,, \\ |f_n(t_1, x_1, c) - f_n(t_2, x_2, c)| &\le (1 + |c|^{\alpha_n})(L|x_1 - x_2| + \rho_f(|t_1 - t_2|) \end{aligned}$$

for all $(t_1, x_1, c), (t_2, x_2, c) \in [0, T] \times Q \times \mathbb{R}^m$ and $n \in \mathbb{N}$, (by modulus *we mean a positive, nondecreasing function, null and continuous at zero);*

$\mathbf{A}_2$ *there exist two nonnegative constants M_1 and M_2 such that*

$$\begin{aligned} |f(t, x, c)| &\le M_1(1 + |c|^{\alpha})(1 + |x|) + M_2(1 + |c|^{\alpha}) \\ |f_n(t, x, c)| &\le M_1(1 + |c|^{\alpha_n})(1 + |x|) + M_2(1 + |c|^{\alpha_n}) \end{aligned}$$

for every $(t, x, c) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^m$;

$\mathbf{A}_3$ *the maps l and l_n are continuous on $[0, T] \times \mathbb{R}^k \times \mathbb{R}^m$ and, for every compact subset $Q \subset \mathbb{R}^k$, there is a modulus ρ_l satisfying*

$$\begin{aligned} |l(t_1, x_1, c) - l(t_2, x_2, c)| &\le (1 + |c|^{\beta})\rho_l(|(t_1, x_1) - (t_2, x_2)|) \\ |l_n(t_1, x_1, c) - l_n(t_2, x_2, c)| &\le (1 + |c|^{\beta})\rho_l(|(t_1, x_1) - (t_2, x_2)|) \end{aligned}$$

for every $(t_1, x_1, c), (t_2, x_2, c) \in [0, T] \times Q \times \mathbb{R}^m$ and $n \in \mathbb{N}$;

A_4 *there exist positive constants Λ_0 and Λ_1 such that the following* coercivity conditions

$$\begin{aligned} l(t,x,c) &\geq \Lambda_0|c|^\beta - \Lambda_1 \\ l_n(t,x,c) &\geq \Lambda_0|c|^\beta - \Lambda_1\,, \end{aligned}$$

are verified for every $(t,x,c) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^m$ and every $n \in \mathbb{N}$;

A_5 *the maps g, g_n are bounded below by a constant $\overline{G}$ and, for every compact $Q \subset \mathbb{R}^k$, there is a modulus ρ_g such that*

$$\begin{aligned} |g_n(x_1) - g_n(x_2)| &\leq \rho_g(|x_1 - x_2|)\,, \\ |g(x_1) - g(x_2)| &\leq \rho_g(|x_1 - x_2|) \end{aligned}$$

for every $x_1, x_2 \in Q$.

When $\alpha = \beta$, we also assume a condition of regularity of f and l at infinity in the variable c. Precisely, we posit the existence of continuous functions f^∞ and l^∞, the *recessions functions* of f and l, respectively, verifying

$$\begin{aligned} \lim_{r\to 0} r^\beta f(t,x,r^{-1}w) &\doteq f^\infty(t,x,w) \\ \lim_{r\to 0} r^\beta l(t,x,r^{-1}w) &\doteq l^\infty(t,x,w)\,, \end{aligned}$$

on compact sets of $[0,T] \times \mathbb{R}^k \times \mathbb{R}^m$ (e.g., if $f(t,x,c) = f_0(t,x) + f_1(t,x)|c| + f_2(t,x)|c|^2$ then $f^\infty(t,x,w) = f_2(t,x)|w|^2$). When $\alpha_n = \beta$ we likewise assume the existence of the recession functions f_n^∞, l_n^∞, respectively.

Theorem 1 below is the main result of the paper and concerns the convergence of the value functions V_n to V. We point out that, unlike previous results on this subject (see [1]), the triples (f_n, l_n, g_n) are allowed to tend to (f, l, g) not uniformly with respect to x and c.

THEOREM 1. *Let us assume that for every set $[0,T] \times Q$, where Q is a compact subset of $\mathbb{R}^k$, there exists a function $\epsilon : \mathbb{N} \to [0,\infty)$ infinitesimal for $n \to \infty$ such that*

$$|f_n(t,x,c) - f(t,x,c)| \leq \epsilon(n)(1 + |c|^\beta)\,, \tag{4}$$

$$|l_n(t,x,c) - l(t,x,c)| \leq \epsilon(n)(1 + |c|^\beta) \tag{5}$$

for $(t,x,c) \in [0,T] \times Q \times \mathbb{R}^m$ and

$$|g_n(x) - g(x)| \leq \epsilon(n)$$

for every $x \in Q$. Then the value functions V_n converge uniformly, as n tends to ∞, to V on compact subsets of $[0,T] \times \mathbb{R}^k$.

This theorem will be proved in Section 4 via some arguments which rely on the fact that the considered value functions are solutions of suitable Hamilton-Jacobi-Bellman equations. Actually, due to the non standard growth properties of the data, the Hamiltonians involved in these equations do not satisfy a uniform growth assumption in the adjoint variable which is shared by most of the uniqueness results existing in literature. In a recent paper [8] we have established some uniqueness and regularity results for this kind of equations. In the next section we recall briefly the points of this investigation that turn out to be essential in the proof of Theorem 1.

3. Reparameterizations and Bellman equations

The contents of this section thoroughly relies on the results of [8]. Let us embed the unperturbed and the perturbed problems in a class of extended problems which have the advantage of involving only bounded controls. There is a reparameterization argument behind this embedding which allows one to transform a L^β constraint (implicitly imposed by the coercivity assumptions) into a L^∞ constraint.

Let us introduce the extended fields

$$\overline{f}(t,x,w_0,w) \doteq \begin{cases} f\left(t,x,\frac{w}{w_0}\right)\cdot w_0^\beta & \text{if } w_0 \neq 0 \\ f^\infty(t,x,v,w) & \text{if } w_0 = 0 \text{ and } \alpha = \beta \end{cases}$$

and

$$\bar{l}(t,x,w_0,w) \doteq \begin{cases} l\left(t,x,\frac{w}{w_0}\right)\cdot w_0^\beta & \text{if } w_0 \neq 0 \\ l^\infty(t,x,v,w) & \text{if } w_0 = 0 \text{ and } \alpha = \beta\,. \end{cases}$$

Similarly, for every n we define the extended fields $\overline{f_n}$ and $\overline{l_n}$ of f_n and l_n, respectively. Hypotheses $\mathbf{A}_1$-$\mathbf{A}_5$ imply the following properties for the maps $\overline{f_n}$, $\overline{l_n}$, $\overline{f}$, and $\bar{l}$.

PROPOSITION 1. **(i)** *The functions $\overline{f_n}$, $\overline{l_n}$, $\overline{f}$, and $\bar{l}$ are continuous on $[0,T]\times\mathbb{R}^k\times[0,+\infty[\times\mathbb{R}^m$ and for every compact $Q\subset\mathbb{R}^k$ we have*

$$(A_{e_1})\quad \begin{aligned} |\overline{f}(t_1,x_1,w_0,w)-\overline{f}(t_2,x_2,w_0,w)| &\le (w_0^\alpha+|w|^\alpha)w_0^{\beta-\alpha}(L|x_1-x_2| \\ &\quad +\rho_f(|t_1-t_2|))\,, \\ |\overline{f}_n(t_1,x_1,w_0,w)-\overline{f}_n(t_2,x_2,w_0,w)| &\le (w_0^{\alpha_n}+|w|^{\alpha_n})w_0^{\beta-\alpha_n}(L|x_1-x_2| \\ &\quad +\rho_f(|t_1-t_2|)) \end{aligned}$$

and

$$(A_{e_3})\quad \begin{aligned} |\bar{l}(t_1,x_1,w_0,w)-\bar{l}(t_2,x_2,w_0,w)| &\le (w_0^\beta+|w|^\beta)\rho_l(|(t_1,x_1)-(t_2,x_2)|)\,, \\ |\bar{l}_n(t_1,x_1,w_0,w)-\bar{l}_n(t_2,x_2,w_0,w)| &\le (w_0^\beta+|w|^\beta)\rho_l(|(t_1,x_1)-(t_2,x_2)|) \end{aligned}$$

$\forall (t_1,x_1,w_0,w),(t_2,x_2,w_0,w)\in[0,T]\times\mathbb{R}^k\times[0,+\infty[\times\mathbb{R}^m$, where α, α_n, β, L, ρ_f, and ρ_l are the same as in assumptions $\mathbf{A}_1$ and $\mathbf{A}_3$.

Moreover,

$$(A_{e_2})\quad \begin{aligned} |\overline{f}(t,x,w_0,w)| &\le (w_0^\alpha+|w|^\alpha)w_0^{\beta-\alpha}(M_1(1+|x|)+M_2)\,, \\ |\overline{f}_n(t,x,w_0,w)| &\le (w_0^{\alpha_n}+|w|^{\alpha_n})w_0^{\beta-\alpha_n}(M_1(1+|x|)+M_2) \end{aligned}$$

and

$$(A_{e_4})\quad \begin{aligned} \bar{l}(t,x,w_0,w) &\ge \Lambda_0|w|^\beta-\Lambda_1|w_0|^\beta\,, \\ \bar{l}_n(t,x,w_0,w) &\ge \Lambda_0|w|^\beta-\Lambda_1|w_0|^\beta \end{aligned}$$

$\forall (t,x,w_0,w)\in[0,T]\times\mathbb{R}^k\times[0,+\infty[\times\mathbb{R}^m$, where M_1, M_2, Λ_0 and Λ_1 are the same as in $\mathbf{A}_2$ and $\mathbf{A}_4$.

(ii) (Positive homogeneity in (w_0, w)). *The map* $\overline{f}$, $\bar{l}$, $\overline{f}_n$, *and* $\bar{l}_n$ *are positively homogeneous of degree* β *in* (w_0, w), *that is,*

$$\overline{f}(t, x, rw_0, rw) = r^\beta \overline{f}(t, x, w_0, w), \quad \bar{l}(t, x, rw_0, rw) = r^\beta \bar{l}(t, x, w_0, w)$$
$$\overline{f}_n(t, x, rw_0, rw) = r^\beta \overline{f}_n(t, x, w_0, w), \bar{l}_n(t, x, rw_0, rw) = r^\beta \bar{l}_n(t, x, w_0, w)$$

$\forall r > 0$, $\forall (t, x, w_0, w) \in [0, T] \times \mathbb{R}^k \times]0, +\infty[\times \mathbb{R}^m$.

For every $\bar{t} \in [0, T]$ let us introduce the following sets of *space-time controls*

$$\Gamma(\bar{t}) \doteq \left\{ (w_0, w) \in \mathcal{B}([0,1], [0, +\infty) \times \mathbb{R}^m) \text{ such that } \bar{t} + \int_0^1 w_0^\beta(s)\, ds = T \right\}$$

and

$$\Gamma^+(\bar{t}) \doteq \left\{ (w_0, w) \in \Gamma(\bar{t}) \text{ such that } w_0 > 0 \text{ a.e.} \right\}$$

where $\mathcal{B}([0,1], [0, +\infty) \times \mathbb{R}^m)$ is the set of L^∞, Borel maps, which take values in $[0, +\infty[\times \mathbb{R}^m$. If $\alpha < \beta$ [resp.$\alpha = \beta$], for every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k$ and every $(w_0, w) \in \Gamma^+(\bar{t})$ [resp. $(w_0, w) \in \Gamma(\bar{t})$], let us denote by $(t, y)_{(\bar{t}, \bar{x})}[w_0, w](\cdot)$ the solution of the (*extended*) Cauchy problem

$$(E_e) \qquad \begin{cases} t'(s) = w_0^\beta(s) \\ y'(s) = \overline{f}(t(s), y(s), w_0(s), w(s)) \\ (t(0), y(0)) = (\bar{t}, \bar{x}), \end{cases}$$

where the parameter s belongs to the interval $[0, 1]$ and the prime denotes differentiation with respect to s. When the initial conditions are meant by the context we shall write $(t, y)[w_0, w](\cdot)$ instead of $(t, y)_{(\bar{t}, \bar{x})}[w_0, w](\cdot)$. Let us consider the following (*extended*) cost functional

$$J_e(\bar{t}, \bar{x}, w_0, w) \doteq \int_0^1 \bar{l}\,((t, y)[w_0, w], w_0, w)\,(s)\, ds + g(y[w_0, w](1))$$

and the corresponding (*extended*) value function

$$V_e : [0, T] \times \mathbb{R}^k \to \mathbb{R}$$
$$V_e(\bar{t}, \bar{x}) \doteq \inf_{(w_0, w) \in \Gamma(\bar{t})} J_e(\bar{t}, \bar{x}, w_0, w).$$

Similarly, for every $n \in \mathbb{N}$, for every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k$ and every $(w_0, w) \in \Gamma(\bar{t})$ let us introduce the system

$$(E_{e_n}) \qquad \begin{cases} t'(s) = w_0^\beta(s) \\ y'(s) = \overline{f}_n(t(s), y(s), w_0(s), w(s)) \quad s \in [0, 1] \\ (t(0), y(0)) = (\bar{t}, \bar{x}), \end{cases}$$

and let us denote its solution by $(t, y)^n_{(\bar{t}, \bar{x})}[w_0, w](\cdot)$. Let us introduce the cost functionals

$$J_{e_n}(\bar{t}, \bar{x}, w_0, w) \doteq \int_0^1 \bar{l}_n\left((t, y)^n_{(\bar{t}, \bar{x})}, w_0, w\right)(s)\, ds + g_n(y_n[w_0, w](1))$$

and the corresponding value functions

$$V_{e_n} : [0,T] \times \mathbb{R}^k \to \mathbb{R}$$
$$V_{e_n}(\bar{t},\bar{x}) \doteq \inf_{(w_0,w)\in\Gamma(\bar{t})} J_{e_n}(\bar{t},\bar{x},w_0,w)\,.$$

Next theorem establishes the coincidence of the value functions of the original problems with those of the extended problems.

THEOREM 2. *Assume* $\mathbf{A}_1$-$\mathbf{A}_5$.

(i) *For every* $(t,x) \in [0,T[\times\mathbb{R}^k$ *and for every* $n \in \mathbb{N}$ *one has* $V_e(t,x) = V(t,x)$*; and* $V_{e_n}(t,x) = V_n(t,x)$*;*

(ii) *the maps* V_e *and* V_{e_n} *are continuous on* $[0,T]\times\mathbb{R}^k$.

Thanks to this theorem – which, in particular, implies that V and V_n can be continuously extended on $[0,T]\times\mathbb{R}^k$ – the problem of the convergence of the V_n is transformed in the analogous problem for the V_{e_n}.

We now recall that each of these value functions is the unique solution of a suitable boundary value problem. This is a consequence of the comparison theorem below. To state these results, let us introduce the *extended Hamiltonians*

(6)
$$H_e(t,x,p_0,p) \doteq \sup_{(w_0,w)\in([0,+\infty[\times\mathbb{R}^m)\cap S_m^+} \{-p_0 w_0^\beta - \langle p, \overline{f}(t,x,w_0,w)\rangle - \overline{l}(t,x,w_0,w)\}$$

where $S_m^+ \doteq \{(w_0,w)\in[0,+\infty[\times\mathbb{R}^m : |(w_0,w)|=1\}$,

$$H_{e_n}(t,x,p_0,p) \doteq \sup_{(w_0,w)\in([0,+\infty[\times\mathbb{R}^m)\cap S_m^+} \{-p_0 w_0^\beta - \langle p, \overline{f_n}(t,x,w_0,w)\rangle - \overline{l_n}(t,x,w_0,w)\}\,,$$

and the corresponding Hamilton-Jacobi-Bellman equations

$$(HJ_e) \qquad H_e(t,x,u_t,u_x) = 0\,,$$

$$(HJ_{e_n}) \qquad H_{e_n}(t,x,u_t,u_x) = 0\,.$$

For the sake of self consistency let us recall the definition of (possibly discontinuous) viscosity solution, which was introduced by H. Ishii in [5].

Given a function $F : \mathcal{Q} \to \mathbb{R}$, $\mathcal{Q} \subseteq \mathbb{R}^k$, let us consider the *upper and lower semicontinuous envelopes*, defined by

$$\begin{aligned} F^*(x) &\doteq \lim_{r\to 0^+} \sup\{F(y) : y\in\mathcal{Q},\ |x-y|\le r\}\,, \\ F_*(x) &\doteq \lim_{r\to 0^+} \inf\{F(y) : y\in\mathcal{Q},\ |x-y|\le r\}\,, \qquad x\in\overline{\mathcal{Q}}\,, \end{aligned}$$

respectively. Of course, F^* is upper semicontinuous and F_* is lower semicontinuous.

DEFINITION 1. *Let* E *be a subset of* $\mathbb{R}^s$ *and let* G *be a real map, the* Hamiltonian*, defined on* $E\times\mathbb{R}\times\mathbb{R}^s$*. An upper[resp. lower]-semicontinuous function* u *is a viscosity subsolution [resp. supersolution] of*

$$(7) \qquad G(y,u,u_y) = 0$$

at $y \in E$ if for every $\phi \in \mathcal{C}^1(\mathbb{R}^s)$ such that y is a local maximum [resp. minimum] point of $u - \phi$ on E one has

$$G_*(y, \phi(y), \phi_y(y)) \leq 0$$

[resp.

$$G^*(y, \phi(y), \phi_y(y)) \geq 0] .$$

A function u is a viscosity solution of (7) at $y \in E$ if u^* is a viscosity subsolution at y and u_* is a viscosity supersolution at y.

THEOREM 3 (COMPARISON). *Assume $\mathbf{A}_1$-$\mathbf{A}_5$. Let $u_1 : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ be an upper semicontinuous, bounded below, viscosity subsolution of (HJ_e) in $]0, T[\times\mathbb{R}^k$, continuous on $(\{0\} \times \mathbb{R}^k) \cup (\{T\} \times \mathbb{R}^k)$. Let $u_2 : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous, bounded below, viscosity supersolution of (HJ_e) in $[0, T[\times\mathbb{R}^k$. For every $x \in \mathbb{R}^k$, assume that*

$$\begin{cases} u_1(T, x) \leq u_2(T, x) \\ \text{or} \\ u_2 \text{ is a viscosity supersolution of } (HJ_e) \text{ at } (T, x) . \end{cases}$$

Then

$$u_1(t, x) \leq u_2(t, x) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k .$$

The same statement holds true for the equations (HJ_{e_n}).

As a consequence of this theorem and of a suitable dynamic programming principle for the extended problems one can prove the following:

THEOREM 4. *The value function V_e is the unique map which*

i) is continuous on $(\{0\} \times \mathbb{R}^k) \cup (\{T\} \times \mathbb{R}^k)$;

ii) is a viscosity solution of (HJ_e) in $]0, T[\times\mathbb{R}^k$;

iii) satisfies the following mixed type boundary condition:

$$(BC_{e_m}) \qquad \begin{cases} V_e(T, x) \leq g(x) \quad \forall x \in \mathbb{R}^k \text{ and} \\ \\ \begin{cases} V_e(T, x) = g(x) \\ \text{or} \\ V_e \text{ is a viscosity supersolution of } (HJ_e) \text{ at } (T, x) . \end{cases} \end{cases}$$

Once we replace (HJ_e) by (HJ_{e_n}), the same statement holds true for the maps V_{e_n}.

Finally let us recall a regularity result which will be useful in the proof of Theorem 1.

THEOREM 5. *Assume $\mathbf{A}_1$-$\mathbf{A}_5$ and fix $R > 0$. Then there exists $R' \geq R$ and positive constants C_1, C_2 such that*

$$|V_e(t, x_1) - V_e(t, x_2)| \leq C_1 \rho_l(C_2|x_2 - x_1|) + \rho_g(C_2|x_2 - x_1|)$$

for every $(t, x_1)\,(t, x_2) \in [0, T] \times B[0; R]$, where ρ_l and ρ_g are the modulus appearing in $\mathbf{A}_3$ and the modulus of uniform continuity of g, respectively, corresponding to the compact $[0, T] \times B[0; R']$. Moreover for every $\bar{t} \in [0, T[$ one has

$$|V_e(t, x) - V_e(\bar{t}, x)| \leq \eta_{\bar{t}}(|t - \bar{t}|)$$

for every $(t, x) \in [0, T[\times B[0; R]$, *where* $\eta_{\overline{t}}$ *is a suitable modulus, and for every* s, $\overline{t} \to \eta_{\overline{t}}(s)$ *is an increasing map. The same statement holds true for the maps* V_{e_n}, *with the same* $\eta_{\overline{t}}$.

REMARK 1. We do not need, for our purposes, an explicit expression of $\eta_{\overline{t}}$, which, however, can be found in [8]. Also in that paper sharper regularity results are established. Finally let us point out that though an estimate like the second one in Theorem 5 is not available for $\overline{t} = T$ the map V_e is continuous on $\{T\} \times \mathbb{R}^k$, (see Theorem 2).

4. Proof of the convergence theorem

Proof of Theorem 1. In view of Theorem 2 it is sufficient to show that the maps V_{e_n} converge to V_e. Observe that the assumptions (4), (5) imply

$$|\overline{f}_n(t, x, w_0, w) - \overline{f}(t, x, w_0, w)| \leq \epsilon(n)(w_0^\beta + |w|^\beta) \tag{8}$$

and

$$|\overline{l}_n(t, x, w_0, w) - \overline{l}(t, x, w_0, w)| \leq \epsilon(n)(w_0^\beta + |w|^\beta)\,, \tag{9}$$

for every $(t, x, w_0, w) \in [0, T] \times Q \times [0, \infty[\times\mathbb{R}^m$ and every $n \in \mathbb{N}$.

Moreover, by the coercivity condition $\mathbf{A}_{e_4}$ and by the obvious local uniform boundedness of V_{e_n}, and V_e when the initial conditions are taken in a ball $B[0, R]$ it is not restrictive to consider only those space time controls such that

$$\int_0^1 (w_0(s) + |w(s)|)^\beta\, ds \leq K_R \tag{10}$$

where K_R is a suitable constant depending on R. By Hölder's inequality we have also that

$$\int_0^1 (w_0(s) + |w(s)|)^{\alpha_n} w_0(s)^{\beta-\alpha_n}\, ds \leq (T+1)(K_R+1)\,.$$

Hence by Gronwall's Lemma, we can assume that there exists a ball $B[0, R'] \subset \mathbb{R}^k$ containing all the trajectories issuing from $B[0, R]$.

Let us fix $\overline{T} < T$: by Theorem 5 the maps V_{e_n} are equicontinuous and equibounded on $[0, \overline{T}] \times B[0, R]$, so we can apply Ascoli-Arzela's Theorem to get a subsequence of V_{e_n}, still denoted by V_{e_n}, converging to a continuous function. Actually by taking R larger and larger, via a standard diagonal procedure we can assume that the V_{e_n} converge to a continuous function $\mathcal{V} : [0, T] \times \mathbb{R}^k \to \mathbb{R}$, uniformly on compact sets of $[0, \overline{T}] \times \mathbb{R}^k$. Now, for every $(t, x) \in [0, T] \times \mathbb{R}^k$, let us consider the *weak limits*

$$\overline{V}(t, x) \doteq \limsup_{\substack{n\to\infty \\ (s,y)\to(t,x) \\ (s,y)\in[0,T]\times\mathbb{R}^k}} V_{e_n}(s, y)$$

and

$$\underline{V}(t, x) \doteq \liminf_{\substack{n\to\infty \\ (s,y)\to(t,x) \\ (s,y)\in[0,T]\times\mathbb{R}^k}} V_{e_n}(s, y)\,.$$

Our goal is to apply a method (see [2]) based on the application of the comparison theorem (see Theorem 3) to these weak limits. Let us observe that both $\overline{V}$ and $\underline{V}$ coincide with $\mathcal{V}$ on the boundary $\{0\} \times \mathbb{R}^k$: in particular they are continuous on $\{0\} \times \mathbb{R}^k$. Since the Hamiltonians H_{e_n} converge to H_e uniformly on compact subsets of $[0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k$, standard arguments imply that $\overline{V}$ is a (upper semicontinuous) viscosity subsolution of (HJ_e) in $[0, T) \times \mathbb{R}^k$, while $\underline{V}$ is a (lower semicontinuous) viscosity supersolution of (HJ_e) in $[0, T) \times \mathbb{R}^k$. Hence the convergence result is proven as soon as one shows that $\overline{V} \le \underline{V}$ in $[0, T] \times \mathbb{R}^k$. For this purpose it is sufficient to show that $\overline{V}$ and $\underline{V}$ verify the hypotheses of Theorem 3. Actually the only hypothesis which is left to be verified is the one concerning the boundary subset $\{T\} \times \mathbb{R}^k$. We claim that

$$\lim_{\substack{n\to\infty \\ (s,y)\to(T,x) \\ (s,y)\in[0,T]\times\mathbb{R}^k}} V_{e_n}(s, y) = V_e(T, x) \tag{11}$$

which implies $\overline{V}(T, \cdot) = \underline{V}(T, \cdot) = V_e(T, \cdot)$. In particular the maps $\overline{V}(T, \cdot)$ and $\underline{V}(T, \cdot)$ turn out to be continuous, so all assumptions of Theorem 3 are verified. The remaining part of this proof is thus devoted to prove (11). Let us consider $x_1, x_2 \in B[0, R]$ and controls $(0, w_n) \in \Gamma(T)$ such that, setting $(t_n, x_n) \doteq (t, y)^n_{(T,x_1)}[0, w_n](\cdot)$, we have

$$V_{e_n}(T, x_1) \ge \int_0^1 \bar{l}_n(t_n, x_n, 0, w_n)(s)\, ds + g_n(x_n(1)) - \epsilon\,.$$

Hence, setting $(\tilde{t}_n, \tilde{x}_n) \doteq (t, y)_{(T,x_2)}[0, w_n](\cdot)$ and noticing that $\tilde{t}_n(s) = t_n(s) = T\ \forall s \in [0, 1]$, we have

$$\begin{aligned} V_e(T, x_2) - V_{e_n}(T, x_1) \quad &\le \quad \int_0^1 \bar{l}(T, \tilde{x}_n, 0, w_n)(s)\, ds + g_n(\tilde{x}_n(1)) \\ &\qquad - \int_0^1 \bar{l}_n(T, x_n, 0, w_n)(s)\, ds - g_n(x_n(1)) + \epsilon \\ &\le \quad \int_0^1 |w_n(s)|^\beta [\epsilon(n) + \rho_l(|\tilde{x}_n(s) - x_n(s)|)]\, ds \\ &\qquad + \rho_g(|\tilde{x}_n(1) - x_n(1)|) + \epsilon(n) + \epsilon\,, \end{aligned}$$

where $\epsilon(n)$, ρ_l and ρ_g (see $\mathbf{A}_3$ and $\mathbf{A}_5$) are determined with reference to the compact subset $Q = B[0, R']$. If $L_{R'}$ is the determination of L in (A_{e_1}) for $B[0, R']$ then

$$|\tilde{x}_n(s) - x_n(s)| \le (|x_1 - x_2| + \epsilon(n)(T + 1)(K_R + 1))e^{L_{R'}(T+1)(K_R+1)}\,.$$

This, together with the fact that a similar inequality can be proved (in a similar way) when the roles of V_e and V_{e_n} are interchanged, implies

$$\begin{aligned} |V_e(T, x_2) - V_{e_n}(T, x_1)| &\le K_R \rho_l \Big[(|x_1 - x_2| \\ &\quad + \epsilon(n)(T + 1)(K_R + 1))e^{L_{R'}(T+1)(K_R+1)}\Big] \\ &\quad + \rho_g \Big[(|x_1 - x_2| + \epsilon(n)(T + 1)(K_R + 1))e^{L_{R'}(T+1)(K_R+1)}\Big] \\ &\quad + (K_R + 1)\epsilon(n)\,. \end{aligned} \tag{12}$$

Now, for $\tau \leq T$, let us estimate the difference $V_{e_n}(\tau, x) - V_e(T, x)$, assuming that this difference is non negative. Let us set $(t_n, x_n)(\cdot) \doteq (t, y)^n_{(\tau,x)}(\tilde{w}_0, 0)(\cdot)$ with $\tilde{w}_0(s) \doteq (T - \tau)^{\frac{1}{\beta}}$ $\forall s \in [0, 1]$. Then the Dynamic Programming Principle

$$V_{e_n}(\tau, x) - V_e(T, x) \leq \int_0^1 \bar{l}_n(t_n, x_n, \tilde{w}_0, 0)(s)\, ds + V_{e_n}(T, x_n(1)) - V_e(T, x)\,.$$

If $M \doteq \max\{M_1 + M_2, 1\}$, by (A_{e_2}) we have $|x_n(1) - x| \leq M(1 + R')|T - \tau|$. Hence, if $K'_R \geq \max_{\substack{(t,x)\in[0,T]\times B[0,R']\\ n\in\mathbb{N}}} \bar{l}_n(t, x, 1, 0)$, by the positive homogeneity of $\bar{l}_n$ and by the first part of the proof we obtain

$$V_{e_n}(\tau, x) - V_e(T, x) \leq K'_R|T - \tau| + \sigma_n(|T - \tau|) \tag{13}$$

where

$$\begin{aligned}\sigma_n(s) =& K_R\rho_l[(M(1 + R')s + \epsilon(n)(T + 1)(K_R + 1))e^{L_{R'}(T+1)(K_R+1)}]\\ &+ \rho_g[(M(1 + R')s + \epsilon(n)(T + 1)(K_R + 1))e^{L_{R'}(T+1)(K_R+1)}] + (K_R + 1)\epsilon(n)\,.\end{aligned}$$

Now let us estimate the difference $V_e(T, x) - V_{e_n}(\tau, x)$, assuming it non negative. Let us consider a sequence of controls $(w_{0_n}, w_n) \in \Gamma(\tau)$ such that, setting $(t_n, x_n) \doteq (t, y)^n_{(\tau,x)}[w_{0_n}, w_n](\cdot)$, one has

$$V_{e_n}(\tau, x) \geq \int_0^1 \bar{l}_n(t_n, x_n, w_{0_n}, w_n)(s)\, ds + g_n(x_n(1)) - \epsilon\,.$$

Then the controls $(0, w_n)$ belong to $\Gamma(T)$, and, setting $(\tilde{t}_n, \tilde{x}_n) \doteq (t, y)[0, w_n](\cdot)$, we obtain

$$\begin{aligned}V_e(T, x) - V_{e_n}(\tau, x) \leq& \int_0^1 \bar{l}(\tilde{t}_n, \tilde{x}_n, 0, w_n)(s)\, ds + g(\tilde{x}_n(1))\\ &- \int_0^1 \bar{l}_n(t_n, x_n, w_{0_n}, w_n)(s)\, ds - g_n(x_n(1)) + \epsilon\end{aligned} \tag{14}$$

for every $n \in \mathbb{N}$. Now one has

$$\begin{aligned}|x_n(s) - \tilde{x}_n(s)| &\leq \int_0^1 |\overline{f}_n(t_n, x_n, w_{0_n}, w_n)(s) - \overline{f}(\tilde{t}_n, \tilde{x}_n, 0, w_n)(s)|\, ds\\ &\leq \int_0^1 |\overline{f}_n(t_n, x_n, w_{0_n}, w_n)(s) - \overline{f}(t_n, x_n, w_{0_n}, w_n)(s)|\, ds\\ &\quad+ \int_0^1 |\overline{f}(t_n, x_n, w_{0_n}, w_n)(s) - \overline{f}(\tilde{t}_n, x_n, w_{0_n}, w_n)(s)|\, ds\\ &\quad+ \int_0^1 |\overline{f}(\tilde{t}_n, x_n, w_{0_n}, w_n)(s) - \overline{f}(\tilde{t}_n, \tilde{x}_n, w_{0_n}, w_n)(s)|\, ds\\ &\quad+ \int_0^1 |\overline{f}(\tilde{t}_n, \tilde{x}_n, w_{0_n}, w_n)(s) - \overline{f}(\tilde{t}_n, \tilde{x}_n, 0, w_n)(s)|\, ds\,.\end{aligned} \tag{15}$$

for all $s \in [0, 1]$. In view of the parameter-free character of the system (see e.g. [7] for the case $\alpha = \beta = 1$), it is easy to show that one can transform the integral bound (10) into the pointwise bound

$$|(w_0, w)(s)| \leq \tilde{K}_R\ \forall s \in [0, 1]\,,$$

where $\tilde{K}_R$ is a constant depending on R. Therefore, in view of basic continuity properties of the composition operator, there exists a modulus ρ such that the last integral in the above inequality is smaller than or equal to $\rho(|T-\tau|)$. Therefore, applying Gronwall's inequality to (15) we obtain

$$
\begin{aligned}
|x_n(s)-\tilde{x}_n(s)| \leq &(T+1)(K_R+1)[\epsilon(n)+\rho_f(|T-\tau|) \\
&+\rho(|T-\tau|)]e^{L_{R'}(T+1)(K_R+1)}\,.
\end{aligned} \tag{16}
$$

Hence (14) yields

$$
\begin{aligned}
V_e(T,x)-V_{e_n}(\tau,x) \leq &\int_0^1 |\bar{l}(\tilde{t}_n,\tilde{x}_n,0,w_n)(s)-\bar{l}(\tilde{t}_n,x_n,0,w_n)(s)|\,ds \\
&+\int_0^1 |\bar{l}(\tilde{t}_n,x_n,0,w_n)(s)|\,ds-\bar{l}(t_n,x_n,0,w_n)(s)|\,ds \\
&+\int_0^1 |\bar{l}(t_n,x_n,0,w_n)(s)-\bar{l}(t_n,x_n,w_{0_n},w_n)(s)\,ds \\
&+\int_0^1 |\bar{l}(t_n,x_n,w_{0_n},w_n)(s)-\bar{l}_n(t_n,x_n,w_{0_n},w_n)(s)|\,ds \\
&+\rho_g(|x_n(1)-\tilde{x}_n(1)|)\,.
\end{aligned} \tag{17}
$$

Again, an argument based on the continuity properties of the composition operator allows one to conclude that there exists a modulus $\tilde{\rho}$ such that

$$
\int_0^1 |\bar{l}(t_n,x_n,0,w_n)(s)-\bar{l}(t_n,x_n,w_{0_n},w_n)(s)|\,ds \leq \tilde{\rho}(|T-\tau|)
$$

Therefore, plugging (15) into (16), we obtain

$$
\begin{aligned}
V_e(T,x)-V_{e_n}(\tau,x) \leq &P_R(\rho_l+\rho_g)[P_R(\epsilon(n)+\rho_f(|T-\tau|)+\rho(|T-\tau|))e^{L_{R'}P_R}] \\
&+P_R[\rho_l(|T-\tau|)+\epsilon(n)]+\tilde{\rho}(|T-\tau|)\,,
\end{aligned} \tag{18}
$$

where $P_R \doteq (T+1)(K_R+1)$.

Estimates (12), (13) and (18) imply the claim, so the theorem is proved.

□

5. Implementing optimal controls in the presence of perturbations

As an application of Theorem 1, Theorems 6 and 7 below provide, for the special case of the linear quadratic problem, an answer to the general questions $\mathbf{Q}_1$ and $\mathbf{Q}_2$, respectively. Let us remark that the perturbation we consider is not the most general among those allowed by Theorem 1. However, it well illustrates the degree of improvement with respect to previous results concerning questions like $\mathbf{Q}_1$ and $\mathbf{Q}_2$ (see Introduction). Let us also remark that the linear-quadratic problem is just a model case. Indeed, it is evident that Theorem 6 below holds also if we replace the linear-quadratic problem with a problem that (satisfy hypotheses $\mathbf{A}_1$-$\mathbf{A}_5$, (4), and (5) and) admits an optimal L^β control, while Theorem 7 is still valid for any problem for which (f is Lipschitz in c uniformly for (t,x) in a compact subset of $[0,T]\times\mathbb{R}^k$ and) a Lipschitz continuous feedback control $c(x)$ does exist.

Let us be more precise by stating that by linear-quadratic problem we mean here an optimal control problem as the ones considered in the previous sections, with

$$f = f(x,c) \doteq Ax + Bc \quad l(x,c) \doteq x^t Dx + c^t Ec \quad g(x) = x^t Sx\,,$$

where D, E, S are symmetric matrices (of suitable dimensions), D and S are nonnegative definite, F is positive definite, while no assumptions are made on A, B, C. Let us observe that the fields f, l, g satisfies hypotheses $\mathbf{A}_1$-$\mathbf{A}_5$. In particular, one has $\alpha = 1$, $\beta = 2$, and Λ_0 is the smallest eigenvalue of E.

Let us consider the following perturbations of the maps f, l, g:

$$\begin{aligned} f_n &\doteq Ax + Bc + \varphi_n(t,x,c) \\ l_n &\doteq x^t Dx + c^t Ec + \theta_n(t,x,c) \\ g_n &\doteq x^t Sx + \psi_n(x)\,. \end{aligned}$$

We assume that for each compact subset $Q \subset \mathbb{R}^k$ there exist a constant λ and moduli ρ and $\overline{\rho}$ such that:

i) *For every n, the map $\varphi_n : [0,T] \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ is continuous and verifies*

$$|\varphi_n(t_1,x_1,c) - \varphi_n(t_2,x_2,c)| \le (1+|c|^{\alpha_n})(\Lambda|x_1 - x_2| + \rho(|t_1 - t_2|)$$

for all $(t_1,x_1,c), (t_2,x_2,c) \in [0,T] \times Q \times \mathbb{R}^m$ and for a suitable $\alpha_n \in [1,2]$ (varying with n and independent of Q).

ii) *There exist constants μ_1, μ_2 such that for every $n \in \mathbb{N}$ one has*

$$|\varphi_n(t,x,c)| \le \mu_1(1+|c|^{\alpha_n})(1+|x|) + \mu_2(1+|c|^{\alpha_n})$$

for every $(t,x,c) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^m$.

iii) *For every $n \in \mathbb{N}$, $\theta_n : [0,T] \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ is continuous and verifies*

$$|\theta_n(t_1,x_1,c) - \theta_n(t_2,x_2,c)| \le (1+|c|^2)\overline{\rho}(|(t_1,x_1) - (t_2,x_2)|)$$

for every $(t_1,x_1,c), (t_2,x_2,c) \in [0,T] \times Q \times \mathbb{R}^m$.

iv) *There exist a (possibly negative) constant λ_0, strictly larger than the opposite of the smallest eigenvalue of E, and a positive constant λ_1 such that*

$$\theta_n(t,x,c) \ge \lambda_0|c|^2 - \lambda_1$$

for every n and every $(t,x,c) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^m$.

v) *$\psi_n : \mathbb{R}^k \to \mathbb{R}$ is continuous and $\psi_n \ge 0$.*

Moreover we assume that for every compact $Q \subset \mathbb{R}^k$ there exists a function $\epsilon : \mathbb{N} \to \mathbb{N}$, infinitesimal as $n \to \infty$ such that

$$\begin{aligned} |\phi_n(x,c)| &\le \epsilon(n)(1+|c|^2)\,, \\ |\theta_n(x,c)| &\le \epsilon(n)(1+|c|^2) \end{aligned}$$

for every $(x,c) \in Q \times \mathbb{R}^m$ and

$$|\psi_n(x)| \le \epsilon(n)$$

for every $x \in Q$.

REMARK 2. Let us observe that the above assumptions imply that the hypotheses of the convergence theorem (Theorem 1) are verified. Let us also point out that we allow α_n to be *equal* to $\beta(=2)$ (see Remark 1).

THEOREM 6 (OPEN LOOP). *Fix $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k$. Assume that $\overline{c}$ is an optimal control for the unperturbed problem that is $J(\bar{t}, \bar{x}, \overline{c}) = V(\bar{t}, \bar{x})$. Then $\overline{c}$ is nearly optimal for the perturbed problem, i.e.,*

$$\lim_{n\to\infty} |J_n(\bar{t}, \bar{x}, \overline{c}) - V_{e_n}(\bar{t}, \bar{x})| = 0\,. \tag{19}$$

Proof. As in the proof of Theorem 1, when the initial condition are taken in a ball $B[0, R]$, by the coercivity condition $\mathbf{A}_4$ we can consider only controls such that

$$\int_0^T (1 + |c(s)|)^2\, ds \le K_R\,,$$

where K_R is a suitable constant depending on R. Then, by Hölder's inequality we have also,

$$\int_0^T (1 + |c(s)|)^{\alpha_n}\, ds \le (K_R + 1)(T + 1)$$

which, by Gronwall's inequality, implies that there is a ball $B[0, R']$ which contains all the trajectories issuing from $B[0, R]$. Setting $x(\cdot) \doteq x_{(\bar{t},\bar{x})}[\overline{c}](\cdot)$ and $x_n(\cdot) \doteq x^n_{(\bar{t},\bar{x})}[\overline{c}](\cdot)$, we have

$$\begin{aligned} |x_n(t) - x(t)| &\le \int_{\bar{t}}^T |f_n(s, x_n(s), \overline{c}(s)) - f(s, x(s), \overline{c}(s))|\, ds \\ &\le \int_{\bar{t}}^T |Ax_n(s) - Ax(s) + \phi_n(x_n, \overline{c})(s)|\, ds \\ &\le (T + 1)(K_R + 1)\epsilon(n) + \|A\| \int_{\bar{t}}^T (|x_n(s) - x(s)|)\, ds \end{aligned} \tag{20}$$

where $\epsilon(n)$ is relative to $B[0, R]$ and $\|A\|$ is the operator norm of the matrix A. Hence, Gronwall's Lemma implies

$$|x_n(s) - x(s)| \le (T + 1)(K_R + 1)\epsilon(n)e^{\|A\|T}\,, \tag{21}$$

for every $t \in [\bar{t}, T]$. Since

$$\begin{aligned} |J_n(t, x, \overline{c}) - J(t, x, \overline{c})| \le &\int_{\bar{t}}^T |x_n(s)^t Dx_n(s) - x(s)^t Dx(s)|\, ds \\ &+ \int_{\bar{t}}^T |\theta_n(x_n(s), \overline{c}(s))|\, ds \\ &+ |x_n(T)^t Sx_n(T) - x(T)^t Sx(T)| + |\psi_n(x_n(T))| \\ \le &\|D\| \int_{\bar{t}}^T |x_n(s) - x(s)|(|x_n(s|) + |x(s)|)\, ds \\ &+ \|S\||x_n(s) - x(s)|(|x_n(T)| + |x(T)|) + \epsilon(n)(1 + K_R)\,, \end{aligned} \tag{22}$$

in view of estimate (22) and of Theorem 1, the theorem is proven.

□

THEOREM 7. *Let $c(x)$ be a locally Lipschitz continuous optimal feedback control for the unperturbed problem. Then this control is nearly optimal for the perturbed problem, that is*

$$\lim_{n\to\infty} |J_n(\bar{t}, \bar{x}, c) - V_{e_n}(\bar{t}, \bar{x})| = 0\,.$$

Proof. If we denote by $x(\cdot)$ and $x_n(\cdot)$ the solutions to (E) to (E_n), respectively, corresponding to the feedback control $c(x)$, we obtain

$$\begin{aligned}
|x_n(t) - x(t)| &\leq \int_{\bar{t}}^{T} | f_n(s, x_n(s), c(x_n(s))) - f(s, x(s), c(x(s)))| \, ds \\
&\leq \int_{\bar{t}}^{T} |Ax_n(s) + Bc(x_n(s)) - Ax(s) - Bc(x(s))| \, ds \\
&\quad + \int_{\bar{t}}^{T} |\phi_n(x_n(s), c(x_n(s))| \, ds \\
&\leq \int_{\bar{t}}^{T} (\|A\| + \|B\|\gamma)(|x_n(s) - x(s)| \, ds + (T+1)(K_R+1)\epsilon(n)
\end{aligned}$$

where γ is the Lipschitz constant of the map $c(x)$ corresponding to the compact set $B[0, R']$. Hence one has

$$|x_n(s) - x(s)| \leq (T+1)(K_R+1)\epsilon(n)e^{T\|A\|+\|B\|\gamma}\,,$$

and from here on one can proceed as in the proof of Theorem 6.

□

REMARK 3. As we have mentioned in the Introduction, when $\alpha_n = \beta$ it may happen that a perturbed problem (P_n) does not possess a minimum in the class of absolutely continuous trajectories. Indeed, due to the fact that the growth ratio $\frac{\beta}{\alpha_n}(= 1)$ is not greater than 1, the minimizing sequences could converge to a *discontinuous trajectory*. In this case, the possibility of implementing a control that is optimal for the unperturbed system – which is now assumed sufficiently coercive, that is, satisfying $\alpha < \beta$ – turns out to be of some interest whenever one is worried to avoid a discontinuous performance of the system under consideration.

To be more concrete, let us consider the very simple (linear-quadratic) minimum-problem where $l = x^2 + \Lambda_0 c^2$ and $f = 0$. In this case $\beta = 2$ and α can be taken equal to 1. Let us perturb this problem by taking $l_n = l$ and $f_n = f + \varphi_n = \varphi_n \doteq \frac{-c^2}{n}$. Observe that these perturbations give rise to *quadratic-quadratic* problems, that is problems where $\alpha_n = \beta = 2$. Let us consider the initial data $\bar{t} = 0$ and $\bar{x} > 0$. The constant map $x(t) = \bar{x}$ is the unique trajectory of the unperturbed system, so the control $\hat{c}(t) = 0 \forall t \in [0, T]$ turns out to be optimal. In view of Theorem 6 this control is nearly optimal for the perturbed problems as well. However, as soon $\bar{x}$ is sufficiently large and Λ_0 is sufficiently small with respect to $\frac{1}{n}$, an application of the Maximum Principle to the space-time extension of the perturbed system shows that the "optimal trajectory" of the perturbed problem is the concatenation of an "initial jump" (from $\bar{x}$ to a point $x_n \in]0, \bar{x}[$) and a suitable absolutely continuous map.

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