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On Certain Properties of A Class of Bivariate Compound Poisson Distributions and an Application to Earthquake Data

Ciertas propiedades de una clase de distribuciones Poisson compuesta bivariadas y una aplicación a datos de terremotos

Gamze Özel[a]

Department of Statistics, Hacettepe University, Ankara, Turkey

Abstract

The univariate compound Poisson distribution has many applications in various areas such as biology, seismology, risk theory, forestry, health science, etc. In this paper, a bivariate compound Poisson distribution is proposed and the joint probability function of this model is derived. Expressions for the product moments, cumulants, covariance and correlation coefficient are also obtained. Then, an algorithm is prepared in Maple to obtain the probabilities quickly and an empirical comparison of the proposed probability function is given. Bivariate versions of the Neyman type A, Neyman type B, geometric-Poisson, Thomas distributions are introduced and the usefulness of these distributions is illustrated in the analysis of earthquake data.

***Key words*:** Bivariate distribution, Coefficient of correlation, Compound Poisson distribution, Cumulant, Moment.

Resumen

La distribución compuesta de Poisson univariada tiene muchas aplicaciones en diversas áreas tales como biología, ciencias de la salud, ingeniería forestal, sismología y teoría del riesgo, entre otras. En este artículo, una distribución compuesta de Poisson bivariada es propuesta y la función de probabilidad conjunta de este modelo es derivada. Expresiones para los momentos producto, acumuladas, covarianza y el coeficiente de correlación respectivos son obtenidas. Finalmente, un algoritmo preparado en lenguaje Maple es descrito con el fin de calcular probabilidades asociadas rápidamente y con el fin de hacer una comparación de la función de probabilidad propuesta. Se introducen además versiones bivariadas de las distribuciones tipo A y tipo B de Neyman, geométrica-Poisson y de Thomas y se ilustra la utilidad de estas distribuciones aplicadas al análisis de datos de terremoto.

***Palabras clave*:** coeficiente de correlación, conjuntas, distribución bivariada, distribución compuesta de Poisson, momento.

[a]Doctor. E-mail: gamzeozl@hacettepe.edu.tr

1. Introduction

Bivariate discrete random variables taking integer non-negative values, have received considerable attention in the literature, in an effort to explain phenomena in various areas of application. For an extensive account of bivariate discrete distributions one can refer to the books by Kocherlakota & Kocherlakota (1992), Johnson, Kotz & Balakrishnan (1997) and the review articles by Papageorgiou (1997) and Kocherlakota & Kocherlakota (1997). There is however, a variety of applications, e.g. in an accident or family studies (see Cacoullos & Papageorgiou 1980, Sastry 1997). The bivariate Poisson distribution (BPD) is probably the best known bivariate discrete distribution (Holgate 1964). It is appropriate for modeling paired count data exhibiting correlation. Paired count data arise in a wide context including marketing (number of purchases of different products), epidemiology (incidents of different diseases in a series of districts), accident analysis (number of accidents in a site before and after infrastructure changes), medical research (the number of seizures before and after treatment), sports (the number of goals scored by each one of the two opponent teams in soccer), econometrics (number of voluntary and involuntary job changes).

Bivariate compound distributions can be especially used in actuarial science to model a business book containing bivariate claim count distributions and bivariate claims severities (Ambagaspitiya 1998). In most actuarial studies, the assumption of independence between classes of business in an insurance business book containing is made. However this assumption is not verified in practice. For example, in the case of a catastrophe such as an earthquake, the damages covered by homeowners and private passenger automobile insurance can not be considered independent (Cossette, Gaillardetz, Marceau & Rioux 2002). In this situation, bivariate compound Poisson distribution (BCPD) is useful when the claim count distribution is bivariate Poisson and the claim size distribution is bivariate.

Although the case of BPD has attracted some attention in the literature, BCPD has not been systematically studied. The studies on such a distribution are sparse due to computational problems involved in its implementation. Hesselager (1996) studied the BCPD but mainly from the recursive evaluation of its joint probability function. On the other hand, non-existence of explicit probabilities and algorithm of the BCPD hinders its use in probability theory itself and its applications in seismology, actuarial science, survival analysis, etc. (see Ozel & Inal 2008, Wienke, Ripatti, Palmgren & Yashin 2010). Consequently, since relative results are sparse and case oriented, the aim of this study is to obtain a general technique for deriving the probabilistic characteristics and obtain an algorithm for the computation of probabilities.

The rest of the paper is organised as follows. In Section 2, some preliminary results are given. In Section 3, the probabilistic characteristics of the BCPD are proposed based on the derivation of the joint probability generating function (pgf). This pgf enables us to obtain the joint probability function of the BCPD. In addition, explicit expressions for the product moments, cumulants, covariance and correlation coefficient are obtained. Then numerical examples and an application to earthquakes in Turkey are presented in Section 4, by means of the proposed algorithm in Maple. The conclusion is given in Section 5.

2. Some Preliminary Results

Let N be a Poisson random variable with parameter $\lambda > 0$ and let X_i, $i = 1, 2, \dots$ be i.i.d. non-negative, integer-valued random variables, independent of N. S has a compound Poisson distribution (CPD), when defined as

$$S = \sum_{i=1}^{N} X_i \tag{1}$$

If $E(X)$ and $V(X)$ are the common mean and variance of the random variables X_1, $i = 1, 2, \dots$, then, the moments of S are given by

$$E(S) = \lambda E(X), \quad V(S) = \lambda[V(X) + [E(X)]^2] \tag{2}$$

The probability function of S is given by

$$p_S(s) = P(S = s) = \sum_{n=0}^{\infty} P(X_1 + X_2 + \cdots + X_n = s \mid N = n)P(N = n), \quad s = 0, 1, 2 \dots \tag{3}$$

However, it is not easy to yield an explicit formula for the probability function of S from (3), and this obstructs use of the CPD completely (see, for example Bruno, Camerini, Manna & Tomassetti 2006, Rolski, Schmidli, Schmidt & Teugels 1999). Panjer (1981) described a procedure for recursive evaluation of the CPD when N is Poisson distributed.

Let N be a Poisson distributed random variable with parameter λ and let S be a compound Poisson distributed random variable. Panjer (1981) showed that when N satisfies a recursion in the form $p_N(n) = \frac{\lambda}{n} p_N(n-1)$, $n = 1, 2, 3 \dots$ than S satisfies

$$\begin{aligned} p_S(0) &= e^{-\lambda[1-p_X(0)]} \\ p_S(s) &= \lambda \sum_{i=1}^{s} \frac{i}{s} p_X(i) p_S(s-i), \quad s = 1, 2, 3 \dots \end{aligned} \tag{4}$$

where $p_X(x)$ is the common probability function of X_i, $i = 1, 2, 3 \dots$ Since (4) is based on a recursive scheme, it causes difficulties in computation time and computer memory for the large values of s (Rolski et al. 1999). The explicit probabilities of S are obtained by Ozel & Inal (2010) as in (6) by using (5).

Let X_i, $i = 1, 2, 3 \dots$, be i.i.d. discrete random variables with the probabilities $P(X_i = j) = p_j$, $j = 0, 1, 2 \dots$ and let define the parameters $\lambda_j = \lambda p_j$. The common probability generating function (pgf) of X_i, $i = 1, 2, 3 \ldots$, is given by $g_X(s) = \sum_{j=0}^{\infty} p_j s^j = p_0 + p_1 s + p_2 s^2 + \cdots$ and the pgf of S is given by

$$\begin{aligned} g_S(z) &= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} [g_X(z)]^n = e^{-\lambda} \left[1 + \frac{\lambda g_X(z)}{1!} + \frac{(\lambda g_X(z))^2}{2!} + \cdots \right] \\ &= e^{\lambda[g_X(z)-1]} = e^{\lambda[(p_0+p_1 z+\cdots+p_m z^m)-1]} \\ &= e^{-\lambda(1-p_0)} e^{\lambda_1 z+\lambda_2 z^2+\cdots+\lambda_m z^m} \end{aligned} \tag{5}$$

Let N be a Poisson distributed random variable with parameter $\lambda > 0$ and $\lambda_j = \lambda p_j$, $j = 1, 2, \ldots, m$. Then, the explicit formula for the probability function of S is determined by using (5) as follows:

$$\begin{aligned} P(S=0) &= e^{-\lambda(1-p_0)} \\ P(S=1) &= e^{-\lambda(1-p_0)} \frac{\lambda_1}{1!} \\ P(S=2) &= e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^2}{2!} + \frac{\lambda_2}{1!} \right] \\ P(S=3) &= e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^3}{3!} + \frac{\lambda_1 \lambda_2}{1!1!} + \frac{\lambda_3}{1!} \right] \\ P(S=4) &= e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^4}{4!} + \frac{\lambda_1^2 \lambda_2}{2!1!} + \frac{\lambda_1 \lambda_3}{1!1!} + \frac{\lambda_2^2}{2!} + \frac{\lambda_4}{1!} \right] \\ P(S=5) &= e^{-\lambda(1-p_0)} \left[\frac{\lambda_1^5}{5!} + \frac{\lambda_1^3 \lambda_2}{3!1!} + \frac{\lambda_1^2 \lambda_3}{2!1!} + \frac{\lambda_1 \lambda_2^2}{1!2!} + \frac{\lambda_1 \lambda_4}{1!1!} + \frac{\lambda_2 \lambda_3}{1!1!} + \frac{\lambda_5}{1!} \right] \\ &\vdots \end{aligned} \tag{6}$$

According to the above probabilities for $s = 1, 2, \ldots$, the on the right terms depend on how s can be partitioned into different forms using integers $1, 2, \ldots$,m. For example, if $s = 5$, it is partitioned in seven ways and all the partitions of five are $\{1, 1, 1, 1, 1\}$, $\{1, 1, 1, 2\}$, $\{1, 2, 2\}$, $\{1, 1, 3\}$, $\{2, 3\}$, $\{1, 4\}$, $\{5\}$. Note that S has a Neyman type A distribution if X_i, $i = 1, 2, \ldots$ are Poisson distributed in (1). Similarly, if X_i, $i = 1, 2, \ldots$ are truncated Poisson distributed, S has a Thomas distribution. S has a Neyman type B distribution if X_i, $i = 1, 2, \ldots$, are binomial distributed. If X_i, $i = 1, 2, \ldots$ are geometric distributed, S has a geometric-Poisson (Pólya-Aeppli) distribution. Let us point out that (6) is also extended by Ozel & Inal (2011) for these special cases of the CPD and by Ozel & Inal (2008) for the compound Poisson process with an application for earthquakes in Turkey. There has also been an increasing interest in bivariate discrete probability distributions and many forms of these distributions have been studied (see, for example, Kocherlakota & Kocherlakota 1992, Johnson et al. 1997). The BPD has been constructed by Holgate (1964) as in (7) using the trivariate reduction method.

Let M_0, M_1, M_2 be independent Poisson variables with parameters $\lambda_0, \lambda_1, \lambda_2$, respectively. Then, $N_1 = M_0 + M_1$ and $N_2 = M_0 + M_1$ follow a BPD and the joint probability function is given by

$$p_{N_1,N_2}(n_1,n_2) = P(N_1 = n_1, N_2 = n_2) = \\ e^{-(\lambda_0+\lambda_1+\lambda_2)} \sum_{i=0}^{\min(n_1,n_2)} \frac{\lambda_1^{n_1-i}\lambda_2^{n_2-i}\lambda_0^i}{(n_1-i)!(n_2-i)!i!}, \quad n_1, n_1 = 0, 1, 2, \dots \quad (7)$$

The formula in (7), allows positive dependence between N_1 and N_2. Marginally, each random variable follows a Poisson distribution with $E(N_1) = V(N_1) = \lambda_0+\lambda_1$ and $E(N_2) = V(N_2) = \lambda_0 + \lambda_2$. Moreover, $Cov(N_1, N_2) = \lambda_0$, and hence λ_0 is a measure of dependence between the two random variables. Then, the correlation coefficient of N_1 and N_2 is given by

$$\rho = \frac{\lambda_0}{\sqrt{(\lambda_0 + \lambda_1)(\lambda_0 + \lambda_2)}}$$

This implies that $\lambda_0 = 0$ is a necessary and sufficient condition for N_1 and N_2 to be independent. Also, $\lambda_0 = 1$, if and only if, N_1 and N_2 are linearly dependent.

In Section 3, the concept of the CPD is extended to the bivariate case.

3. Main Results

3.1. The Joint Probability Function

Let M_0, M_1, M_2 be independent Poisson variables with parameters $\lambda_0, \lambda_1, \lambda_2$, respectively, and let $N_1 = M_0+M_1$, $N_2 = M_0+M_2$ be bivariate Poisson distributed random variables with parameters $\lambda_0+\lambda_1$ and $\lambda_0+\lambda_2$. Then, (S_1, S_2) has a BCPD when defined as

$$\left(S_1 = \sum_{i=1}^{N_1} X_i, S_2 \sum_{i=1}^{N_2} Y_i\right) \quad (8)$$

where X_i and Y_i, $i = 1, 2, \dots$ i.i.d. integer-valued random variables and independent of N_1 and N_2.

In particular, if X_i and Y_i, $i = 1, 2, \dots$ are Poisson distributed with parameters μ_1 and μ_2 in (8), S_1 and S_2 have a bivariate Neyman type A distribution. If X_i and Y_i, $i = 1, 2, \dots$ are binomial distributed with parameters (m_1, p_1) and (m_2, p_2), S_1 and S_2 have a bivariate Neyman type B distribution. Let X_i and Y_i, $i = 1, 2, \dots$ are truncated Poisson distributed with the probability functions $p_j = P(X_i = j) = e^{-\alpha_1}\frac{\alpha_1^{j-1}}{(j-1)!}$, $j = 1, 2, 3, \dots$ and $q_k = P(Y_i = k) = e^{-\alpha_2}\frac{\alpha_2^{j-1}}{(j-1)!}$, $k = 1, 2, 3, \dots$ for $\alpha_1, \alpha_2 > 0$, respectively. Then, the pair of (S_1, S_2) has a bivariate Thomas distribution. If X_i and Y_i, $i = 1, 2, \dots$ are geometric distributed with parameters θ_1 and θ_2, S_1 and S_2 have a bivariate geometric-Poisson distribution.

The joint probability function of S_1 and S_2 takes the following form

$$
\begin{aligned}
&p_{S_1,S_2}(s_1,s_2) = \\
&\qquad \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} p(n_1,n_2)P(X_1+\cdots+X_{n_1}=s_1 \mid N_1=n_1) \\
&\qquad\qquad P(Y_1+\cdots+Y_{n_2}=s_2 \mid N_2=n_2), \quad s_1,s_2=0,1,\dots \qquad (9)
\end{aligned}
$$

where $p_{S_1,S_2}(s_1,s_2) = P(S_1=s_1, S_2=s_2)$. Since the probability function given in (9) contains a summation over i from 0 to ∞, it is not suitable to obtain probabilities quickly (Ambagaspitiya 1998). More generally, for large n_1 and n_2, it is difficult to use (9) because of the high order of convolutions involved.

Hesselager (1996), in his pioneering work on recursive computation of the bivariate compound distributions, considered three classes of Poisson distributions and related compound distributions. A brief description of related recursive relations is given as follows:

Let M_0, M_1, M_2 be independent Poisson variables with parameters $\lambda_0, \lambda_1, \lambda_2$. Let $p_X(x)$ and $p_Y(y)$ be the common probability function of X_i, Y_i, $i=1,2,\dots$, respectively. Then, the joint probability function of S_1 and S_2 satisfies the recursive relations

$$
\begin{aligned}
p_{S_1,S_2}(s_1,s_2) &= \frac{\lambda_1}{s_1}\sum_{x=1}^{s_1} x p_X(x) p_{S_1,S_2}(s_1-x,s_2) + \\
&\quad \frac{\lambda_0}{s_1}\sum_{x=1}^{s_1}\sum_{y=0}^{s_2} x p_X(x) p_Y(y) p_{S_1,S_2}(s_1-x,s_2-y) \\
p_{S_1,S_2}(s_1,s_2) &= \frac{\lambda_2}{s_1}\sum_{x=1}^{s_1} y p_Y(y) p_{S_1,S_2}(s_1,s_2-y) + \qquad\qquad (10) \\
&\quad \frac{\lambda_0}{s_2}\sum_{x=0}^{s_1}\sum_{y=1}^{s_2} y p_X(x) p_Y(y) p_{S_1,S_2}(s_1-x,s_2-y) \\
&\qquad\qquad s_1,s_2=1,2,\dots
\end{aligned}
$$

Although the use of these recursions considerably reduces the number of computations to obtain probabilities $P(S_1=s_1, S_2=s_2)$, $s_1,s_2=0,1,2,\dots$ compared with the traditional method based on convolutions in (9), these computations are still time consuming since each probability depends on all the preceding ones. It occurs in underflow problems which are not always easy to overcome and therefore restrict its applicability further (Sundt 1992). Thus, it can be applied only in some practical circumtances or in an approximate manner.

Finally to establish the probabilistic characteristics of the BCPD. We first compute the joint pgf of S_1 and S_2 as follows:

Let X_i, Y_i, $i=1,2,\dots$ be i.i.d. discrete random variables with the probabilities $P(X_i=j)=p_j$, $j=0,1,2,\dots,m$ and $P(Y_i=k)=q_k$, $k=0,1,2,\dots,r$. Then, the joint pgf of S_1 and S_2 is found to be

$$\begin{aligned}
g_{S_1,S_2}(z_1,z_2) &= \sum_{s_1}^{\infty}\sum_{s_2}^{\infty} P\left(\sum_{i=1}^{N_1} X_i = s_1, \sum_{i=1}^{N_2} Y_i = s_2\right) z_1^{s_1} z_2^{s_2} \\
&= \sum_{s_1}^{\infty}\sum_{s_2}^{\infty}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \\
& P\left(\sum_{i=1}^{n_1} X_i = s_1, \sum_{i=1}^{n_2} Y_i = s_2 \mid N_1 = n_1, N_2 = n_2\right) \\
& p_{N_1,N_2}(n_1,n_2) z_1^{s_1} z_2^{s_2} \\
&= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} p_{N_1,N_2}(n_1,n_2) \sum_{s_1}^{\infty}\sum_{s_2}^{\infty} \\
& P\left(\sum_{i=1}^{n_1} X_i = s_1, \sum_{i=1}^{n_2} Y_i = s_2 \mid N_1 = n_1, N_2 = n_2\right) z_1^{s_1} z_2^{s_2}
\end{aligned}$$

Since X_i, Y_i, $i = 1, 2, \dots$ are i.i.d. random variables, we have

$$\begin{aligned}
g_{S_1,S_2}(z_1,z_2) &= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} p_{N_1,N_2}(n_1,n_2) \sum_{s_1}^{\infty} P(X_1 + \cdots + X_{n_1} = s_1) z_1^{s_1} \\
& \sum_{s_2}^{\infty} P(Y_1 + \cdots + Y_{n_2} = s_1) z_2^{s_2} \\
&= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} p_{N_1,N_2}(n_1,n_2) g_{X_1+\cdots+X_{n_1}}(z_1) g_{Y_1+\cdots+Y_{n_2}}(z_2) \\
&= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} p_{N_1,N_2}(n_1,n_2) [g_X(z_1)]^{n_1} [g_Y(z_2)]^{n_2} \\
&= g_{N_1,N_2}[g_X(z_1), g_Y(z_2)]
\end{aligned} \tag{11}$$

where $g_X(z_1)$, $g_Y(z_2)$ are the common pgfs of X_i, Y_i, $i = 1, 2, \dots$, respectively.

Let $N_1 = M_0 + M_1$, $N_2 = M_0 + M_2$ be a BPD with parameters $\lambda_0 + \lambda_1$ and $\lambda_0 + \lambda_2$, then the joint pgf of N_1 and N_2 is given by

$$\begin{aligned}
g_{N_1,N_2}(z_1,z_2) &= g_{M_0+M_1,M_0+M_2}(z_1,z_2) \\
&= E(z_1^{M_0+M_1} z_2^{M_0+M_2}) \\
&= E(z_1^{M_1}) E(z_2^{M_2}) E(z_1 z_2)^{M_0} \\
&= \exp[\lambda_1(z_1-1) + \lambda_2(z_2-1) + \lambda_0(z_1 z_2 - 1)]
\end{aligned} \tag{12}$$

From (11) and (12), the joint pgf of S_1 and S_2 is obtained by the following expression

$$
\begin{aligned}
g_{S_1,S_2}(z_1,z_2) &= \exp\big(\lambda_1[g_X(z_1)-1]+\lambda_2[g_Y(z_2)-1] \\
&\quad +\lambda_0[g_X(z_1)g_Y(z_2)-1]\big) \\
&= \exp\big(\lambda_1[p_0+p_1z_1+p_2z_1^2+\cdots+p_mz_1^m-1] \\
&\quad +\lambda_2[q_0+q_1z_1+q_2z_2^2+\cdots+q_rz_2^r-1] \\
&\quad +\lambda_0\big[(p_0+p_1z_2+p_2z_1^2+\cdots+p_mz_1^m) \\
&\qquad (q_0+q_1z_2+q_2z_2^2+\cdots+q_rz_2^r)-1\big]\big) \\
&= \exp\big(-(\lambda_0+\lambda_1+\lambda_2)\big) \\
&\quad \exp\big(\lambda_1(p_0+p_1z_1+\cdots+p_mz_1^m) \\
&\quad +\lambda_2(q_0+q_1z_2+\cdots+q_rz_2^r) \\
&\quad +\lambda_0[(p_0+p_1z_1+\cdots+p_mz_1^m)(q_0+q_1z_2+\cdots+q_rz_2^r)]\big)
\end{aligned} \tag{13}
$$

Now we are interested in studying the joint probability function of the pair S_1 and S_2. The joint pgf in (13) can be differentiated any number of times with respect to s_1 and s_2 and evaluated at $(0,0)$ yielding

$$
\begin{aligned}
P(S_1=0,S_2=0) &= g_{S_1,S_2}(0,0) \\
P(S_1=s_1,S_2=s_2) &= \frac{\left.\frac{\partial^{S_1+S_2}g_{S_1,S_2}(z_1,z_2)}{\partial z_1^{s_1}z_2^{s_2}}\right|_{z_1=z_2=0}}{s_1!s_2!}, \quad s_1s_2=0,1,2,\ldots
\end{aligned} \tag{14}
$$

Differentiating the joint pgf given by (13) and substituting in (14) and after some algebraic manipulations, the probabilities $p_{S_1,S_2}(s_1,s_2)=P(S_1=s_1,S_2=s_2)$, $s_1s_2=0,1,2,\ldots$ are obtained as

$$
\begin{aligned}
p_{S_1,S_2}(0,0) &= e^{-(\lambda_0+\lambda_1+\lambda_2)}e^{(\lambda_1p_0+\lambda_2q_0+\lambda_0p_0q_0)} \\
p_{S_1,S_2}(1,0) &= p_{S_1,S_2}(0,0)\left[p_1\frac{\Lambda_x}{1!}\right] \\
p_{S_1,S_2}(2,0) &= p_{S_1,S_2}(0,0)\left[p_1^2\frac{\Lambda_x^2}{2!}+p_2\frac{\Lambda_x}{1!}\right] \\
p_{S_1,S_2}(3,0) &= p_{S_1,S_2}(0,0)\left[p_1^3\frac{\Lambda_x^3}{3!}+p_1p_2\frac{\Lambda_x^2}{2!}+p_3\frac{\Lambda_x}{1!}\right] \\
p_{S_1,S_2}(0,1) &= p_{S_1,S_2}(0,0)\left[q_1\frac{\Lambda_y}{1!}\right]
\end{aligned}
$$

$$
\begin{aligned}
p_{S_1,S_2}(0,2) &= p_{S_1,S_2}(0,0)\left[q_1^2\frac{\Lambda_y^2}{2!} + q_2\frac{\Lambda_y}{1!}\right]\\
p_{S_1,S_2}(0,3) &= p_{S_1,S_2}(0,0)\left[q_1^3\frac{\Lambda_y^3}{3!} + q_1q_2\frac{\Lambda_y^2}{2!} + q_3\frac{\Lambda_y}{1!}\right]\\
p_{S_1,S_2}(1,1) &= p_{S_1,S_2}(0,0)\left[p_1q_1\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\right]\\
p_{S_1,S_2}(1,2) &= p_{S_1,S_2}(0,0)\left[p_1q_1^2\left(\frac{\Lambda_x\Lambda_y^2}{1!2!} + \frac{\Lambda_y}{1!}\right) + p_1q_2\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\right]\\
p_{S_1,S_2}(1,3) &= p_{S_1,S_2}(0,0)\bigg[p_1q_1^3\left(\frac{\Lambda_x\Lambda_y^3}{1!3!} + \frac{\Lambda_y^2}{2!}\right) + p_1q_1q_2\left(\frac{\Lambda_x\Lambda_y^2}{1!2!} + \frac{\Lambda_y}{1!}\right)\\
&\quad + p_1q_3\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\bigg]\\
p_{S_1,S_2}(2,1) &= p_{S_1,S_2}(0,0)\left[p_1^2q_1\left(\frac{\Lambda_x^2\Lambda_y}{1!2!} + \frac{\Lambda_x}{1!}\right) + p_2q_1\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\right]\\
p_{S_1,S_2}(2,2) &= p_{S_1,S_2}(0,0)\bigg[p_1^2q_1^2\left(\frac{\Lambda_x^2\Lambda_y^2}{2!2!} + \frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0^2\right)\\
&\quad + p_1^2q_2\left(\frac{\Lambda_x^2\Lambda_y}{2!1!} + \frac{\Lambda_x}{2!1!}\right) + p_2q_1^2\left(\frac{\Lambda_x\Lambda_y^2}{1!2!} + \frac{\Lambda_y}{1!}\right)\\
&\quad + p_2q_2\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\bigg]\\
p_{S_1,S_2}(2,3) &= p_{S_1,S_2}(0,0)\bigg[p_1^2q_1^3\left(\frac{\Lambda_x^2\Lambda_y^3}{2!3!} + \frac{\Lambda_x\Lambda_y^2}{1!2!} + \frac{\Lambda_y}{1!}\right)\\
&\quad + p_1^2q_1q_2\left(\frac{\Lambda_x^2\Lambda_y^2}{2!2!} + \frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0^2\right)\\
&\quad + p_2q_1^3\left(\frac{\Lambda_x\Lambda_y^3}{3!1!} + \frac{\Lambda_y^2}{2!}\right) + p_2q_1q_2\left(\frac{\Lambda_x\Lambda_y^2}{1!2!} + \frac{\Lambda_y^2}{2!} + \frac{\Lambda_y}{1!}\right)\\
&\quad + p_1^2q_3\left(\frac{\Lambda_x^2\Lambda_y}{2!1!} + \frac{\Lambda_x}{1!}\right) + p_2q_3\left(\frac{\Lambda_x\Lambda_y}{1!1!} + \lambda_0\right)\bigg]
\end{aligned}
\tag{15}
$$

where $\Lambda_x = (\lambda_1 + \lambda_0 q_0)$ and $\Lambda_y = (\lambda_2 + \lambda_0 p_0)$. According to above probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 1, 2, 3, \dots$ the on the right side terms p_j, $j = 1, 2, \dots, m$ and q_k, $k = 1, 2, \dots, r$ depend on how s_1 and s_2 can be partitioned into different forms using integers $1, 2, \dots$ Similarly, the terms Λ_x and Λ_y also have an order related with the powers of p_j, $j = 1, 2, \dots, m$ and q_k, $k = 1, 2, \dots, r$ based on the integer partitions. Furthermore, the denominators of Λ_x and Λ_y suitable to these partitions. For example, if $(s_1 = 1, s_2 = 3)$, the partitions of p_j for $j = 1$ and q_k, $k = 1, 2, 3$ are $(p_1, q_1^3), (p_1, q_1q_2), (p_1, q_3)$ and the partitions of Λ_x and Λ_y

are $\left[\left(\frac{\Lambda_x}{1!}, \frac{\Lambda_y^3}{3!}, \frac{\Lambda_y^2}{2!}\right)\right]$ for p_1, q_1^3, $\left[\left(\frac{\Lambda_x}{1!}, \frac{\Lambda_y^2}{2!}, \frac{\Lambda_y}{1!}\right)\right]$ for $p_1, q_1 q_2$, $\left[\left(\frac{\Lambda_x}{1!}, \frac{\Lambda_y^1}{1!}\right)\right]$ for p_1, q_3. Using these properties, an algorithm is prepared in Maple for the joint probability function of the BCPD.

A general formula is given in (15) for the joint probability function of the BCPD. $P(X_i = j) = p_j$, $j = 0, 1, 2, \ldots .m$ and $P(Y_i = k) = q_k$, $k = 0, 1, 2, \ldots, r$ are defined in (15) to obtain joint probabilities of bivariate Neyman type A and B, Thomas and geometric-Poisson distribution respectively,

$$
\begin{aligned}
p_j &= e^{-\mu_1}\mu_1^j/j!, && j = 0, 1, 2, \ldots \\
q_k &= e^{-\mu_2}\mu_2^k/k!, && k = 0, 1, 2, \ldots \\
p_j &= \binom{m_1}{j} p_1^j (1-p_1)^{m_1-j}, && j = 0, 1, 2, \ldots, m_1 \\
q_k &= \binom{m_2}{k} p_2^k (1-p_2)^{m_2-k}, && k = 0, 1, 2, \ldots, m_2 \\
p_j &= e^{-\alpha_1}\alpha_1^{(j-1)}/(j-1)!, && j = 1, 2, \ldots \\
q_k &= e^{-\alpha_2}\alpha_2^{(k-1)}/(k-1)!, && k = 1, 2, \ldots \\
p_j &= \theta_1(1-\theta_1)^j, && j = 0, 1, 2, \ldots \\
q_k &= \theta_2(1-\theta_2)^k, && k = 0, 1, 2, \ldots
\end{aligned}
$$

3.2. Joint Moment Characteristics

We turn now to the consideration of moments and coefficient of correlation for the BCPD. As far as we know, product moments, cumulants, coefficient of correlation and covariance of the BCPD have never been investigated before (Homer 2006). We start with finding (a, b)-th product moment $\mu'(a, b) = E(S_1^a S_2^b)$. We derive the product moments of S_1 and S_2 by calculating the joint moment generating function

$$
\begin{aligned}
M(z_1, z_2) = \exp(-(\lambda_0 + \lambda_1 + \lambda_2)) \exp\big(&\lambda_1[p_0 + p_1\exp(z_1) + \cdots + p_m \exp(z_1^m)] \\
&+ \lambda_2[q_0 + q_1\exp(z_2) + \cdots + q_r\exp(z_2^r)] \\
&+ \lambda_0[(p_0 + p_1\exp(z_1) + \cdots + p_m\exp(z_1^m)) \\
&\ (q_0 + q_1\exp(z_2) + \cdots + q_r\exp(z_2^r))]\big)
\end{aligned}
$$

Differentiating $M(z_1, z_2)$ at $z_1 = z_2 = 0$, the (a, b)-th product moments are given by

$$
\begin{aligned}
\mu'(1,1) &= \mu_X^{[1]}\mu_Y^{[1]}(\Lambda_1 + \Lambda_2 + \Lambda_0)\\
\mu'(2,1) &= \left(\mu_X^{[1]}\right)^2 \mu_Y^{[1]}(\Lambda_1^2\Lambda_2 + \Lambda_1) + \mu_X^{[2]}\mu_Y^{[1]}(\Lambda_1\Lambda_2 + \Lambda_0)\\
\mu'(3,1) &= \left(\mu_X^{[1]}\right)^3 \mu_Y^{[1]}(\Lambda_1^3\Lambda_2 + \Lambda_1^2) + \mu_X^{[1]}\mu_X^{[2]}\mu_Y^{[1]}(\Lambda_1^2\Lambda_2 + \Lambda_1)\\
&\quad + \mu_X^{[3]}\mu_Y^{[1]}(\Lambda_1\Lambda_2 + \Lambda_0)\\
\mu'(2,2) &= \left(\mu_X^{[1]}\right)^2 \left(\mu_Y^{[1]}\right)^2 (\Lambda_1^2\Lambda_2^2 + \Lambda_1\Lambda_2 + \Lambda_0^2)\\
&\quad + \mu_X^{[2]}\left(\mu_Y^{[1]}\right)^2 (\Lambda_1 + \Lambda_2^2 + \Lambda_2) + \left(\mu_X^{[1]}\right)^2 \mu_Y^{[2]}(\Lambda_1^2\Lambda_2 + \Lambda_1)\\
&\quad + \mu_X^{[2]}\mu_Y^{[2]}(\Lambda_1\Lambda_2 + \Lambda_0)\\
\mu'(2,3) &= \left(\mu_X^{[2]}\right)^2 \left(\mu_Y^{[1]}\right)^3 (\Lambda_1^2\Lambda_2^3 + \Lambda_1\Lambda_2^2 + \Lambda_2)\\
&\quad + \mu_X^{[2]}\left(\mu_Y^{[1]}\right)^3 (\Lambda_1\Lambda_2^3 + \Lambda_2^2) + \left(\mu_X^{[1]}\right)^2 \mu_Y^{[1]}\mu_Y^{[2]}(\Lambda_1^2\Lambda_2^2 + \Lambda_1\Lambda_2 + \Lambda_0^2)\\
&\quad + \mu_X^{[2]}\mu_Y^{[1]}\mu_Y^{[2]}(\Lambda_1\Lambda_2^2 + \Lambda_2)\\
&\quad + \left(\mu_X^{[1]}\right)^2 \mu_Y^{[3]}(\Lambda_1^2\Lambda_2 + \Lambda_1)\mu_X^{[2]}\mu_Y^{[3]}(\Lambda_1\Lambda_2 + \Lambda_0)
\end{aligned}
\tag{16}
$$

3.3. Cumulants

The joint cumulant generating function of S_1 and S_2 is the logarithm of the joint moment generating function $M(z_1, z_2)$ and is given by

$$
\begin{aligned}
\kappa_{S_1,S_2}(z_1, z_2) = -(\lambda_0 + \lambda_1 + \lambda_2)\lambda_1[p_0 + p_1\exp(z_1) + \cdots + p_m\exp(z_1^m)]\\
+ \lambda_2[q_0 + q_1\exp(z_2) + \cdots + q_r\exp(z_2^r)] + \lambda_0[(p_0 + p_1\exp(z_1) + \cdots + p_m\exp(z_1^m))\\
(q_0 + q_1\exp(z_2) + \cdots + p_r\exp(z_2^r))]
\end{aligned}
\tag{17}
$$

From (17) we have

$$
\begin{aligned}
\kappa_{1,1} &= \lambda_1\mu_X + \lambda_2\mu_Y + \lambda_0\mu_X\mu_Y\\
\kappa_{1,2} &= \lambda_1\mu_X + \lambda_2\mu_Y^2 + \lambda_0\mu_X\mu_Y^2\\
\kappa_{2,2} &= \lambda_1\mu_X^2 + \lambda_2\mu_Y^2 + \lambda_0\mu_X^2\mu_Y^2\\
\kappa_{2,3} &= \lambda_1\mu_X^2 + \lambda_2\mu_Y^3 + \lambda_0\mu_X^2\mu_Y^3
\end{aligned}
$$

where μ_X and μ_Y are the expected values of X_i and Y_i, $i = 1, 2, \ldots$, respectively.

3.4. Independence of S_1 and S_2

The covariance of S_1 and S_2 is obtained using (2) and (16)

$$\begin{aligned} Cov(S_1, S_2) &= E(S_1 S_2) - E(S_1)E(S_2) \\ &= E(X)E(Y)[(\lambda_0+\lambda_1)(\lambda_0+\lambda_2)+\lambda_0] \\ &- [(\lambda_0+\lambda_1)E(X)][(\lambda_0+\lambda_2)E(Y)] \\ &= \lambda_0 E(X)E(Y) \end{aligned} \tag{18}$$

Let σ_{s_1} and σ_{s_2} be standard deviations of the random variables S_1 and S_2, then the coefficient of correlation of S_1 and S_2 is obtained from (2) and (18) as follows

$$\begin{aligned} \rho = Corr(S_1, S_2) &= \frac{Cov(S_1, S_2)}{\sigma_{s_1}\sigma_{s_2}} \\ &= \frac{\lambda_0 E(X)E(Y)}{\sqrt{(\lambda_0+\lambda_1)[V(X)+[E(X)]^2](\lambda_0+\lambda_2)[V(Y)+[E(Y)]^2]}} \end{aligned} \tag{19}$$

Note that the correlation of S_1 and S_2 assumes only positive values. This implies that $\rho = 0$ is a necessary condition for S_1 and S_2 to be independent. Also, $\rho = 1$ if and only if S_1 and S_2 are linearly dependent.

3.5. Asymptotics

If $(\lambda_0+\lambda_1) \to \infty$, $(\lambda_0+\lambda_2) \to \infty$, then

$$(Z_1, Z_2) = \left(\frac{S_1 - (\lambda_0+\lambda_1)E(X)}{\sqrt{(\lambda_0+\lambda_1)[V(X)+[E(X)]^2]}}, \frac{S_2 - (\lambda_0+\lambda_2)E(Y)}{\sqrt{(\lambda_0+\lambda_2)[V(Y)+[E(Y)]^2]}} \right) \tag{20}$$

follows a standardized normal bivariate distribution and asymptotically, $\frac{(Z_1^2 - 2\rho Z_1 Z_2 + Z_2^2)}{1-\rho^2}$ is a Chi-squared distribution with two degrees of freedom.

4. Some Numerical Examples

As an illustration of the BCPD and algorithm, a variety of special cases for the BCPD is considered. An algorithm is prepared in Maple for the joint probability function of the BCPD. This algorithm can also be used for the special cases of the BCPD. The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ are presented in Table 1, which are calculated from (15) for the bivariate Neyman type A distribution. In these calculations, X_i, $i = 1, 2, \ldots$ have a Poisson distribution with parameter $\mu_1 = 0.35$ and Y_i, $i = 1, 2, \ldots$ have a Poisson distribution with parameter $\mu_2 = 0.65$; M_0, M_1, M_2 are independent Poisson distributed random variables with parameters $\lambda_0 = 0.5, \lambda_1 = 0.7, \lambda_2 = 0.1$, respectively.

Table 2 presents $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ for the bivariate Neyman type B distribution where X_i, $i = 1, 2, 3, \ldots$ are binomial distributed with

Table 1: The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$, with the parameters ($\mu_1 = 0.35, \mu_2 = 0.65$) and ($\lambda_0 = 0.5, \lambda_1 = 0.7, \lambda_2 = 0.1$).

	s_1					
s_2	0	1	2	3	4	5
0	0.2836	0.1163	0.0674	0.0436	0.0212	0.0192
1	0.0985	0.0776	0.0167	0.0091	0.0149	0.0064
2	0.0867	0.0113	0.0095	0.0074	0.0097	0.0052
3	0.0065	0.0095	0.0074	0.0037	0.0087	0.0049
4	0.0042	0.0082	0.0062	0.0019	0.0063	0.0037
5	0.0038	0.0075	0.0057	0.0011	0.0041	0.0024

parameters ($m_1 = 5, p_1 = 0.02$) and Y_i, $i = 1, 2, \ldots$ are binomial distributed with parameters ($m_2 = 15, p_2 = 0.3$); M_0, M_1, M_2 are independent Poisson distributed random variables with parameters $\lambda_0 = 0.4, \lambda_1 = 0.6, \lambda_2 = 0.2$, respectively.

Table 2: The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$, with the parameters ($m_1 = 5, p_1 = 0.02$), ($m_2 = 15, p_2 = 0.3$) and ($\lambda_0 = 0.4, \lambda_1 = 0.6, \lambda_2 = 0.2$).

	s_1					
s_2	0	1	2	3	4	5
0	0.2836	0.1163	0.0674	0.0436	0.0212	0.0192
1	0.0985	0.0776	0.0167	0.0091	0.0149	0.0064
2	0.0867	0.0113	0.0095	0.0074	0.0097	0.0052
3	0.0065	0.0095	0.0074	0.0037	0.0087	0.0049
4	0.0042	0.0082	0.0062	0.0019	0.0063	0.0037
5	0.0038	0.0075	0.0057	0.0011	0.0041	0.0024

The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ are shown in Table 3, for the bivariate Thomas distribution. In these calculations X_i, $i = 1, 2, 3, \ldots$, have a truncated Poisson distribution with parameter $\alpha_1 = 0.75$ and Y_i, $i = 1, 2, 3, \ldots$ have a truncated Poisson distribution with parameter $\alpha_2 = 2$; M_0, M_1, M_2 are independent Poisson distributed random variables with parameters $\lambda_0 = 0.5, \lambda_1 = 0.4, \lambda_2 = 0.2$, respectively.

The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ are presented in Table 4, for the bivariate geometric-Poisson distribution. In these calculations, X_i, $i = 1, 2, 3, \ldots$ have a geometric distribution with parameter $\theta_1 = 0.25$ and Y_i, $i = 1, 2, 3, \ldots$, have a geometric distribution with parameter $\theta_2 = 0.5$; M_0, M_1, M_2 are independent Poisson distributed random variables with parameters $\lambda_0 = 0.9, \lambda_1 = 0.5, \lambda_2 = 0.2$, respectively.

The results are also illustrated with an analysis of the earthquake data in Turkey. The data is obtained from the database of the Kandilli Observatory, Turkey. Earthquakes are an unavoidable natural disasters for Turkey since a significant portion of Turkey is subject to frequent destructive mainshocks, their foreshock and aftershock sequences. In this study, mainshocks that occured in

Table 3: The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$, with the parameters ($\alpha_1 = 0.75, \alpha_2 = 2$) and ($\lambda_0 = 0.5, \lambda_1 = 0.4, \lambda_2 = 0.2$).

	s_1						
s_2	0	1	2	3	4	5	6
0	0.4266	0.0540	0.0533	0.0306	0.0225	0.0094	0.0082
1	0.0707	0.0288	0.0131	0.0090	0.0061	0.0085	0.0069
2	0.0468	0.0114	0.0096	0.0074	0.0056	0.0067	0.0053
3	0.0421	0.0094	0.0089	0.0052	0.0042	0.0052	0.0047
4	0.0019	0.0061	0.0072	0.0043	0.0035	0.0048	0.0034
5	0.0003	0.0043	0.0064	0.0038	0.0027	0.0032	0.0028
6	0.0002	0.0036	0.0056	0.0029	0.0018	0.0025	0.0019

Table 4: The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$, with the parameters ($\theta_1 = 0.25, \theta_2 = 0.5$) and ($\lambda_0 = 0.9, \lambda_1 = 0.5, \lambda_2 = 0.2$).

	s_1							
s_2	0	1	2	3	4	5	6	7
0	0.3122	0.0374	0.0430	0.0449	0.0387	0.0212	0.0145	0.0093
1	0.0285	0.0173	0.0323	0.0146	0.0214	0.0109	0.0098	0.0086
2	0.0097	0.0115	0.0237	0.0099	0.0138	0.0093	0.0083	0.0074
3	0.0149	0.0092	0.0116	0.0084	0.0097	0.0082	0.0045	0.0062
4	0.0099	0.0083	0.0092	0.0063	0.0085	0.0073	0.0037	0.0053
5	0.0076	0.0064	0.0092	0.0055	0.0073	0.0064	0.0021	0.0047
6	0.0068	0.0035	0.0086	0.0048	0.0062	0.0056	0.0001	0.0036
7	0.0052	0.0023	0.0062	0.0027	0.0053	0.0043	0.0001	0.0027

Turkey between 1900 and 2010, having surface wave magnitudes $M_s \geq 5.0$, their foreshocks within five days with $M_s \geq 3.0$ and aftershocks within one month with $M_s \geq 4.0$, are considered. In this area, 132 mainshocks with surface magnitude $M_s \geq 5.0$ have occured between 1900 and 2010.

(Kocyigit & Ozacar 2003)

A BCPD is constructed to explain the total number of foreshocks and aftershocks in Turkey. For this purpose, the neotectonic subdivision of Turkey is considered for the first time with the BCPD. To better understand the neotectonic features and active tectonics of Turkey, the simplied tectonic map of Turkey is given in Figure 1.

As seen in Figure 1, Turkey is divided into three main neotectonic domains: area of extensional neotectonic regime, area of strike-slip neotectonic regime with normal component and area of strike-slip neotectonic regime with thrust component. The mainshocks in Turkey are separated according to these neotectonic zones to obtain more reliable results. Let M_0 be the number of mainshocks in the area of extensional neotectonic regimes, M_1 be the number of mainshocks in the area of strike-slip neotectonic regime with normal component and M_2 be the area of strike-slip neotectonic regime with thrust component. Then X_i,

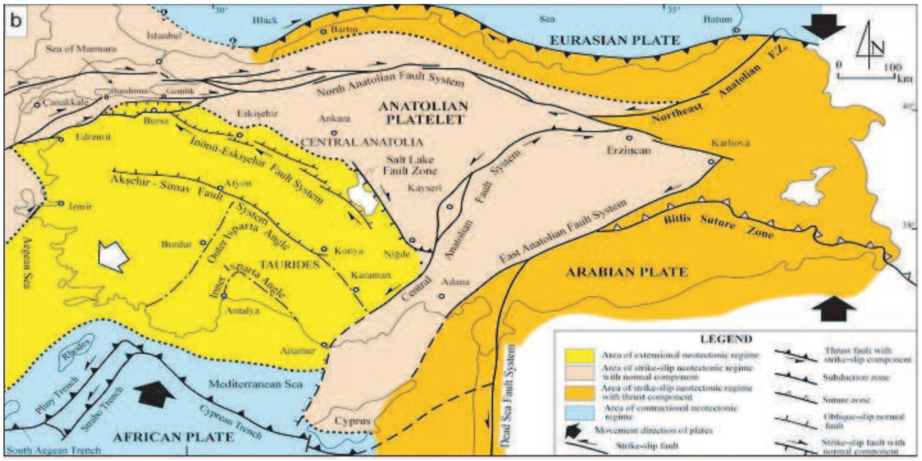

FIGURE 1: Neotectonic subdivision of Turkey and adjacent areas (Kocyigit & Ozacar 2003).

$i = 1, 2, 3, \dots$ are defined as the number of foreshocks of i^{th} mainshock and Y_i, $i = 1, 2, 3, \dots$ are defined as the number of aftershocks of i^{th} mainshock. Hence, $\left(S_1 = \sum_{i=1}^{N_1} X_i, S_2 = \sum_{i=1}^{N_2} Y_i\right)$ shows the total number of foreshocks and aftershocks for the mainshocks. If the following conditions hold, the pair of (S_1, S_2) has a BCPD:

Condition 1 Fit of the Poisson distribution to the mainshocks: Several studies have modelled earthquakes in Turkey as a Poisson distribution (Kalyoncuoglu 2007, Ozel & Inal 2008). The test for goodness of fit is performed to compare the observed frequency distributions of the mainshocks to the theoretical Poisson distribution. Chi-square values of M_0, M_1, M_2 are calculated as (0.082 with *df* = 9, *p*-value= 0.248), (0.068 with *df* = 15, *p*-value = 0.563), and (0.875 with *df* = 10, *p*-value = 0.351, respectively. These values indicate that M_0, M_1, M_2 fit the Poisson distribution with parameters $\lambda_0 = 2.83, \lambda_1 = 0.862, \lambda_2 = 0.145$ at the level of 0.05, respectively.

Condition 2 Independency tests of the random variables N_1, N_2, X_i and Y_i, $i = 1, 2, \dots$: Previous studies have indicated that there is no correlation between the number of mainshocks, foreshocks and aftershocks (Agnew & Jones 1991). Spearman's ρ test verifies the absence of correlation between N_1 and X_i, $i = 1, 2, \dots$ (Spearman's $\rho = 0.092$; *p*-value = 0.759). No correlation is also found between N_2 and Y_i, $i = 1, 2, \dots$ (Spearman's $\rho = 0.017$; *p*-value = 0.473). Similarly, it is shown that there is no statistically significant dependence between X_i and Y_i, $i = 1, 2, \dots$ (Spearman's $\rho = 0.098$; *p*-value = 0.764).

Condition 3 Fit of the binomial distribution to the foreshocks: As discussed in Jones (1985), if the occurrence of foreshock sequences is assumed as independent from the occurrence of mainshocks without foreshocks, then the distribution of foreshocks in the set of all earthquakes can be treated as a binomial distribution. The percentage, p, of foreshocks is an estimate of the probability that a future earthquake will be a foreshock. After obtaining the frequency distribution of foreshocks and the result of the test for goodness of fit ($\chi^2 = 1.437$, with $df = 36$, p-value $= 0.925$), it is seen that X_i, $i = 1, 2, \ldots$ have a binomial distribution with parameters $m = 35, p = 0.953$ at the level of 0.05.

Condition 4 Fit of the geometric distribution to the aftershocks: It is pointed that in the literature the number of aftershocks of a shock has a geometric distribution (Christophersen & Smith 2000). The test for goodness of fit is carried out to compare the theoretical geometric distribution to the experimental geometric distribution for the number of aftershocks. The test for goodness of fit ($\chi^2 = 1.184$, with $df = 30$, p-value $= 0.273$) shows that Y_i, $i = 1, 2, \ldots$ have a geometric distribution with parameter $\theta = 0.086$.

Because all conditions hold, it can be written $\left(S_1 = \sum_{i=1}^{N_1} X_i, S_2 = \sum_{i=1}^{N_2} Y_i\right)$ and suggested that (S_1, S_2) has a BCPD. Then, $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ are computed using (15) for the parameters $\lambda_0 = 2.83, \lambda_1 = 0.862, \lambda_2 = 0.145$; $(m = 35, p = 0.953)$; $\theta = 0.086$ and presented in Table 5.

Table 5: The probabilities $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$, with the parameters $\theta = 0.086$ and $(m = 35, p = 0.953)$ and $(\lambda_0 = 2.83, \lambda_1 = 0.862, \lambda_2 = 0.145)$.

	s_1										
s_2	0	1	2	3	4	5	6	7	8	9	10
0	0.3630	0.0071	0.0001	0.0049	0.0041	0.0040	0.0037	0.0026	0.0013	0.0010	0.0009
1	0.0075	0.0063	0.0053	0.0045	0.0038	0.0036	0.0035	0.0034	0.0013	0.0009	0.0009
2	0.0001	0.0056	0.0048	0.0041	0.0034	0.0035	0.0034	0.0032	0.0012	0.0008	0.0007
3	0.0058	0.0050	0.0043	0.0037	0.0031	0.0035	0.0032	0.0032	0.0009	0.0008	0.0006
4	0.0051	0.0044	0.0037	0.0033	0.0022	0.0021	0.0021	0.0019	0.0009	0.0007	0.0005
5	0.0041	0.0040	0.0034	0.0031	0.0020	0.0019	0.0019	0.0017	0.0008	0.0006	0.0005
6	0.0035	0.0032	0.0031	0.0030	0.0019	0.0019	0.0016	0.0015	0.0006	0.0005	0.0003
7	0.0021	0.0020	0.0020	0.0029	0.0016	0.0015	0.0014	0.0014	0.0006	0.0004	0.0003
8	0.0018	0.0018	0.0013	0.0015	0.0015	0.0013	0.0009	0.0011	0.0004	0.0003	0.0001
9	0.0013	0.0013	0.0009	0.0013	0.0010	0.0009	0.0007	0.0009	0.0004	0.0003	0.0001
10	0.0010	0.0008	0.0008	0.0009	0.0008	0.0008	0.0007	0.0098	0.0002	0.0001	0.0001

It can be seen from Table 5 that the joint probability recurrence of zero foreshock and zero aftershock is approximately 0.363. The expected values, variances, joint moments, cumulants for S_1 and S_2 are given in Table 6.

Table 6: Expected values, variances and some joint moments and cumulants of S_1 and S_2.

$E(S_1)$	$E(S_2)$	$V(S_1)$	$V(S_2)$	$\mu'(1,1)$	$\mu'(2,1)$	$\kappa_{1,1}$	$\kappa_{1,2}$
123.14	34.59	4113.34	804.49	6384.32	22430.97	1128.05	12811.28

As shown in Table 6 that approximately to 123 foreshocks with $M_s \geq 3.0$ and 35 aftershocks with $M_s \geq 4.0$ are expected in Turkey. It can be concluded from Table 5 that the expected value of total number of foreshocks is less than the expected value of total number of aftershocks. The coefficient of correlation between S_1 and S_2 is found as 0.60 using (19). This result seemed to indicate that increases on the incidence of foreshocks might lead to a more occurences of aftershocks.

5. Conclusion

In this paper the joint probability function, moments, cumulants, covariance and coefficient of correlation of BCPD are obtained. It is concluded that $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ can be computed easily for the BCPD if p_j, $j = 1, 2, \ldots, m$ and q_k, $k = 1, 2, \ldots, r$ are known. As seen in Section 3, (9) and (10) need long and tedious computations but $P(S_1 = s_1, S_2 = s_2)$, $s_1, s_2 = 0, 1, 2, \ldots$ can be computed accurately from (15) and its proposed algorithm in Maple. Then, some important probabilistic characteristics such as moments, cumulants, covariance, and correlation coefficient of the BCPD are provided. Some numerical examples and an application to the earthquake data have been also presented to illustrate the usage of the bivariate geometric-Poisson, Thomas, Neyman type A and B distributions. The results can be informative regarding BCPD and its applications

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Appendix A. Maple Code for the Joint Probability Function of the BCPD

```
    # $Source: /u/maple/research/lib/bcpd/jpf, v $

bcpd/jpf‘:=proc(L::{set,nvl},q::posint)
local p0, q0, i, lambda1,lambda2, f_final, R, F,
LambdaP_i, LambdaP_n, j, S1, S2, subscript, k, a, b, c,
        n, p, m, z, us, say, y_denom, denom v;

Partitionproduct := proc( n, f, g, statistic) local j, R, visit;
visit:= proc(L) local i, A, S, U, V, W;
   A:= add(pow (x, L[i]), i= 1.. nops (L));
   S:= [seq(coeff (A,x,i), i=1..n)];
   V:= mul (pow (f(i), S[i],i=1,..,n);
   W:= mul (pow (g(i), S[i])*S[i]!, i=1..n);
   U:= abs (n!*V/W);
    i statistic = "sum" then R := R+U
   elif statistic = "part" then R := [op (R), U]
   elif statistic = "len" then R [nops (L)] := R[nops (L)] + U
   elif statistic = "big" then R [L(1)] := R[ K[1]] + U;
   fi;
end;
if n = 0 then if statistic = "sum"
then RETURN (1) else RETURN ([1]) fi fi;
  if statistic = "sum" then R := 0
elif statistic = "part" then R : = []
else R := [ seq (0, j=1..n)] fi;
GeneratePartitions (n, visit);
R end:
 F0 := exp(-lambda1);
   if k =1 then
        F := lambda1;
    else
         i := k-2;
  n := 2;
  F := lambda1^k/k!;
        W := lambda2^k/k!;
    else
  F := F + (lambda1^i/i!)* (LambdaP_n);
        W := W + (lambda2^i/i!)* (LambdaP_n);
  i := i-1;
  n := n+1;
    fi;
    if i < 0
    fi;
```

```
  F := 0;
  k := k+1;
   if k > = subscript then k := 4 ;
        else
         i := k-2;
         m := 1;
         n := 2;
      nL := nops (L);
           F := LambdaP_i * LambdaP_n;
           nL := n;
           us := 1;
           if  i =n  then  us := us+1;
           else
           y_denom := seq(us!,1);
           if F >0 then
             do b = nvl(b,0) + F/y_denom while denom =k
           m := m+1;
           fi
        fi;
for z from 1 to n do
  p:=0;
   if subscript >0 then
            p:=p+1;
   fi
  z:=z-1;
   elif z=1 or p>0;
      od;
if p=0 then
subscript := F
                   F := p*LambdaP_n;
                   denom := subscript/(n+1);
                   y_denom:= denom*denom!/(denom-1)!
                     if F and w>0 then
                      do b = NVL(b,0) + F/y_denom while denom =k
                      y_denom :=1;
                      m:=m+1;
                     fi
       fi;
    i := i-1;
    n := n+1;
    p := 0;
    while i < k-trunc(k/2)
           k := k + 1;
                while k > :fNumber;
                            F := 0;
                            i := 1;
```

```
                      fi
         j := i-1;
         n := 1;
     for j from 1 to n do
       F := F + lambda1^n/n!;
       W := W + lambda1^n/n!;
      od;
      j := j-1;
      n := n+1;
      while j < 3;
     fi
     F := 0;
     W := 0;
     i := i+1;
     while i > :say
        set value=nvl(nvl (a,0)+ nvl(b,0)+nvl(c,0),0)*F0
     while denom> 0;
   end:
   return F
for i from 1 to n do
            f_final:= 1;
      f_final:= f_final*i;
           i:=i+1;
      while i>n
   fi
   return f_final
fi;
say :=0;
   for i from 1 to n do
      v_value(i):=substr (p_string, i, 1);
         if v_value(i):=p_string then
              say :=say+1;
               fi
               i:= i+1;
               while i-1> length(p);
             od;
   fi
end:

#savelib(’‘bcpd/jpf‘’):
```