PORTUGALIAE MATHEMATICA
Vol. 63 Fasc. 4 – 2006
Nova Série

ON THE FROBENIUS NUMBER OF A PROPORTIONALLY MODULAR DIOPHANTINE INEQUALITY

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Abstract: We give an algorithm to compute the greatest integer that is not solution of a Diophantine inequality of the form $ax \bmod b \leq cx$. As a consequence we obtain, for various cases, a formula, function of a, b and c, for that number.

1 – Introduction

Given two non negative integers a and b, with $b \neq 0$, we denote by $a \bmod b$ the remainder of the division of a by b. A *proportionally modular Diophantine inequality* is an expression of the form $ax \bmod b \leq cx$, where a is a non negative integer, b and c are positive integers. In [9] it is shown that the set $\mathrm{S}(a,b,c)$ of integer solutions of the former inequality is a numerical semigroup, that is, it is a subset of the set $\mathbb{N}$ of the non negative integers that is closed under addition, contains 0 and whose complement in $\mathbb{N}$ is empty or finite. This type of of numerical semigroups (that are not all, as is shown in [9]) are known as *proportionally modular numerical semigroups*. In [8] it is shown that the numerical semigroups having a Toms decomposition (see [13]) are precisely those numerical semigroups that can be expressed as a finite intersection of proportionally modular numerical semigroups.

If S is a numerical semigroup, then the greatest integer that is not in S is an important invariant of S which is called the *Frobenius number* of S and is denoted by $\mathrm{g}(S)$. This invariant has been widely studied in the literature (see,

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Received: August 30, 2005; *Revised*: March 10, 2006.
The first author gratefully acknowledges support of FCT through the CMUP. He also acknowledges a sabbatical grant of FCT used to visit the Algebra Department of the University of Granada. The second author is supported by the project MTM2004-01446 and FEDER founds.

for example, [1, 2, 3, 5, 11]). The reader may find many other bibliographic references as well as results related to the problem of determining the Frobenius number of a numerical semigroup in [6].

It is an open problem to find a formula that determines $\mathrm{g}(\mathrm{S}(a,b,c))$ as a function of a, b and c. For the case $c=1$ some progress was made in [10] and [7]. For instance, [7, Corollary 20] provides us a convenient method to to compute $\mathrm{g}(\mathrm{S}(a,b,1))$, although it remains an open problem to find a formula, as a function of a and b, to compute $\mathrm{g}(\mathrm{S}(a,b,1))$, even when a and b are relatively prime. Our main goal in this paper is to give an algorithm (Algorithm 17) to compute $\mathrm{g}(\mathrm{S}(a,b,c))$ from a, b and c. The theoretical basis of that algorithm is Theorem 5 which gives a particular form for $\mathrm{g}(\mathrm{S}(a,b,c))$. As a consequence of our work we obtain Theorem 22 which gives a formula for the Frobenius number of a large family of proportionally modular Diophantine inequalities.

As usual, for a rational number r, $\lceil r\rceil$ denotes the least integer not smaller than r and $\lfloor r\rfloor$ denotes the greatest integer not bigger than r.

We benefited from the computations done with a GAP [12] package on numerical semigroups still under development [4] to get the necessary intuition to conjecture some of the results proved here. Furthermore, this software was used to produce some of the examples presented.

2 – The main result

As the inequality $ax \bmod b \le cx$ has precisely the same integer solutions than the inequality $(a \bmod b)x \bmod b \le cx$, we do not loose generality by supposing that $a<b$. If $c\ge a$, then $\mathrm{S}(a,b,c)=\mathbb{N}$, thus we may also suppose that $c<a$.

From now on, unless otherwise stated, we will suppose that a,b and c are positive integers satisfying $c<a<b$.

The following proposition has as a consequence that we may also suppose that $a\le \frac{b+c}{2}$.

Proposition 1. $\mathrm{S}(a,b,c)=\mathrm{S}(b+c-a,b,c)$.

Proof: We start proving that $\mathrm{S}(a,b,c)\subseteq \mathrm{S}(b+c-a,b,c)$. If $x\in \mathrm{S}(a,b,c)$, then $ax \bmod b\le cx$. Therefore, there exist $q,r\in\mathbb{Z}$ such that $ax=qb+r$, with $0\le r\le cx$. Thus $(b+c-a)x=(b+c)x-qb-r=(x-q)b+cx-r$, with $0\le cx-r\le cx$. It follows that $(b+c-a)x \bmod b\le cx$ and consequently $x\in \mathrm{S}(b+c-a,b,c)$.

Now note that $c < b+c-a < b$. Repeating the preceding reasoning we have that $\mathrm{S}(b+c-a, b, c) \subseteq \mathrm{S}(a,b,c)$. ∎

Our next goal is to prove Theorem 5. To this end we need some preliminary results.

Lemma 2. *Let a and b be positive integers such that $a < b$. Then, for each integer k there exists exactly one integer α_k such that $b(a-k)-a \le a\,\alpha_k < b(a-k)$. Moreover $\alpha_k = b - \lfloor \frac{kb}{a} \rfloor - 1$.*

Proof: As $0 \le kb \bmod a < a$, we have that $b(a-k)-a \le b(a-k)-a+kb \bmod a < b(a-k)$. Therefore $b(a-k)-a \le ab-a-\lfloor \frac{kb}{a} \rfloor a < b(a-k)$, since $kb = \lfloor \frac{kb}{a} \rfloor a + kb \bmod a$. Consequently $b(a-k)-a \le a\bigl(b - \lfloor \frac{kb}{a} \rfloor - 1\bigr) < b(a-k)$. Uniqueness follows from the fact that the set $\bigl\{b(a-k)-a,\ b(a-k)-a+1, \ldots, b(a-k)-1\bigr\}$ consists of a consecutive integers and therefore contains exactly one multiple of a. ∎

As an immediate consequence of the uniqueness of α_k in the preceding lemma we get the following:

Lemma 3. *Let a and b be positive integers such that $a < b$ and let k be an integer. Then $a\bigl(b - \lfloor \frac{kb}{a} \rfloor - 1\bigr)$ is the greatest multiple of a in the set $\bigl\{b(a-k-1),\ b(a-k-1)+1, \ldots, b(a-k)-1\bigr\}$.* ∎

Remark 4. Observe that for an integer x, if $b(a-k-1) \le ax < b(a-k)$, then $\lfloor \frac{ax}{b} \rfloor = a-k-1$. Thus $ax \bmod b = ax - b(a-k-1)$. In particular, by using the proof of Lemma 2, we have the following equalities which will be used in the sequel:

$$\begin{aligned} a\left(b - \left\lfloor \frac{kb}{a} \right\rfloor - 1\right) \bmod b &= a\left(b - \left\lfloor \frac{kb}{a} \right\rfloor - 1\right) - b(a-k-1) \\ &= (k+1)b - a\left(\left\lfloor \frac{kb}{a} \right\rfloor + 1\right) \\ &= kb \bmod a + b - a\ . \ \square \end{aligned}$$

Now we are ready to prove the announced result concerning the form of the Frobenius number of $\mathrm{S}(a,b,c)$.

Theorem 5. $\mathrm{g}(\mathrm{S}(a,b,c)) \in \bigl\{ b - \lfloor \frac{kb}{a} \rfloor - 1 \mid k \in \{1, \ldots, a-1\} \bigr\}$.

Proof: Let $g = \mathrm{g}(\mathrm{S}(a,b,c))$. Then $g \notin \mathrm{S}(a,b,c)$ and therefore $g < b$, since every integer greater than or equal to b clearly belongs to $\mathrm{S}(a,b,c)$. Thus there exists $k \in \{0,1,\ldots,a-1\}$ such that $b(a-k-1) \le ag < b(a-k)$. If $g < b - \lfloor \frac{kb}{a} \rfloor - 1$, then $g+1 \le b - \lfloor \frac{kb}{a} \rfloor - 1$ and applying Lemma 2 we have that $b(a-k-1) \le a(g+1) < b(a-k)$. By Remark 4 we have that $a(g+1) \bmod b = a(g+1) - b(a-k-1) = ag - b(a-k-1) + a = ag \bmod b + a$. As $g \notin \mathrm{S}(a,b,c)$, $ag \bmod b > cg$ and therefore $a(g+1) \bmod b > cg + a > c(g+1)$, since $a > c$. Consequently $g+1 \notin \mathrm{S}(a,b,c)$, contradicting the fact that g is the Frobenius number of $\mathrm{S}(a,b,c)$. Thus $g \ge b - \lfloor \frac{kb}{a} \rfloor - 1$ and applying Lemma 3 we have that $g = b - \lfloor \frac{kb}{a} \rfloor - 1$.

Finally, note that if $k = 0$, then $g = b-1$. This implies that $a(b-1) \bmod b > c(b-1) \ge b-1$, which is absurd. Therefore k must not be 0 and this completes the proof. ∎

From the preceding theorem it follows that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$, where $\xi = \min\left\{k \in \{1,\ldots,a-1\} \mid b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)\right\}$. The following lemma will allow us to reformulate this fact.

Lemma 6. *Let $k \in \{1,\ldots,a-1\}$. Then $b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)$ if and only if $kb \bmod a + \lfloor \frac{kb}{a} \rfloor c > (c-1)b + a - c$.*

Proof: Observe that $b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)$ if and only if $a\big(b - \lfloor \frac{kb}{a} \rfloor - 1\big) \bmod b > c\big(b - \lfloor \frac{kb}{a} \rfloor - 1\big)$. By Remark 4 we know that $a\big(b - \lfloor \frac{kb}{a} \rfloor - 1\big) \bmod b = kb \bmod a + b - a$. Thus $b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)$ if and only if $kb \bmod a + b - a > c\left(b - \lfloor \frac{kb}{a} \rfloor - 1\right)$ and this is equivalent to $kb \bmod a + \lfloor \frac{kb}{a} \rfloor c > (c-1)b + a - c$. ∎

As a consequence we have that the set $\left\{k \in \{1,\ldots,a-1\} \mid b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)\right\}$ is equal to the set $\left\{k \in \{1,\ldots,a-1\} \mid kb \bmod a + \lfloor \frac{kb}{a} \rfloor c > (c-1)b + a - c\right\}$ and therefore we can reformulate the observation made after Theorem 5 as follows:

Corollary 7. $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$, *where*

$$\xi = \min\left\{k \in \{1,\ldots,a-1\} \mid kb \bmod a + \left\lfloor \frac{kb}{a} \right\rfloor c > (c-1)b + a - c\right\} . \blacksquare$$

The preceding corollary gives an algorithm to compute the Frobenius number of a proportionally modular Diophantine inequality. Note that one has to do at most $a-1$ tests and that we may, by Proposition 1, suppose that $a \le \frac{b+c}{2}$. Next we will work towards an improvement of this algorithm. In next section we give lower bounds for ξ.

3 – The algorithm

In what follows we will use the notation

$$\xi \;=\; \min\left\{k \in \{1,\dots,a-1\} \;\middle|\; kb \bmod a + \left\lfloor \frac{kb}{a} \right\rfloor c > (c-1)b + a - c\right\} .$$

Recall that by Corollary 7 we know that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$ and it follows from Lemma 6 that $\xi = \min\bigl\{k \in \{1,\dots,a-1\} \mid b - \lfloor \frac{kb}{a} \rfloor - 1 \notin \mathrm{S}(a,b,c)\bigr\}$.

Proposition 8. $\xi > c - 1 + \frac{2(a-c)}{b}$.

Proof: By Remark 4 we have that $a\bigl(b - \lfloor \frac{\xi b}{a} \rfloor - 1\bigr) \bmod b = (\xi+1)b - a\bigl(\lfloor \frac{\xi b}{a} \rfloor + 1\bigr)$. As $\bigl(b - \lfloor \frac{\xi b}{a} \rfloor - 1\bigr) \notin \mathrm{S}(a,b,c)$, we have that $(\xi+1)b - a\bigl(\lfloor \frac{\xi b}{a} \rfloor + 1\bigr) > c\bigl(b - \lfloor \frac{\xi b}{a} \rfloor - 1\bigr)$ and therefore $(\xi + 1 - c)b > (a-c)\bigl(\lfloor \frac{\xi b}{a} \rfloor + 1\bigr)$. From $\xi \geq 1$ and $b > a$ it follows that $(\xi + 1 - c)b > 2(a-c)$, thus $\xi > c + \frac{2(a-c)}{b} - 1$. ∎

Corollary 9. $\xi \geq c$.

Proof: It suffices to apply Proposition 8 and observe that $c + \frac{2(a-c)}{b} - 1$ is strictly greater than $c - 1$. ∎

Note that by Proposition 1 we may suppose that $a \leq \frac{b+c}{2}$. Under this condition we have that $\frac{2(a-c)}{b} \leq \frac{2(a-c)}{2a-c} = 1 - \frac{c}{2a-c} < 1$. Consequently if $a \leq \frac{b+c}{2}$, then the bound given by Proposition 8 is not better than the bound given by Corollary 9, which has the advantage of being very simple.

As an immediate consequence of Corollaries 7 and 9 we have the following:

Corollary 10. $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \lfloor \frac{cb}{a} \rfloor - 1$ *if and only if* $cb \bmod a + \lfloor \frac{cb}{a} \rfloor c > (c-1)b + a - c$. ∎

Example 11. Let $(a,b,c) = (7,34,6)$. Then $cb \bmod a + \lfloor \frac{cb}{a} \rfloor c = 175$ and $(c-1)b + a - c = 171$. We thus have $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \lfloor \frac{cb}{a} \rfloor - 1 = 4$. One can confirm this value by observing that $\mathrm{S}(a,b,c) = \{0\} \cup \{x \in \mathbb{Z} \mid x \geq 5\}$. □

Combining Corollaries 7 and 9 we have $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$, where $\xi = \min\bigl\{k \in \{c,\dots,a-1\} \mid kb \bmod a + \lfloor \frac{kb}{a} \rfloor c > (c-1)b + a - c\bigr\}$. Note that when $c = a-1$ we have $\xi = a-1$ and when $c = a-2$ we have $\xi = a-2$ or $\xi = a-1$. We get then immediately the following result.

Corollary 12.

(**1**) $\mathrm{g}(\mathrm{S}(a,b,a-1)) = b - \left\lfloor \frac{(a-1)b}{a} \right\rfloor - 1.$

(**2**) $\mathrm{g}(\mathrm{S}(a,b,a-2)) =$

$$= \begin{cases} b - \left\lfloor \frac{(a-2)b}{a} \right\rfloor - 1 & \textit{if } (a-2)b \bmod a + \left\lfloor \frac{(a-2)b}{a} \right\rfloor (a-2) > (a-3)b + 2 \\ b - \left\lfloor \frac{(a-1)b}{a} \right\rfloor - 1 & \textit{otherwise}. \ \blacksquare \end{cases}$$

Our next goal is to prove Proposition 14 where we will give a new lower bound for ξ. As we will then see in Fact 16 that bound is better than the one given by Corollary 9. We start with a simple observation given by the following lemma.

Lemma 13. $\mathrm{g}(\mathrm{S}(a,b,c)) \leq \frac{b-2}{c}$.

Proof: Let $g = \mathrm{g}(\mathrm{S}(a,b,c))$. As $g \notin \mathrm{S}(a,b,c)$, $ag \bmod b > cg$. Therefore $cg \leq b-2$. ∎

Proposition 14. $\xi \geq \left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil$.

Proof: By Corollary 7 we have that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$. Applying Lemma 13 we have that $b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1 \leq \frac{b-2}{c}$ and consequently $b - \frac{\xi b}{a} - 1 \leq \frac{b-2}{c}$. It follows that $\xi \geq a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb}$. ∎

As an immediate consequence of Corollary 7 and Proposition 14 we have the following result.

Corollary 15. *Let* $\alpha = \left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil$. *Then* $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\alpha b}{a} \right\rfloor - 1$ *if and only if* $\alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c > (c-1)b + a - c$. ∎

Recall that Corollary 12 gives us a formula for the Frobenius number of $\mathrm{S}(c+1,b,c)$. We observe next that, when $a \neq c+1$, the bound given by Proposition 14 improves the bound given by Corollary 9.

Fact 16. *If* $a \neq c+1$, *then* $\left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil \geq c$.

Proof: If $c = 1$, then $a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} = a - a - \frac{a}{b} + \frac{2a}{b} = \frac{a}{b}$. Therefore $\left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil = 1$.

Let us now suppose that $c \geq 2$ and $a \geq c+2$. Then $a(c-1) \geq (c+2)(c-1) = c^2 + c - 2 \geq c^2$ and therefore $a - \frac{a}{c} \geq c$. As $\frac{a}{b} - \frac{2a}{cb} = \frac{(c-2)a}{cb} < 1$, we have that $a - \frac{a}{c} - (\frac{a}{b} - \frac{2a}{cb}) > c - 1$, whence $\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \rceil \geq c$. ∎

As a summary of the results shown, we give the following algorithm to compute the Frobenius number of a proportionally modular Diophantine inequality.

Algorithm 17. INPUT: *positive integers* a, b *and* c.
OUTPUT: $\mathrm{g}(\mathrm{S}(a, b, c))$.

(**1**) *If* $a \leq c$, *then* return -1.

(**2**) $a := a \bmod b$.

(**3**) *If* $a = 0$, *then* return -1.

(**4**) *If* $a > \frac{b+c}{2}$, *then* $a := b + c - a$.

(**5**) *If* $a = c + 1$, *then* return $g = b - \left\lfloor \frac{(a-1)b}{a} \right\rfloor - 1$.

(**6**) *Compute* $\alpha = \left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil$.

(**7**) while $\alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c \leq (c-1)b + a - c$ do $\alpha := \alpha + 1$.

(**8**) return $b - \left\lfloor \frac{\alpha b}{a} \right\rfloor - 1$.

The observations made at the beginning of Section 2 and the fact that $\mathrm{g}(\mathbb{N})=-1$ justify the steps (1), (2), (3) and (4). Step (5) is a consequence of Corollary 12. Finally steps (6) and (7) are consequences of Corollary 7 and Proposition 14.

This algorithm works quite fast. The following self-explanatory example produced with the already mentioned package on numerical semigroups [4] should be convincing. The time is measured in GAP units. We can observe that for this example (where a, b and c were (pseudo)-randomly chosen) the time consumed using our algorithm to compute the Frobenius number is insignificant. This is not the case for the time consumed to compute the list of elements of the semigroup up to b.

Example 18. SmallElementsOfNumericalSemigroup(S) returns the list of elements of the semigroup S not greater than the Frobenius number $+1$. (It is readily obtained from the list of elements not greater than b, which is first computed.)

```
gap> a := 7957;;b := 733778;;c := 1257;;
gap> S := NumericalSemigroup("propmodular",a,b,c);
```

```
<Proportionally modular numerical semigroup satisfying
7957x mod 733778 <= 1257x >
gap> FrobeniusNumberOfNumericalSemigroup(S);
553
gap> time;
0
gap> L := SmallElementsOfNumericalSemigroup(S);;time;
2244
gap> L[Length(L)]-1;
553
```

4 – Some consequences

This section starts with a proposition that gives an upper bound for ξ. It will be used to prove Theorem 22 which gives a formula to compute the Frobenius number of a large family of proportionally modular Diophantine inequalities.

Proposition 19. $\xi \le \left\lfloor a - \frac{a}{c} + \frac{a(a-c)}{bc} \right\rfloor + 1$.

Proof: By Remark 4, $a\big(b-\lfloor\frac{(c+t)b}{a}\rfloor-1\big) \bmod b = (c+t+1)b-a\big(\lfloor\frac{(c+t)b}{a}\rfloor+1\big)$. Thus $b-\lfloor\frac{(c+t)b}{a}\rfloor-1 \notin \mathrm{S}(a,b,c)$ if and only if $(c+t+1)b-a\big(\lfloor\frac{(c+t)b}{a}\rfloor+1\big) > c\big(b-\lfloor\frac{(c+t)b}{a}\rfloor-1\big)$, which is equivalent to $(t+1)b > (a-c)\big(\lfloor\frac{(c+t)b}{a}\rfloor+1\big)$.

Let us now see that if $t > a-c+\frac{a(a-c)}{bc}-\frac{a}{c}$, then $(t+1)b > (a-c)\big(\lfloor\frac{(c+t)b}{a}\rfloor+1\big)$. It suffices to show that $(t+1)b > (a-c)\big(\frac{(c+t)b}{a}+1\big)$, since $\lfloor\frac{(c+t)b}{a}\rfloor \le \frac{(c+t)b}{a}$. If $t > a-c+\frac{a(a-c)}{bc}-\frac{a}{c}$, then $t\frac{c}{a} > \frac{(a-c)c}{a}+\frac{a-c}{b}-1$. Hence $t\left(1-\frac{a-c}{a}\right) > \frac{(a-c)c}{a}+\frac{a-c}{b}-1$ and consequently $t+1 > \frac{a-c}{a}t+\frac{(a-c)c}{a}+\frac{a-c}{b}$. So, $t+1>(a-c)\left(\frac{t}{a}+\frac{c}{a}+\frac{1}{b}\right)$ and therefore $(t+1)b > (a-c)\big(\frac{(c+t)b}{a}+1\big)$.

We have shown that if $t > a-c+\frac{a(a-c)}{bc}-\frac{a}{c}$, then $b-\lfloor\frac{(c+t)b}{a}\rfloor-1 \notin \mathrm{S}(a,b,c)$. Therefore if $k > a+\frac{a(a-c)}{bc}-\frac{a}{c}$, then $b-\lfloor\frac{kb}{a}\rfloor-1 \notin \mathrm{S}(a,b,c)$. By the observation made after Theorem 5 we have that $\xi \le \left\lfloor a-\frac{a}{c}+\frac{a(a-c)}{bc}\right\rfloor+1$. ∎

Combining Propositions 14 and 19 we have the following result.

Corollary 20. $\left\lceil a-\frac{a}{c}-\frac{a}{b}+\frac{2a}{cb}\right\rceil \le \xi \le \left\lfloor a-\frac{a}{c}+\frac{a(a-c)}{bc}+1\right\rfloor$. ∎

Remark 21. Let $\alpha = \left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil$ and $\beta = \left\lfloor a - \frac{a}{c} + \frac{a(a-c)}{bc} + 1 \right\rfloor$. Observe that $\beta - \alpha \leq \left(a - \frac{a}{c} + \frac{a(a-c)}{bc} + 1\right) - \left(a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb}\right) = \frac{a(a-c)}{bc} + 1 + \frac{a}{b} - \frac{2a}{cb} < \frac{a(a-c)}{bc} + 2$ since $\frac{a}{b} - \frac{2a}{cb} = \frac{a(c-2)}{cb} < 1$. Thus $\left\lfloor \frac{a(a-c)}{bc} \right\rfloor + 2$ is an upper bound to the number of tests that have to be realized in Step (7) of Algorithm 17. Note also that if $a(a-c) < bc$, then $\beta - \alpha < 3$. By Corollary 20 we deduce that $\xi \in \{\alpha, \alpha+1, \alpha+2\}$. As a consequence of Corollary 7 we may then state the theorem that follows which gives formulas for the Frobenius number of a large number of proportionally modular numerical semigroups. Notice that $a(a-c) < bc$ happens quite frequently; for instance, for $c = 1$, it suffices to take $b > a(a-1)$. □

Theorem 22. *If* $a(a-c) < bc$ *and* $\alpha = \left\lceil a - \frac{a}{c} - \frac{a}{b} + \frac{2a}{cb} \right\rceil$, *then*

$$\mathrm{g}(\mathrm{S}(a,b,c)) =$$

$$\begin{cases} b - \left\lfloor \frac{\alpha b}{a} \right\rfloor - 1 & \text{if } \alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c > (c-1)b + a - c; \\ b - \left\lfloor \frac{(\alpha+1)b}{a} \right\rfloor - 1 & \text{if } \alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c \leq (c-1)b + a - c \\ & \qquad < (\alpha+1)b \bmod a + \left\lfloor \frac{(\alpha+1)b}{a} \right\rfloor c; \\ b - \left\lfloor \frac{(\alpha+2)b}{a} \right\rfloor - 1 & \text{otherwise}. \end{cases}$$ ■

Example 23. In this example we use the notation of the above theorem.

(**1**) Let $(a,b,c) = (13,70,4)$. Then $\mathrm{S}(a,b,c) = \{0,6,7,11,12,13,14,15\} \cup \{x \in \mathbb{Z} \mid x \geq 17\}$.
We have $\alpha = 10$. As $\alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c = 223 > 219 = (c-1)b + a - c$, it follows that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\alpha b}{a} \right\rfloor - 1 = 16$.

(**2**) Let $(a,b,c) = (20,92,6)$. Then $\mathrm{S}(a,b,c) = \{0,5,6\} \cup \{x \in \mathbb{Z} \mid x \geq 10\}$.
Now we have $\alpha = 17$. As $\alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c = 472 \leq 474 = (c-1)b + a - c$ and $(c-1)b + a - c = 474 < 508 = (\alpha+1)b \bmod a + \left\lfloor \frac{(\alpha+1)b}{a} \right\rfloor c$, we have that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{(\alpha+1)b}{a} \right\rfloor - 1 = 9$.

(**3**) Let $(a,b,c) = (20,40,7)$. Then $\mathrm{S}(a,b,c) = \{0\} \cup \{x \in \mathbb{Z} \mid x \geq 2\}$.
Now, $\alpha = 17$, $\alpha b \bmod a + \left\lfloor \frac{\alpha b}{a} \right\rfloor c = 238$, $(c-1)b + a - c = 253$, $(\alpha+1)b \bmod a + \left\lfloor \frac{(\alpha+1)b}{a} \right\rfloor c = 252$. We therefore have $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{(\alpha+2)b}{a} \right\rfloor - 1 = 1$. □

The following result shows another situation in which we are able to give a formula for the Frobenius number.

Proposition 24. *If b is a multiple of a, then* $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\gamma b}{a} \right\rfloor - 1$, *where* $\gamma = \left\lfloor a - \frac{a}{c} + \frac{a^2}{bc} - \frac{a}{b} + 1 \right\rfloor$.

Proof: As a divides b, we have that $kb \bmod a + \left\lfloor \frac{kb}{a} \right\rfloor c > (c-1)b + a - c$ if and only if $\frac{kb}{a} c > (c-1)b + a - c$, which is equivalent to $k > a - \frac{a}{c} + \frac{a^2}{bc} - \frac{a}{b}$. Applying Corollary 7 we have that $\xi = \left\lfloor a - \frac{a}{c} + \frac{a^2}{bc} - \frac{a}{b} + 1 \right\rfloor$. ∎

Observing that for $(a,b,c) = (20,40,7)$ one has $\left\lfloor a - \frac{a}{c} + \frac{a^2}{bc} - \frac{a}{b} + 1 \right\rfloor = 19$, Example 23(3) can also be seen as an example of application of previous proposition.

This section ends with a new lower bound for the Frobenius number of $\mathrm{S}(a,b,c)$. Combined with the upper bound given by Lemma 13 we may state the following.

Proposition 25. $\left\lceil \frac{b-a}{c} - \frac{b}{a} \right\rceil \le \mathrm{g}(\mathrm{S}(a,b,c)) \le \left\lfloor \frac{b-2}{c} \right\rfloor$.

Proof: By Corollary 7 we know that $\mathrm{g}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\xi b}{a} \right\rfloor - 1$ and by Proposition 19 we know that $\xi \le a - \frac{a}{c} + \frac{a(a-c)}{bc} + 1$. Therefore $\mathrm{g}(\mathrm{S}(a,b,c)) \ge b - \frac{\xi b}{a} - 1 \ge b - \frac{b\left(a - \frac{a}{c} + \frac{a(a-c)}{bc} + 1\right)}{a} - 1 = \frac{b}{c} - \frac{a-c}{c} - \frac{b}{a} - 1 = \frac{b}{c} - \frac{a}{c} - \frac{b}{a}$. The other inequality was given in Lemma 13. ∎

Example 26. The last theorem guarantees that the $\mathrm{g}(\mathrm{S}(33,219,6))$ lies between 25 and 36. Since $a(a-c) < bc$, we could use Theorem 22 and get $\mathrm{g}(\mathrm{S}(33,219,6)) = 33$. □

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