PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 3 – 2005
Nova Série

# DISTRIBUTION OF DIGITS IN PERIODIC DECIMALS

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*Recommended by M. Nathanson*

**Abstract:** Consider fractions $\frac{a}{N}$, $0<a<N$, and their periodic expansions in base $B$, $(B,N)=1$. Our interest is in the number of times a $B$-digit $b$ occurs among the periods of all such fractions, for fixed $N$, and also in the number of occurances of $b$ among the proper periods, those where $(a,N)=1$. Some historical perspective on this problem is also included.

## 1 – Introduction

Let $N>1$ be an integer relatively prime to 10 and consider fractions $\frac{a}{N}$ between 0 and 1, $a=1,2,...,N-1$. It is well known [3, p. 111] that such a fraction has a purely periodic decimal expansion, and if the fraction is reduced, $(a,N)=1$, the length of the period is the order of 10 mod $N$. We are interested in how often each of the digits $0,1,...,9$ occurs in the periods of these fractions. To introduce the topic we consider a simple example. Take $N=21$; $\frac{2}{21}=0.09523809523809...$ has period 095238 with length 6. For brevity we write this decimal as 0.[095238]; others use the device of placing dots or bars over the period. In elementary school one learns how to find the period by long division which is equivalent to the following system of equations:

$$
\begin{aligned}
10\times 2 &= 0\times 21+20\\
10\times 20 &= 9\times 21+11\\
10\times 11 &= 5\times 21+5\\
10\times 5 &= 2\times 21+8\\
10\times 8 &= 3\times 21+17\\
10\times 17 &= 8\times 21+2
\end{aligned}
\tag{1}
$$

*Received*: December 15, 2003; *Revised*: September 2, 2004.

Having reached the remainder 2 with which we started, the equations now repeat and the period of the decimal is 095238. If we had begun with $\frac{20}{21}$ the process would have started with the second equation, showing $\frac{20}{21} = 0.[952380]$; the period has undergone a 'cyclic' permutation or left shift, the initial digit moving to the final position. The six fractions $\frac{2}{21}, \frac{20}{21}, \frac{11}{21}, \frac{5}{21}, \frac{8}{21}, \frac{17}{21}$ correspond to the period 095238 and its cyclic permutations which we now think of as a single period belonging to 21. For $\frac{1}{21}$ the period is 047619 corresponding to the fractions $\frac{1}{21}, \frac{10}{21}, \frac{16}{21}, \frac{13}{21}, \frac{4}{21}, \frac{19}{21}$. These twelve fractions are all the reduced fractions with denominator 21; we call the two periods they determine the proper periods for 21. Observe that 0,9 both occur twice among the twelve digits in the proper periods for 21 while $1, 2, ..., 8$ each occur once. Could this have been predicted? Is there a general rule?

The remaining fractions with denominator 21 need not be reduced to carry out the process. For example, $\frac{7}{21}$ produces $10 \times 7 = 3 \times 21 + 7$, so $\frac{7}{21} = 0.[3]$, the period is 3 with length 1. No surprise, since $\frac{7}{21} = \frac{1}{3}$. We call 3 an improper period for 21. The remaining improper periods are 6, determined by $\frac{14}{21} = 0.[6]$, and 142857 determined by $\frac{3}{21} = 0.[142857]$. To summarize, the periods belonging to 21 are 047619, 095238, 142857, 3, 6, comprising a total of 20 digits corresponding to the 20 fractions $\frac{a}{21}$, $a = 1, 2, ..., 20$. Now observe that each of the digits $0, 1, ..., 9$ occurs twice among all the periods.

There is no need to restrict to base 10. Let $B > 1$ be an integer taken as base; the numbers $b = 0, 1, ..., B - 1$ are the digits for base $B$; call them $B$-digits for short, or just digits when $B$ is understood. For an integer $N > 1$, relatively prime to $B$, the fractions $\frac{a}{N}$ have periodic (analogous to decimal) $B$-expansions, found the same way as above with $B$ replacing 10. For example, with $N = 21$ again and base $B = 4$, the fraction $\frac{2}{21}$ leads to equations

$$\begin{aligned} 4 \times 2 &= 0 \times 21 + 8 \\ 4 \times 8 &= 1 \times 21 + 11 \\ 4 \times 11 &= 2 \times 21 + 2 \end{aligned} \tag{2}$$

showing $\frac{2}{21} = 0.[012]$ in base 4, the period 012 corresponding to the fractions $\frac{2}{21}, \frac{8}{21}, \frac{11}{21}$. As an exercise the reader may wish to calculate all the proper periods for 21 in base 4; they are 003, 012, 033, 321. Here 0,3 both occur 4 times while 1,2 occur 2 times each. The remaining improper periods are 1, 2, 021, 123 and we see that each base 4 digit occurs 5 times among all the base 4 periods for 21.

To study the general case, fix a base $B$ and $N > 1$ relatively prime to $B$ and a $B$-digit $b$. We define $f(b, N, B)$ as the frequency (number of occurrences) of $b$ among the proper periods for $N$ in base $B$, and $F(b, N, B)$ as the frequency of $b$

among all the periods for $N$ in base $B$. For example, we've seen $f(0, 21, 10) = 2$, $f(1, 21, 10) = 1$, $F(b, 21, 10) = 2$ for $0 \le b \le 9$; $f(0, 21, 4) = 4, F(b, 21, 4) = 5$, for $0 \le b \le 3$. For later use it is convenient to define $f, F$ for $N = 1$ also by setting $f(b, 1, B) = 0 = F(b, 1, B)$. This is consistent with the original meaning since there are no periods for $N = 1$, as there are no fractions with denominator 1 between 0 and 1.

Let $\mathcal{A} = \{1, 2, ..., N-1\}$, $\mathcal{B} = \{0, 1, ..., B-1\}$. In the process of writing all the equations to find the periods for $N$ in base $B$, as we began in (1) and (2), there are a total of $N - 1$ equations of the form

$$Ba = bN + a' \ , \tag{3}$$

one for each $a \in \mathcal{A}$ with, necessarily, $a' \in \mathcal{A}$. Since each $a \in \mathcal{A}$ determines a digit $b \in \mathcal{B}$, the total number of $B$-digits occurring in all the periods for $N$ is $N - 1$. The proper periods arise from those equations with $(a, N) = 1$ and there are $\phi(N)$ of these ($\phi$ being Euler's function). In other words

$$\sum_{b=0}^{B-1} F(b, N, B) = N - 1 \quad \text{and} \quad \sum_{b=0}^{B-1} f(b, N, B) = \phi(N) \ . \tag{4}$$

In the next section we present theorems providing formulas for $f$ and $F$. But first it is appropriate to see what there is in the mathematical literature about this topic. My source is Dickson's History [1], whose ch. VI is titled Periodic Decimal Fractions, where I find only three references to our specific topic: Sardi [5], Glaisher [2], and Reynolds [4]. About the first Dickson [1, p. 167] says, "C. Sardi noted that if 10 is a primitive root of a prime $p = 10n+1$, the period for $\frac{1}{p}$ contains each digit $0, ..., 9$ exactly $n$ times. For $p = 10n + 3$, this is true of the digits other than 3 and 6, which occur $n + 1$ times. Analogous results are given for $10n + 7$ and $10n + 9$." I have not been able to obtain [5], but since Dickson says 'noted' rather than 'proved', which is used elsewhere, it seems Sardi had no proof, only empirical evidence. Note that when $N$ is a prime, $p$, all periods are proper and $f(b, p, B) = F(b, p, B)$. Glaisher [2, p. 202] remarks that "from Mr. Goodwyn's tables it may be noticed that for a prime $p = 10m+1$ each of the digits $0, 1, ..., 9$ occurs $m$ times among the periods." Reynolds [4] extends these observations to include $N$ not necessarily prime. He discusses bases other than 10, specifically 7 and 11, and tabulates the pattern of distribution of the digits. In conclusion he states, "These results have been empirically obtained, and the writer does not profess to be able to prove them. Yet he offers them with great confidence as, having once discovered what the laws were, he has not met with a single exception. Still... it may be rash to positively assert the results to be universally true, until someone has proved them." Beyond these I have found only one reference in the past century, [6], in which the author states and proves the rule of distribution of the digits in the periods only for $N = p$, prime, and base 10, without reference to any previous literature. No one seems to have discussed the distribution of digits among the proper periods, when $N$ is not prime.

## 2 – Theorems

Here we present theorems and proof, and in the next section deduce some consequences and illustrate with numerical examples. We keep the notations $N, B, F, f, \mathcal{A}, \mathcal{B}$ already introduced. The greatest integer function plays an important role so we collect here some of its properties that will be useful. Recall that for real $x$, $[x]$ denotes the greatest integer $\leq x$; $[x] = k$ iff $k$ is an integer and $k \leq x < k+1$. It is immediate that for any integer $n$, $[n+x] = n+[x]$. The following lemma states two properties of $[x]$ that will be used frequently. The proofs are straightforward, hence omitted.

**Lemma.**

**(i)** *If $x$ is not an integer, $[-x] = -[x] - 1$.*

**(ii)** *If $x < y$ are both not integers, then the number of integers between $x$ and $y$ is $[y] - [x]$.*

We need a bit more notation. In what follows, in a context where $N$, $B$ are fixed, we write simply $f(b)$ in place of $f(b, N, B)$ and $F(b)$ in place of $F(b, N, B)$. Also, for a $B$-digit $b \in \mathcal{B}$, we define the complementary digit $b^*$ to be $B - 1 - b$, and for $a \in \mathcal{A}$ we define $\tilde{a}$ to be $N - a$, Note that $b \to b^*, a \to \tilde{a}$ are involutions on $\mathcal{B}$, $\mathcal{A}$ respectively.

**Theorem 1.** *Write $N = gB + r$, $g$ the quotient, $r$ the remainder when $N$ is divided by $B$. $g = [\frac{N}{B}]$ and $1 \leq r \leq B-1$, $(r, B) = 1$.*

**(i)** *For any $B$-digit $b$, $F(b^*) = F(b)$ and $f(b^*) = f(b)$.*

**(ii)** *$F(0) = g$ and $F(B-1) = g$.*

**(iii)** *For $0 < b < B-1$, $F(b) = g$ except for the $r-1$ digits $b_i = \left[\frac{Bi}{r}\right]$, $i = 1, 2, ..., r-1$ where $F(b_i) = g+1$.*

**Proof:** Our assumptions $N > 1$, $B > 1$, $(N, B) = 1$ prove the restrictions on $r$. As we have seen, $F(b)$ is the number of $a \in \mathcal{A}$ for which an equation, (3) above, $Ba = bN + a'$, holds for some $a' \in \mathcal{A}$. (3) implies $B\tilde{a} = BN - Ba = BN - bN - a' = (B - b - 1)N + N - a' = b^*N + \tilde{a'}$, thus

$$B\tilde{a} = b^*N + \tilde{a'} \ . \tag{5}$$

Thus for every occurrence of $b$ in the periods for $N$ there is a corresponding occurrence of $b^*$. And if $(a, N) = 1$, so the occurrence of $b$ given by (3) is in a proper period, then the corresponding occurrence of $b^*$ given by (5) is in a proper period also, since $(\tilde{a}, N) = 1$. This proves (i).

The equation (3) with $a, a' \in \mathcal{A}$ is equivalent to the inequality $bN < Ba < bN + N = (b + 1)N$. For a set $X$, $|X|$ denotes the number of elements in the set $X$. Thus we have $F(b) = |X(b)|$ where

$$X(b) = \left\{ a \,\middle|\, \frac{bN}{B} < a < \frac{(b+1)N}{B} \right\} \ . \tag{6}$$

Note that the inequality in (6) already ensures that $a \in \mathcal{A}$. When $b = 0$, $X(0) = \left\{ a \,\middle|\, 0 < a < \dfrac{N}{B} \right\}$ and so $F(0) = [\frac{N}{B}] = g$. Since $0^* = B - 1$, $F(B - 1) = g$ follows from (i), proving (ii). Assume now $0 < b < B - 1$. Then $\frac{bN}{B}$ and $\frac{(b+1)N}{B}$ both are not integers so, by the lemma,

$$F(b) = |X(b)| = \left[\frac{(b+1)N}{B}\right] - \left[\frac{bN}{B}\right] \ .$$

Substituting $N = gB + r$ this simplifies to $F(b) = g + \left[\dfrac{(b+1)r}{B}\right] - \left[\dfrac{br}{B}\right]$. Define

$$E(b, N, B) = \left[\frac{(b+1)r}{B}\right] - \left[\frac{br}{B}\right] \tag{7}$$

or $E(b)$ for short, so we now have

$$F(b) = g + E(b) \ . \tag{8}$$

Note that $E(b, N, B) = E(b, r, B)$ depends only on $r$, and $E(b, 1, B) = 0$. In particular, $E(b, N, B)$ depends only on the congruence class of $N$ mod $B$. Again by the lemma, $E(b, N, B)$ is the number of integers between $\frac{br}{B}$ and $\frac{(b+1)r}{B}$. But $\frac{(b+1)r}{B} - \frac{br}{B} = \frac{r}{B} < 1$ so there can be at most one integer between them. Thus $E(b) = 0$, $F(b) = g$, except when there is an integer $i$ such that

$$\frac{br}{B} < i < \frac{(b+1)r}{B} \tag{9}$$

in which case $E(b) = 1$, $F(b) = g + 1$. Now (9) is equivalent to $b < \frac{Bi}{r} < b + 1$, or $b = [\frac{Bi}{r}]$. But we require $0 < b < B - 1$ which one easily sees is exactly for $i = 1, 2, ...r - 1$. This completes the proof of the theorem. ∎

It is convenient to call the digits $b$ for which $E(b) = 1$, $F(b) = g+1$, abundant for $N$ in base $B$, since they occur more frequently, albeit only by 1, in the periods for $N$. We see that if $N \equiv 1 \pmod{B}$, $r = 1$, then $r - 1 = 0$ and there are no abundant digits, while at the opposite extreme, $N \equiv B - 1 \pmod{B}$, $r = B - 1$, then $r - 1 = B - 2$ and all digits $0 < b < B - 1$ are abundant. We can now explain the distribution rules for $B = 10$ observed by the authors previously cited. For $(N, 10) = 1$, $N = 10g + r$, $r$ must be one of 1,3,7,9. If $r = 1$, no digits are abundant and $F(b) = g$ for all $b$. If $r = 9$, all digits $0 < b < 9$ are abundant and $F(b) = g + 1$ for all such $b$. If $r = 3$, the abundant digits are $[\frac{10i}{3}]$ for $i = 1, 2$ which give 3,6 as abundant. Finally, for $r = 7$, the abundant digits are $[\frac{10i}{7}]$ for $i = 1, 2, ..., 6$ which give 1,2,4,5,7,8 as abundant. In general, let $AD(r, B)$, for $(r, B) = 1$, $1 \leq r \leq B - 1$, be the abundant digits for $r$ in base $B$. $AD(r, B)$ is a subset of $\{1, 2, ..., B - 2\}$. Note that for $B = 10$, $AD(1, 10)$, $AD(9, 10)$ are mutually disjoint and their union is $\{1, 2, ..., 8\}$, and the same is true for $AD(3, 10)$ and $AD(7, 10)$. This is no coincidence – there is another bit of symmetry, actually anti-symmetry, lurking here.

**Theorem 2.** *Let $(r, B) = 1$, $1 \leq r \leq B - 1$. For $0 < b < B - 1$, $E(b, r, B) + E(b, B - r, B) = 1$.*

**Proof:** By (7), $E(b, B - r, B) = [\frac{(b+1)(B-r)}{B}] - [\frac{b(B-r)}{B}] = [(b + 1) - \frac{(b+1)r}{B}] - [b - \frac{br}{B}] = 1 + [-\frac{(b+1)r}{B}] - [-\frac{br}{B}]$. Now by (i) of the lemma, this last expression is $1 - \left[\frac{(b+1)r}{B}\right] - 1 - \left(-\left[\frac{br}{B}\right] - 1\right) = 1 - E(b, r, B)$. Since the values of $E$ are 0,1 it follows that $E(b, r, B) = 1$ iff $E(b, B - r, B) = 0$ and $E(b, r, B) = 0$ iff $E(b, B - r, B) = 1$. Thus the sets $AD(r, B)$, $AD(B - r, B)$ form a partition of $\{1, 2, ..., B - 2\}$. ∎

The formula for $f(b, N, B)$ is more complicated. It involves the prime structure of $N$, but in a curious way. Suppose $N = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$ is the prime factorization of $N$; $m \geq 1$, $p_1, ...p_m$ are the distinct prime divisors of $N$, $k_1 \geq 1, ..., k_m \geq 1$. Let $\mathcal{M} = \{1, 2, ..., m\}$. For a subset $I = \{i_1, ..., i_t\}$ of $\mathcal{M}$, $|I| = t$, set $p(I) = p_{i_1} p_{i_2} ... p_{i_t}$; $p(I) = 1$ when $t = 0$, $I$ the empty set.

**Theorem 3.**

(**i**) $f(0,N,B)=\sum_{t=0}^{m}(-1)^t\sum_{|I|=t}\left[\frac{N/p(I)}{B}\right]$, *where in the inner sum $I$ ranges over all $t$-element subsets of $\mathcal{M}$.*

(**ii**) $f(B-1,N,B)=f(0,N,B)$

(**iii**) *For $0<b<B-1$, $f(b,N,B)=f(0,N,B)+d(b,N,B)$, where*

$$d(b,N,B)=\sum_{t=0}^{m}(-1)^t\sum_{|I|=t}E\left(b,\frac{N}{p(I)},B\right)\ .$$

**Proof:** Every period for $N$ is a proper period for a unique divisor $N_1$ of $N$. In fact, if $a\in\mathcal{A}$, $(a,N)=c$, then write $a=ca_1$, $N=cN_1$, so $\frac{a}{N}=\frac{a_1}{N_1}$ and the period determined by $\frac{a}{N}$ is a proper period for $N_1$. It follows that the frequency of a digit $b$ among all periods for $N$ is the sum of its frequencies among all proper periods for $N_1$ ranging over the divisors of $N$. Thus

$$F(b,N,B)=\sum_{N_1|N}f(b,N_1,B)\ . \tag{10}$$

Holding $b,B$ fixed and considering $f,F$ only as functions of the middle argument, (10) can be inverted by the Mobius inversion formula [3, p. 236] to obtain

$$f(b,N,B)=\sum_{N_1|N}\mu(N_1)F\left(b,\frac{N}{N_1},B\right)\ . \tag{11}$$

Recall that $\mu(1)=1$, $\mu(n)=(-1)^t$ if $n$ is a product of $t$ distinct primes and $\mu(n)=0$ otherwise. So the only $N_1|N$ in (11) with $\mu(N_1)\neq 0$ are $N_1=p(I)$ where $I$ ranges over all subsets of $\mathcal{M}$. When $|I|=t$, $\mu(N_1)=(-1)^t$, so that (11) can be rewritten as

$$f(b,N,B)=\sum_{t=0}^{m}(-1)^t\sum_{|I|=t}F\left(b,\frac{N}{p(I)},B\right)\ . \tag{12}$$

By Theorem 1, $F\left(0,\frac{N}{p(I)},B\right)=\left[\frac{N/p(I)}{B}\right]=F\left(B-1,\frac{N}{p(I)},B\right)$ while, by (8), for $0<b<B-1$, $F\left(b,\frac{N}{p(I)},B\right)=\left[\frac{N/p(I)}{B}\right]+E\left(b,\frac{N}{p(I)},B\right)$. Substituting these expressions into (12) proves the theorem. ∎

## 3 – Consequences and examples

As a first example we take $N = 21$, $B = 4$ considered earlier. Since $21 \equiv 1 \pmod 4$ and $[\frac{21}{4}] = 5$, Theorem 1 says $F(b) = 5$ for all digits $b$ in base 4. Since $21 = 3 \times 7$ we have, in the notation of Theorem 3, $f(0) = \left[\frac{21}{4}\right] - \left(\left[\frac{21/3}{4}\right] + \left[\frac{21/7}{4}\right]\right) + \left[\frac{21/(3\times 7)}{3}\right] = 5 - (1+0) + 0 = 4$, and also $f(3) = 4$. The quantity $d(b,N,B)$ of Theorem 3 is $f(b,N,B) - f(0,N,B)$, the amount by which $f(b)$ differs from $f(0)$. So here we have for $b = 1,2$, $d(b,21,4) = E(b,21,4) - (E(b,\frac{21}{3},4) + E(b,\frac{21}{7},4)) + E(b,\frac{21}{(3\times 7)},4)$. Since $21 \equiv 1 \pmod 4$, $E(b,21,4) = 0$, and the last term $E(b,1,4) = 0$. The two middle terms $E(b,7,4)$, $E(b,3,4)$ are equal, since $7 \equiv 3 \pmod 4$. Since $3 = 4-1$, $E(b,3,4) = 1$ for both $b = 1,2$. So finally, $d(b,21,4) = 0-(1+1)+0 = -2$, so $f(b,21,4) = 4-2 = 2$, for $b = 1,2$, in agreement with what we saw by actually writing down the periods.

Theorem 1 shows that the $B$-digits are distributed as fairly (uniformly) as possible among the periods for $N$, and the distribution is actually uniform iff $N \equiv 1 \pmod B$, for only in this case are there no abundant digits. When is the distribution among all the proper periods uniform, i.e., when is $d(b) = f(b) - f(0) = 0$ for all $b$? We do not have a necessary and sufficient condition but claim:

**Theorem 4.** *If $N$ has at least one prime factor $p \equiv 1 \pmod B$ then $f(b,N,B) = f(0,N,B)$ for all $b$.*

**Proof:** Suppose, without loss of generality, that $p_1 \equiv 1 \pmod B$. In calculating $d(b)$ as in Theorem 3, the sum $\displaystyle\sum_{|I|=t} E\left(b,\frac{N}{p(I)},B\right)$ contains two types of terms; those where $1 \in I$, so $I$ can be written as $I = \{1\} \cup J$, where $|J| = t-1$ is a subset of $\mathcal{M}_1 = \{2,...,m\}$ and those where $1 \notin I$, so $I = J$ is a subset of $\mathcal{M}_1$ and $|J| = t$. In the first case, $p(I) = p_1 \cdot p(J) \equiv p(J) \pmod B$ and $\frac{N}{p(I)} \equiv \frac{N}{p(J)} \pmod B$, so $E\left(b,\frac{N}{p(I)},B\right) = E\left(b,\frac{N}{p(J)},B\right)$. So the entire sum is $\displaystyle\sum_{|J|=t-1} E\left(b,\frac{N}{p(J)},B\right) + \sum_{|J|=t} E\left(b,\frac{N}{p(J)},B\right)$, $J$ ranging over subsets of $\mathcal{M}_1$. Call these 2 sums $S(t-1), S(t)$ respectively. Note that for $t = 0$, $S(0-1) = 0$, since there are no 'empty sets' $I$ with $1 \in I$. Also for $t = m$, $S(m) = 0$ since there is no set $J \subset \mathcal{M}_1$ with $|J| = m$. Taking into account the coefficient $(-1)^t$ and summing over $t$ gives:

$$d(b) = S(0) + \sum_{t=1}^{m-1}(-1)^t(S(t-1)+S(t)) + (-1)^m S(m-1) = 0$$

as all the terms cancel, and the proof is complete. ∎

Here is a simple illustration. Take $N{=}15 = 3{\times}5$ and $B{=}4$. Since $5 \equiv 1 \pmod 4$ we have $f(b,15,4) = f(0,15,4)$ for $0 \le b \le 3$. By direct calculation, the proper periods for 15 in base 4 are 01,02,13,23 and we see $f(b) = f(0) = 2$. Actually, whenever there is uniform distribution there is no need to use Theorem 3 to evaluate $f(0)$. Indeed, by (4), when $f(b) = f(0)$ for all $b$, $B \cdot f(0) = \phi(N)$ so $f(0) = \frac{\phi(N)}{B}$. For example, take $N = 693 = 3^2 \times 7 \times 11$ and $B = 10$; since $11 \equiv 1 \pmod{10}$ we know there is uniform distribution, so $f(b,693,10) = \frac{\phi(693)}{10} = \frac{3(3-1)(7-1)(11-1)}{10} = 36$, for $0 \le b \le 9$. In case of uniform distribution, $f(0) = \frac{\phi(N)}{B}$, so we must have $B|\phi(N)$, or, since $(B,N) = 1$, $B|(p_1-1)(p_2-1)...(p_m-1)$. But this is not sufficient for uniform distribution, since for $N = 21$, $B = 4$ we have $4|(3-1)(7-1)$ but we've seen in this case $f(0) = 4$, $f(1) = 2$.

At the opposite extreme, how large can $f(b) - f(0) = d(b)$ be? The formula of Theorem 3 shows $d(b)$ as a double sum $\sum_{t=0}^{m}(-1)^t \sum_{|I|=t} E\left(b, \frac{N}{p(I)}, B\right)$. The inner sum over $t$-element subsets $I$ of $\mathcal{M}$ is a sum of $\binom{m}{t}$ (binomial coefficient) terms each of which has value 0 or 1. So the inner sum is between 0 and $\binom{m}{t}$, inclusive. To make $d(b)$ as large as possible, those sums for $t$ even, should be $\binom{m}{t}$, and those for $t$ odd, where something is being subtracted, should be 0. This gives $d(b) \le \sum_{\substack{t=0 \\ t \ \ even}}^{m} \binom{m}{t} = 2^{m-1}$. By a similar argument $d(b) \ge -2^{m-1}$, so we have the bounds: If $N$ has $m$ distinct prime divisors then

$$|f(b,N,B) - f(0,N,B)| \le 2^{m-1} \ . \tag{13}$$

It is easy to construct examples where these bounds are attained for all $b$, $0 < b < B-1$. Take $N$ to be a product $p_1^{k_1}...p_m^{k_m}$ with all prime factors $p_i \equiv B-1$ (mod $B$). Let $K = k_1 + k_2 + ... + k_m$. Suppose $K$ is odd; then $N \equiv (B-1)^K \equiv B-1$ (mod $B$). Then for any $I$ with $|I| = t$ even, $p(I) \equiv (B-1)^t \equiv 1$ (mod $B$), hence $\frac{N}{p(I)} \equiv (B-1) \pmod B$ and $E\left(b, \dfrac{N}{p(I)}, B\right) = 1$, for all $b$, since we've seen that $E(b, B-1, B) = 1$ for all $b$. Thus each term in $\displaystyle\sum_{\substack{|I|=t \\ t \text{ even}}}$ contributes a 1. On the other hand, for $t$ odd, each term in $\displaystyle\sum_{\substack{|I|=t \\ t \text{ odd}}}$ contributes 0, since now $p(I) \equiv (B-1)^t \equiv (B-1) \pmod B$, so $\frac{N}{p(I)} \equiv 1 \pmod B$ and $E(b, 1, B) = 0$ for all $b$. So in this case $d(b) = 2^{m-1}$. If $K$ is even, the whole argument is reversed and one has $d(b) = -2^{m-1}$. Actually the case $N = 21 = 3 \times 7$, $B = 4$ is a trivial example of this, since $7 \equiv 3 \equiv (4-1) \pmod 4$, $m = 2$, $K = 2$ and we've seen $f(0, 21, 4) = 4$, $f(b, 21, 4) = 2 = 4 - 2^{2-1}$ for $b = 1, 2$.

In general, in a case where $d(b, N, B) = 2^{m-1}$ for $0 < b < B-1$ then (4) yields a simple expression for $f(0)$, without going through the calculation of the formula in Theorem 3. Namely, our assumptions and (4) imply $2f(0) + (B-2)(f(0) + 2^{m-1}) = \phi(N)$ so that $f(0) = \dfrac{\phi(N) - (B-2)2^{m-1}}{B}$. Similarly if $d(b, N, B) = -2^{m-1}$, for $0 < b < B-1$, one has $f(0) = \dfrac{\phi(N) + (B-2)2^{m-1}}{B}$.

Incidentally, it should be obvious, but worth mentioning, that when $B = 2$, $0^* = 1$, and the distributions are uniform. (4) then implies that for $N$ odd, $B = 2$, $f(0) = f(1) = \frac{\phi(N)}{2}$ and $F(0) = F(1) = \frac{N-1}{2}$.

For any $N, B$ the formula of Theorem 3 shows that $f(0, N, B)$ depends very much on the actual prime divisors of $N$. But $d(b, N, B) = f(b, N, B) - f(0, N, B)$ depends only on the congruence classes mod $B$ of the prime divisors in an unusual way. Namely let $N = p_1^{k_1} \ldots p_m^{k_m}$ as previously, and suppose $M = q_1^{h_1} \ldots q_m^{h_m}$ with $p_1 \equiv q_1, \ldots p_m \equiv q_m \pmod B$ and $M \equiv N \pmod B$, with no further restriction on the exponents. Then $d(b, N, B) = d(b, M, B)$ for $0 < b < B-1$. For the double sums involved in each differ only in that for $N$ one has terms $E\left(b, \dfrac{N}{p_{i_1} \ldots p_{i_m}}, B\right)$ while for $M$ the terms are $E\left(b, \dfrac{M}{q_{i_1} \ldots q_{i_m}}, B\right)$. But these terms are equal since by our assumptions the middle arguments are congruent mod $B$. Example: $B = 10$, $N = 3 \times 13 \times 17 \times 19$, $M = 23^2 \times 43^5 \times 7^6 \times 29^3$. Then $3 \equiv 23$, $13 \equiv 43$, $17 \equiv 7$, $19 \equiv 29 \pmod{10}$ and also, easily checked, $N \equiv M \pmod{10}$. Then $d(b, N, 10) = d(b, M, 10)$ for $0 < b < 9$.

There are other questions about the possible values of $f(b, N, B)$ that one may ask, but we must stop here and let the interested reader investigate them.

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