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ALMOST NORMALITY AND MILD NORMALITY OF THE TYCHONOFF PLANK

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Abstract: The Tychonoff Plank is a popular example of the fact normality is not hereditary. We will show that it is mildly normal but not almost normal.

The Tychonoff plank $X = (\omega_1 + 1 \times \omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$ is a famous example of a $T_{3\frac{1}{2}}$-space which is not normal, see [1]. It is also a famous example of the fact that normality is not hereditary, see [1]. In this paper, we will show that the Tychonoff plank is mildly normal but not almost normal. We will denote an order pairs by $\langle x, y \rangle$, the set of positive integers by $\mathbb{N}$ and the set of all real numbers by $\mathbb{R}$.

Definition 1. A subset A of a topological space X is called *regularly closed* (called also, *closed domain*) if $A = \overline{\operatorname{int} A}$. Two subsets A and B in a topological space X are said to be separated if there exist two disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$. ▫

Definition 2. A topological space X is called *mildly normal* (called also *κ-normal)* if any two disjoint regularly closed subsets A and B of X, can be separated. ▫

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In [2], Shchepin introduced the notion of κ-normal property. He required regularity in his definition. In [3], Singal and Singal introduced the notion of mildly normal property. They did not require regularity.

Let ω be the first infinite ordinal and ω_1 be the first uncountable ordinal with their usual order topology. Consider the product space $\omega_1+1\times\omega+1$. The Tychonoff Plank is the subspace $X=(\omega_1+1\times\omega+1)\setminus\{\langle\omega_1,\omega\rangle\}$. Write $X=A\cup B\cup C$, where $A=\{\omega_1\}\times\omega$, $B=\omega_1\times\{\omega\}$, and $C=X\setminus(A\cup B)$. Let $p_1\colon \omega_1+1\times\omega+1\longrightarrow\omega_1+1$ and $p_2\colon \omega_1+1\times\omega+1\longrightarrow\omega+1$ be the natural projections. To show that X is mildly normal, we need the following lemma:

Lemma 1. *If H and K are closed disjoint unseparated subsets of X, then either ($p_1(H\cap B)$ is unbounded and $p_2(K\cap A)$ is unbounded) or ($p_1(K\cap B)$ is unbounded and $p_2(H\cap A)$ is unbounded).*

Proof: Let H and K be any closed disjoint unseparated subsets of X. Suppose that the conclusion is false. This gives us that ($p_1(H\cap B)$ is bounded or $p_2(K\cap A)$ is bounded) and ($p_1(K\cap B)$ is bounded and $p_2(H\cap A)$ is bounded). This gives us the following four cases:

1. $p_1(H\cap B)$ is bounded and $p_2(H\cap A)$ is bounded.

2. $p_1(H\cap B)$ is bounded and $p_1(K\cap B)$ is bounded.

3. $p_2(K\cap A)$ is bounded and $p_2(H\cap A)$ is bounded.

4. $p_2(K\cap A)$ is bounded and $p_1(K\cap B)$ is bounded.

Case 1: $p_1(H\cap B)$ is bounded and $p_2(H\cap A)$ is bounded. Let γ be the least upper bound of $p_1(H\cap B)$ and m be the least upper bound of $p_2(H\cap A)$. In the space $Y=\omega_1+1\times\omega+1\supset X$ we have that $\langle\omega_1,\omega\rangle\notin\overline{H}^Y$. Because if $\langle\omega_1,\omega\rangle\in\overline{H}^Y$, then for each $\alpha<\omega_1$ and for each $n<\omega$, we have $((\alpha,\omega_1]\times(n,\omega])\cap H\neq\emptyset$. Pick $k>m$ and $\alpha>\gamma$. Pick $\langle\alpha_1,k_1\rangle\in((\alpha,\omega_1]\times(k,\omega])\cap H$. Pick $\langle\alpha_2,k_2\rangle\in((\alpha_1,\omega_1]\times(k_1,\omega])\cap H$. Observe that $\alpha_1<\alpha_2$ and $k_1<k_2$. If $l\geq 3$, $l<\omega$, and $\langle\alpha_1,k_1\rangle,...,\langle\alpha_l,k_l\rangle$ are all picked such that $\alpha_1<\alpha_2<...<\alpha_l$ and $k_1<k_2<...<k_l$. Then pick $\langle\alpha_{l+1},k_{l+1}\rangle\in((\alpha_l,\omega_1]\times(k_l,\omega])\cap H$. By induction, we get a countably infinite sequence $\{\langle\alpha_i,k_i\rangle\colon i\in\mathbb{N}\}$ such that $\alpha_i<\alpha_{i+1}$ and $k_i<k_{i+1}$ for each $i\in\mathbb{N}$. Since ω_1 has uncountable cofinality, then there exists a limit ordinal $\beta<\omega_1$ such that $\langle\beta,\omega\rangle$ is a limit point of the sequence $\{\langle\alpha_i,k_i\rangle\colon i\in\mathbb{N}\}\subseteq H$. Hence $\langle\beta,\omega\rangle\in\overline{H}^X=H$. This means that $\langle\beta,\omega\rangle\in H\cap B$ with $\gamma<\beta$ which is a contradiction because γ is the least upper bound. Therefore, H is closed in Y. Now, let $K^\star=K\cup\{\langle\omega_1,\omega\rangle\}$. Then $K^\star$ is closed in Y which is disjoint from H.

Since Y is normal, being a T_2-compact space, then H and $K^\star$ can be separated in Y by two disjoint open sets, say U and V with $H \subseteq U$ and $K^\star \subseteq V$. Now, the two X-open sets U and $V \cap X$ are disjoint with $H \subseteq U$ and $K \subseteq V \cap X$. So, H and K are separated, which is a contradiction.

Case 4: $p_2(K \cap A)$ is bounded and $p_1(K \cap B)$ is bounded. This case is similar to Case 1.

Case 2: $p_1(H \cap B)$ is bounded and $p_1(K \cap B)$ is bounded. Let γ_1 be the least upper bound for $p_1(H \cap B)$ and γ_2 be the least upper bound for $p_1(K \cap B)$. For each $n \in p_2(K \cap A)$, there exists an $\alpha_n < \omega_1$ such that the open set $V_n = (\alpha_n, \omega_1] \times \{n\}$ is disjoint from H. For each $m \in p_2(H \cap A)$, there exists a $\beta_m < \omega_1$ such that the open set $U_m = (\beta_m, \omega_1] \times \{m\}$ is disjoint from K. Now, the set $\{\gamma_1, \gamma_2, \alpha_n, \beta_m \colon n \in p_2(K \cap A),\ m \in p_2(H \cap A)\}$ is a countable subset of ω_1. Pick an upper bound ξ of it. Now, observe that the set $D = \{\langle \alpha, k\rangle \in H \cup K \colon \xi \leq \alpha < \omega_1 \text{ and } k \notin p_2(K \cap A) \cup p_2(H \cap A)\}$ is countable. So, pick an upper bound ζ of the set $\{\alpha \colon \langle \alpha, k\rangle \in D \text{ for some } k < \omega\}$ with $\xi \leq \zeta$. Let $\eta = \zeta + 1$. We have that $(\eta, \omega_1] \times \{n\} \subseteq V_n$ for each $n \in p_2(K \cap A)$ and $(\eta, \omega_1] \times \{m\} \subseteq U_m$ for each $m \in p_2(H \cap A)$. Thus $\bigcup_{n \in p_2(K \cap A)} (\eta, \omega_1] \times \{n\} = N$ is open and disjoint from H. Also, $\bigcup_{m \in p_2(H \cap A)} (\eta, \omega_1] \times \{m\} = M$ is open and disjoint from K. Now, consider the clopen (closed-and-open) subspace $Z = \eta + 1 \times \omega + 1$ of X which is normal, being T_2-compact. So, the disjoint Z-closed subsets $Z \cap H$ and $Z \cap K$ can be separated in Z by, say, G and L with $Z \cap H \subseteq G$ and $Z \cap K \subseteq L$. Now, let $U = M \cup G$ and $V = N \cup L$. Then U and V are disjoint X-open subsets with $H \subseteq U$ and $K \subseteq V$. Thus H and K are separated in X which is a contradiction.

Case 3: $p_2(K \cap A)$ is bounded and $p_2(H \cap A)$ is bounded. In this case, we must have that either $p_1(H \cap B)$ is bounded or $p_1(K \cap B)$ is bounded since closed unbounded subsets of ω_1 have nonempty intersection and H and K are disjoint. Since either $p_1(H \cap B)$ is bounded or $p_1(K \cap B)$ is bounded, then this case is reduced to either Case 1 or Case 4.

In each case we got a contradiction. Therefore, the Lemma is true. ∎

Theorem 1. *The Tychonoff Plank X is mildly normal.*

Proof: Suppose that there exist two disjoint non-empty regularly closed subsets H and K of X which are unseparated. We have that $\operatorname{int} H \neq \emptyset \neq \operatorname{int} K$. Since any regularly closed set is closed, then, by Lemma 1, assume, without loss of generality, that $p_1(H \cap B)$ is unbounded and $p_2(K \cap A)$ is unbounded.

Claim 1: For each $n \in p_2(K \cap A)$ and for each $\alpha < \omega_1$ there exists $\beta > \alpha$ with $\langle \beta, n \rangle \in \operatorname{int} K \cap (\omega_1 \times \omega)$.

The statement is clear if $\langle \omega_1, n \rangle \in \operatorname{int} K$. If $\langle \omega_1, n \rangle \notin \operatorname{int} K$, then for any basic open neighborhood of $\langle \omega_1, n \rangle$ which is of the form $(\alpha, \omega_1] \times \{n\}$, where $\alpha < \omega_1$, will meet $\operatorname{int} K$ because $\langle \omega_1, n \rangle \in K = \overline{\operatorname{int} K}$.

Claim 2: For each $\gamma \in p_1(H \cap B)$, for each $\zeta_\gamma < \gamma$, and for each $m < \omega$ there exist $n > m$ and β with $\zeta_\gamma < \beta \leq \gamma$ and $\langle \beta, n \rangle \in \operatorname{int} H \cap (\omega_1 \times \omega)$.

The statement is clear if $\langle \gamma, \omega \rangle \in \operatorname{int} H$. If $\langle \gamma, \omega \rangle \notin \operatorname{int} H$, then for any basic open neighborhood of $\langle \gamma, \omega \rangle$ which is of the form $(\zeta_\gamma, \gamma] \times (m, \omega]$, where $\zeta_\gamma < \gamma$ and $m < \omega$, will meet $\operatorname{int} H$ because $\langle \gamma, \omega \rangle \in H = \overline{\operatorname{int} H}$.

Now, pick $n_1 \in p_2(K \cap A)$ and $\alpha_1 < \omega_1$. By Claim 1, pick $\langle \beta_1, n_1 \rangle \in \operatorname{int} K \cap (\omega_1 \times \omega)$. Since $p_1(H \cap B)$ is unbounded, pick $\gamma_1 \in p_1(H \cap B)$ with $\beta_1 < \gamma_1$. Since $p_2(K \cap A)$ is unbounded, pick $m_1 \in p_2(K \cap A)$ with $n_1 < m_1$. Using Claim 2, pick $\langle \alpha_1, k_1 \rangle \in \operatorname{int} H \cap (\omega_1 \times \omega) \cap ((\beta_1, \gamma_1] \times (m_1, \omega])$. We continue by induction. If for $l \geq 2$, $\langle \beta_1, n_1 \rangle, ..., \langle \beta_l, n_l \rangle \in \operatorname{int} K \cap (\omega_1 \times \omega)$ and $\langle \alpha_1, k_1 \rangle, ..., \langle \alpha_l, k_l \rangle \in \operatorname{int} H \cap (\omega_1 \times \omega)$ are all picked with $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < ... < \beta_l < \alpha_l$ and $n_1 < k_1 < n_2 < k_2 < ... < n_l < k_l$. Then, since $p_2(K \cap A)$ is unbounded, pick $n_{l+1} \in p_2(K \cap A)$. Pick $\langle \beta_{l+1}, n_{l+1} \rangle \in \operatorname{int} K \cap (\omega_1 \times \omega) \cap ((\alpha_l, \omega_1] \times \{n_{l+1}\})$. Since $p_1(H \cap B)$ is unbounded, pick $\gamma_{l+1} \in p_1(H \cap B)$ such that $\beta_{l+1} < \alpha_{l+1}$ and m_{l+1} with $n_{l+1} < m_{l+1}$. Pick $\langle \alpha_{l+1}, k_{l+1} \rangle \in \operatorname{int} H \cap (\omega_1 \times \omega) \cap ((\beta_{l+1}, \gamma_{l+1}] \times (m_{l+1}, \omega])$. So, by induction, we got two sequences $\{\langle \beta_i, n_i \rangle \in \operatorname{int} K \cap (\omega_1 \times \omega) : i \in \mathbb{N}\}$ and $\{\langle \alpha_i, k_i \rangle \in \operatorname{int} H \cap (\omega_1 \times \omega) : i \in \mathbb{N}\}$ with $\beta_i < \alpha_i < \beta_{i+1} < \alpha_{i+1}$ for each $i \in \mathbb{N}$ and $n_i < k_i < n_{i+1} < k_{i+1}$ for each $i \in \mathbb{N}$. Now, the set $\{\beta_i, \alpha_i : i \in \mathbb{N}\}$ is a countably infinite subset of ω_1. Let η be its least upper bound. By our construction, any basic open neighborhood of $\langle \eta, \omega \rangle$ will meet $\operatorname{int} H$ and $\operatorname{int} K$. Thus $\langle \eta, \omega \rangle \in \overline{\operatorname{int} H} = H$ and $\langle \eta, \omega \rangle \in \overline{\operatorname{int} K} = K$. Therfore, $H \cap K \neq \emptyset$, which is a contradiction. Thus there are no unseparated disjoint regularly closed sets. Thus X is mildly normal. ∎

Definition 3 (Singal and Singal, [4])**.** A topological space X is called *almost normal* if any two disjoint closed subsets A and B of X one of which is regularly closed can be separated. □

It is clear from the definition that any almost normal space is mildly normal. In [4], Singal and Singal gave a non-regular space which is mildly normal bot not almost normal. The next theorem will give a $T_{3\frac{1}{2}}$-space which is mildly normal but not almost normal.

Theorem 2. *The Tychonoff Plank X is not almost normal.*

Proof: Let $O = \{2n+1\colon\ n < \omega\}$ and $E = \omega \setminus O$. Let

$$K = \Big\{\langle \omega_1, n\rangle\colon\ n \in O\Big\}$$

and

$$H = \left(\bigcup_{m\in E} \Big\{\langle \alpha, m\rangle\colon\ \alpha \le \omega_1,\ m \in E\Big\}\right) \cup B\ .$$

Now, $\operatorname{int} H = \bigcup_{m\in E}\{\langle \alpha, m\rangle\colon\ \alpha \le \omega_1,\ m \in E\}$, and hence $\overline{\operatorname{int} H} = \overline{\bigcup_{m\in E}\{\langle \alpha, m\rangle\colon\ \alpha \le \omega_1,\ m \in E\}} = (\bigcup_{m\in E}\{\langle \alpha, m\rangle\colon\ \alpha \le \omega_1,\ m \in E\}) \cup B = H$. Thus H is regularly closed. It is clear that K is closed and disjoint from H. Since $K \subset A$ is infinite and $B \subset H$, then H and K cannot be separated. Thus X is not almost normal. ∎

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