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STRONGLY NONLINEAR PARABOLIC EQUATIONS WITH NATURAL GROWTH TERMS AND L^1 DATA IN ORLICZ SPACES

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Abstract: We prove compactness and approximation results in inhomogeneous Orlicz–Sobolev spaces and look at, as an application, the Cauchy–Dirichlet problem $u' + A(u) + g(x,t,u,\nabla u) = f \in L^1$. We also give a trace result allowing to deduce the continuity of the solutions with respect to time.

1 – Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let Q be the cylinder $\Omega\times(0,T)$ with some given $T>0$ and let

$$A(u) = -\mathrm{div}\Big(a(x,t,u,\nabla u)\Big)$$

be a Leray–Lions operator defined on $L^p(0,T;W^{1,p}(\Omega))$.

Dall'aglio–Orsina [9] and Porretta [19] proved the existence of solutions for the following Cauchy–Dirichlet problem

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & \text{in } Q,\\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x) & \text{in } \Omega\ , \end{cases}$$

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where g is a nonlinearity with the following "natural" growth condition (of order p):

$$|g(x,t,s,\xi)| \leq b(|s|)\left(c(x,t)+|\xi|^p\right)$$

and which satisfies the classical sign condition $g(x,t,s,\xi)s \geq 0$. The right hand side f is assumed to belong to $L^1(Q)$. This result generalizes analogous one of Boccardo–Gallouet [4]. See also [5] and [6] for related topics. In all of these results, the function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

When trying to relax this restriction on a (for example, if a has exponential or logarithmic growth with respect to ∇u) we are led to replace $L^p(0,T;W^{1,p}(\Omega))$ with an inhomogeneous Sobolev space $W^{1,x}L_M(Q)$ built from an Orlicz space L_M instead of L^p where the N-function M which defines L_M is related to the actual growth of a. The solvability of (1) in this setting is only proved in the variational case i.e. where f belongs to the Orlicz space $W^{-1,x}E_{\overline{M}}(Q)$, see Donaldson [8] for $g \equiv 0$ and Robert [20] for $g \equiv g(x,t,u)$ when A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2 condition and also Elmahi [11] for $g = g(x,t,u,\nabla u)$ when M satisfies a Δ' condition and $M(t) \ll t^{N/(N-1)}$ and finally the recent work Elmahi–Meskine [13] for the general case.

It is our purpose in this paper to prove, in the case where f belongs to $L^1(Q)$, the existence of solutions for parabolic problems of the form (1) in the setting of Orlicz spaces by using a classical approximating method. Thus, and in order to study the behaviour of the approximate solutions we call upon compactness tools, so that, we first establish (in section 3) L^1 compactness results nearly similar to those of Simon [21] and Boccardo–Murat [6] and Elmahi [10].

Next, and when going to the limit in approximating problems, we have to regularize an arbitrary test function by smooth ones with converging distributional time derivatives. This becomes possible thanks to the approximate theorem 3 which is slightly different from theorems 3 and 4 of [15] and will be also applied to get a trace result giving the continuity of such test functions with respect to time.

The plan of the paper is as follows: in Section 2 we recall some preliminaries concerning Orlicz–Sobolev spaces while in Section 3 we prove the compactness results in inhomogeneous Orlicz–Sobolev spaces.

Section 4 will be devoted to approximation results which allow us to overcome the difficulties which arise on time derivatives while in Section 5, we look at, as an application of all previous results, the solvability, in the framework of entropy solutions, of strongly nonlinear parabolic initial-boundary value problems of the form (1), whose simplest model is the following

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u)\,\dfrac{m(|\nabla u|)}{|\nabla u|}\nabla u\Big) + g(x,t,u)\,m(|\nabla u|)\,|\nabla u| \;= f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega\;, \end{cases}$$

where $0 < \alpha \le a(x,t,s) \le \beta$ and where m is any continuous function on $[0,+\infty)$ which strictly increases from 0 to $+\infty$.

Note that, our existence result generalizes analogous ones of [9] and [19] (take indeed $m(t) = t^{p-1}$, with $p > 1$). Moreover, and contrary to [9] and [19], the proof is achieved without extending the initial problem or assuming the positiveness of either the data f or the initial condition u_0.

2 – Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t m(\tau)d\tau$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau$, where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup\{s : m(s) \le t\}$ (see [1], [16] and [17]).

We will extend these N-functions into even functions on all $\mathbb{R}$.

The N-function M is said to satisfy a Δ_2 condition if, for some $k > 0$:

$$M(2t) \le k\,M(t) \qquad \forall\, t \ge 0\;. \tag{2}$$

when (2) holds only for $t \ge$ some $t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).

$L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx \le 1\right\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)\, v(x)\, dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition (near infinity only if Ω has finite measure).

Two N-functions M and P are said to be equivalent (resp. near infinity), if there exist reals numbers $k_1, k_2 > 0$ such that $P(k_2 t) \le M(t) \le P(k_2 t)$ for all $t \ge 0$ (resp. for all $t \ge$ some $t_0 > 0$).

$P \ll M$ means that P grows essentially less rapidly than M, i.e. for each $\varepsilon > 0$, $P(t)/(M(\varepsilon\, t)) \to 0$ as $t \to \infty$. This is the case if and only if $M^{-1}(t)/P^{-1}(t) \to 0$ as $t \to \infty$, therefore, we have the following continuous imbedding $L_M(\Omega) \subset E_P(\Omega)$ when Ω has finite measure.

2.3. We now turn to the Orlicz–Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha| \le 1} \|D^\alpha u\|_{M,\Omega} .$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M((D^\alpha u_n - D^\alpha u)/\lambda) dx \to 0$ for all $|\alpha| \le 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Note that, if $u_n \to u$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_{\overline{M}}(\Omega)$ for the modular convergence, we have

$$\int_\Omega u_n v_n\, dx \to \int_\Omega uv\, dx \qquad \text{as } n \to \infty . \tag{3}$$

Indeed, let $\lambda > 0$ and $\mu > 0$ such that

$$\int_\Omega M\Big(\frac{u_n - u}{\lambda}\Big)dx \;\to\; 0 \quad \text{ and } \quad \int_\Omega \overline{M}\Big(\frac{v_n - v}{\mu}\Big)\,dx \;\to\; 0$$

and, since $u_n v_n - uv = (u_n - u)\,(v_n - v) + u_n v + u v_n - 2uv$, we obtain

$$\frac{1}{\lambda\mu}\Big|\int_\Omega (u_n v_n - uv)\,dx\Big| \;\le$$
$$\le\; \int_\Omega M\Big(\frac{u_n - u}{\lambda}\Big)dx \;+ \int_\Omega \overline{M}\Big(\frac{v_n - v}{\mu}\Big)dx \;+ \frac{1}{\lambda\mu}\Big|\int_\Omega (u_n v + u v_n - 2uv)\,dx\Big|$$

therefore, by letting $n \to \infty$ in the last side, we get the result.

If M satisfies the Δ_2 condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [14], [15]). Consequently, the action of a distribution T in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1 L_M(\Omega)$ is well defined, it will be denoted by $\langle T, u\rangle$.

2.5. Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega\times]0,T[$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz–Sobolev spaces of order 1 are defined as follows

$$W^{1,x}L_M(Q) \;=\; \Big\{u \in L_M(Q)\colon\; D_x^\alpha u \in L_M(Q),\;\; \forall\,|\alpha| \le 1\Big\}$$

and

$$W^{1,x}E_M(Q) \;=\; \Big\{u \in E_M(Q)\colon\; D_x^\alpha u \in E_M(Q),\;\; \forall\,|\alpha| \le 1\Big\}\;.$$

The latter space is a subspace of the former. Both are Banach spaces under the norm

$$\|u\| \;=\; \sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{M,Q}\;.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$ then the function: $t \longmapsto u(t) = u(.,t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$ then $u(.,t)$ is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0,T;W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable from $(0,T)$ into $W^1L_M(\Omega)$. However, the scalar function $t \longmapsto \|D^\alpha_x u(t)\|_{M,\Omega}$ is in $L^1(0,T)$ for all $|\alpha| \le 1$.

2.6. The space $W^{1,x}_0 E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [15] (see the proof of theorem 3 below) that when Ω has the segment property then each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak $*$ topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is limit, in $W^{1,x}L_M(Q)$, of some sequence $(u_n) \subset \mathcal{D}(Q)$ for the modular convergence i.e. there exists $\lambda > 0$ such that, for all $|\alpha| \le 1$, $\int_Q M((D^\alpha_x u_n - D^\alpha_x u)/\lambda)dx\,dt \to 0$ when $n \to \infty$, this implies that (u_n) converges to u in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently, $\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M,\Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M,\Pi L_{\overline{M}})}$, this space will be denoted by $W^{1,x}_0 L_M(Q)$. Furthermore, $W^{1,x}_0 E_M(Q) = W^{1,x}_0 L_M(Q) \cap \Pi E_M$.

Poincaré's inequality also holds in $W^{1,x}_0 L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W^{1,x}_0 L_M(Q)$ one has

$$\sum_{|\alpha|\le 1} \|D^\alpha_x u\|_{M,Q} \le C \sum_{|\alpha|=1} \|D^\alpha_x u\|_{M,Q} \;,$$

thus both sides of the last inequality are equivalent norms on $W^{1,x}_0 L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W^{1,x}_0 L_M(Q) & F \\ W^{1,x}_0 E_M(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W^{1,x}_0 E_M(Q)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W^{1,x}_0 E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q)$ and it is shown that $W^{-1,x}L_{\overline{M}}(Q) = \{f = \sum_{|\alpha|\le 1} D^\alpha_x f_\alpha : f_\alpha \in L_{\overline{M}}(Q)\}$. This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\overline{M},Q}$$

where the inf is taken over all possible decompositions $f = \sum_{|\alpha|\le 1} D^\alpha_x f_\alpha$, $f_\alpha \in L_{\overline{M}}(Q)$. The space F_0 is then given by $F_0 = \{f = \sum_{|\alpha|\le 1} D^\alpha_x f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

3 – Compactness results

In this section, we shall prove some compactness theorems in inhomogeneous Orlicz–Sobolev spaces which will be applied to study the behaviour of the approximating solutions for parabolic problems. These results, which are nearly similar to those of Simon [21], Boccardo–Murat [6] and Elmahi [10], give only L^1 (and not L_M) compactness for sets in $W^{1,x}L_M(Q)$. They are, however, sufficient for applications to solve parabolic problems in Orlicz spaces of variational type or with L^1 data.

For each $h > 0$, define the usual translated $\tau_h f$ of the function f by $\tau_h f(t) = f(t+h)$. If f is defined on $[0,T]$ then $\tau_h f$ is defined on $[-h, T-h]$.

First of all, recall the following compactness result proved by Simon [21].

Theorem 1. *See [21]. Let B be a Banach space and let $T > 0$ be a fixed real number. If $F \subset L^1(0,T;B)$ is such that*

$$\left\{\int_{t_1}^{t_2} f(t)\,dt\right\}_f \text{ is relatively compact in } B, \qquad \text{for all } \ 0 < t_1 < t_2 < T \ . \tag{4}$$

$$\|\tau_h f - f\|_{L^1(0,T;B)} \to 0 \ \text{ uniformly in } \ f \in F, \qquad \text{when } \ h \to 0 \ . \tag{5}$$

Then F is relatively compact in $L^1(0,T;B)$.

Next, we prove the following lemma, which it can be seen as a "Orlicz" version of the well known interpolation inequality related to the space $L^p(0,T;W_0^{1,p}(\Omega))$.

Lemma 1. *Let M be an N-function. Let Y be a Banach space such that the following continuous imbedding holds: $L^1(\Omega) \subset Y$. Then, for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$, with $|\nabla u|/\lambda \in \mathcal{L}_M(Q)$,*

$$\|u\|_{L^1(Q)} \ \le \ \varepsilon\,\lambda\left(\int_Q M\left(\frac{|\nabla u|}{\lambda}\right)dx\,dt \,+\, T\right) + C_\varepsilon \|u\|_{L^1(0,T;Y)} \ .$$

Proof: Since $W_0^1 L_M(\Omega) \subset L^1(\Omega)$ with compact imbedding, see [1], then, for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all $v \in W_0^1 L_M(\Omega)$:

$$\|v\|_{L^1(\Omega)} \ \le \ \varepsilon\|\nabla v\|_{L_M(\Omega)} + C_\varepsilon\|v\|_Y \ . \tag{6}$$

Indeed, if the above assertion holds false, there is $\varepsilon_0 > 0$ and $v_n \in W_0^1 L_M(\Omega)$ such that

$$\|v_n\|_{L^1(\Omega)} \;\geq\; \varepsilon_0 \|\nabla v_n\|_{L_M(\Omega)} + n\|v_n\|_Y \;.$$

This gives, by setting $w_n = v_n/\|\nabla v_n\|_{L_M(\Omega)}$:

$$\|w_n\|_{L^1(\Omega)} \geq \varepsilon_0 + n\|w_n\|_Y \,, \qquad \|\nabla w_n\|_{L_M(\Omega)} = 1 \;.$$

Since (w_n) is bounded in $W_0^1 L_M(\Omega)$ then for a subsequence,

$$w_n \rightharpoonup w \;\text{ in } W_0^1 L_M(\Omega) \quad \text{ for } \;\sigma(\Pi L_M, \Pi E_{\overline{M}}) \;\text{ and strongly in }\; L^1(\Omega) \;.$$

Thus $\|w_n\|_{L^1(\Omega)}$ is bounded and $\|w_n\|_Y \to 0$ as $n \to \infty$. We deduce that $w_n \to 0$ in Y and that $w = 0$ implying that $\varepsilon_0 \leq \|w_n\|_{L^1(\Omega)} \to 0$, a contradiction.

Using $v = u(t)$ in (6) for all $u \in W_0^{1,x} L_M(Q)$ with $|\nabla u|/\lambda \in \mathcal{L}_M(Q)$ and a.e. t in $(0,T)$, we have

$$\|u(t)\|_{L^1(\Omega)} \;\leq\; \varepsilon \|\nabla u(t)\|_{L_M(\Omega)} + C_\varepsilon \|u(t)\|_Y \;.$$

Since $\int_Q M(|\nabla u(x,t)|/\lambda)dx\,dt < \infty$ we have thanks to Fubini's theorem,

$$\int_\Omega M\Big(\frac{|\nabla u(x,t)|}{\lambda}\Big) dx \;<\; \infty \qquad \text{for a.e. }\; t \;\text{ in } (0,T)$$

and then

$$\|\nabla u(t)\|_{L_M(\Omega)} \;\leq\; \lambda \bigg(\int_\Omega M\Big(\frac{|\nabla u(x,t)|}{\lambda}\Big) dx \;+\; 1 \bigg)$$

which implies that

$$\|u(t)\|_{L^1(\Omega)} \;\leq\; \varepsilon\,\lambda \bigg(\int_\Omega M\Big(\frac{|\nabla u(x,t)|}{\lambda}\Big) dx \;+\; 1 \bigg) \;+\; C_\varepsilon \|u(t)\|_Y \;.$$

Integrating this over $(0,T)$ yields

$$\|u\|_{L^1(Q)} \;\leq\; \varepsilon\,\lambda \bigg(\int_Q M\Big(\frac{|\nabla u(x,t)|}{\lambda}\Big) dx\,dt \;+\; T \bigg) \;+\; C_\varepsilon \int_0^T \|u(t)\|_Y \,dt$$

and finally

$$\|u\|_{L^1(Q)} \;\leq\; \varepsilon\,\lambda \bigg(\int_Q M\Big(\frac{|\nabla u|}{\lambda}\Big) dx\,dt \;+\; T \bigg) \;+\; C_\varepsilon \|u\|_{L^1(0,T;Y)} \;. \;\blacksquare$$

We also prove the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 2. *Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous imbedding.*

If F is bounded in $W_0^{1,x} L_M(Q)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q)$ (and also in $E_P(Q)$ for all N-function $P \ll M$).

Proof: Let $\varepsilon > 0$ be given. Let $C > 0$ be such that $\int_Q M(|\nabla f|/C) dx\, dt \le 1$ for all $f \in F$.

By the previous lemma, there exists $C_\varepsilon > 0$ such that, for all $u \in W_0^{1,x} L_M(Q)$ with $|\nabla u|/(2C) \in \mathcal{L}_M(Q)$,

$$\|u(t)\|_{L^1(Q)} \;\le\; \frac{2\,\varepsilon C}{4\,C(1+T)} \left(\int_Q M\left(\frac{|\nabla u|}{2C}\right) dx\, dt \,+\, T \right) + C_\varepsilon \|u\|_{L^1(0,T;Y)} \;.$$

Moreover, there exists a finite sequence (f_i) in F satisfying:

$$\forall\, f \in F, \quad \exists\, f_i \quad \text{ such that } \; \|f - f_i\|_{L^1(0,T;Y)} \le \frac{\varepsilon}{2\, C_\varepsilon}$$

so that

$$\begin{aligned} \|f - f_i\|_{L^1(Q)} \;&\le\; \frac{\varepsilon}{2(1{+}T)} \left(\int_Q M\left(\frac{|\nabla f - \nabla f_i|}{2\,C}\right) dx\, dt \,+\, T \right) + C_\varepsilon \|f - f_i\|_{L^1(0,T;Y)} \\ &\le\; \varepsilon \end{aligned}$$

and hence F is relatively compact in $L^1(Q)$.

Since $P \ll M$ then by using Vitali's theorem, it is easy to see that F is relatively compact in $E_P(Q)$. ∎

Lemma 3. *(See [21]). Let B be a Banach space.*

If $f \in \mathcal{D}'(]0,T[;B)$ is such that $\frac{\partial f}{\partial t} \in L^1(0,T;B)$ then $f \in C(]0,T[,B)$ and for all $h > 0$

$$\|\tau_h f - f\|_{L^1(0,T;B)} \;\le\; h \left\| \frac{\partial f}{\partial t} \right\|_{L^1(0,T;B)} .$$

Remark 1. By lemma 4, if $F \subset L^1(0,T;B)$ is such that $\left\{ \frac{\partial f}{\partial t} : f \in F \right\}$ is bounded in $L^1(0,T;B)$ then

$$\|\tau_h f - f\|_{L^1(0,T;B)} \to 0 \;\text{ as }\; h \to 0 \quad \text{ uniformly with respect to } \; f \in F \;. \;\square$$

Lemma 4. *(See [8]). The following continuous imbedding hold:* $W_0^{1,x}E_M(Q) \subset L^1(0,T;W_0^1E_M(\Omega))$ *and* $W^{-1,x}E_{\overline{M}}(Q) \subset L^1(0,T;W^{-1}E_{\overline{M}}(\Omega))$.

We shall now apply the previous results to prove some compactness theorems in inhomogeneous Orlicz–Sobolev spaces.

Theorem 2. *Let M be an N-function. If F is bounded in $W_0^{1,x}L_M(Q)$ and $\left\{\frac{\partial f}{\partial t} : f \in F\right\}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q)$ then F is relatively compact in $L^1(Q)$.*

Proof: Let P and R be N-functions such that $P \ll M$ and $R \ll \overline{M}$ near infinity.

For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have

$$\begin{aligned} \left\|\int_{t_1}^{t_2} f(t)\,dt\right\|_{W_0^1E_P(\Omega)} &\leq \int_0^T \|f(t)\|_{W_0^1E_P(\Omega)}\,dt \\ &\leq C_1\|f\|_{W_0^{1,x}E_P(Q)} \leq C_2\|f\|_{W_0^{1,x}L_M(Q)} \leq C \end{aligned}$$

where we have used the following continuous imbedding

$$W_0^{1,x}L_M(Q) \subset W_0^{1,x}E_P(Q) \subset L^1(0,T;W_0^1E_P(\Omega)) .$$

Since the imbedding $W_0^1E_P(\Omega) \subset L^1(\Omega)$ is compact we deduce that $\left(\int_{t_1}^{t_2} f(t)\,dt\right)_{f\in F}$ is relatively compact in $L^1(\Omega)$ and in $W^{-1,1}(\Omega)$ as well.

On the other hand $\left\{\frac{\partial f}{\partial t} : f \in F\right\}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q)$ and in $L^1(0,T;W^{-1,1}(\Omega))$ as well, since

$$W^{-1,x}L_{\overline{M}}(Q) \subset W^{-1,x}E_R(Q) \subset L^1(0,T;W^{-1}E_R(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega)) ,$$

with continuous imbedding.

By Remark 1, we deduce that $\|\tau_h f - f\|_{L^1(0,T;W^{-1,1}(\Omega))} \to 0$ uniformly in $f \in F$ when $h \to 0$ and by using theorem 1, F is relatively compact in $L^1(0,T;W^{-1,1}(\Omega))$.

Since $L^1(\Omega) \subset W^{-1,1}(\Omega)$ with continuous imbedding we can apply lemma 2 to conclude that F is relatively compact in $L^1(Q)$. ∎

Corollary 1. *Let M be an N-function.*
Let (u_n) be a sequence of $W^{1,x}L_M(Q)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,x}L_M(Q) \qquad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}})$$
$$\text{and} \quad \frac{\partial u_n}{\partial t} = h_n + k_n \qquad \text{in} \quad \mathcal{D}'(Q)$$

with (h_n) bounded in $W^{-1,x}L_{\overline{M}}(Q)$ and (k_n) bounded in the space $\mathcal{M}(Q)$ of measures on Q.
Then $u_n \to u$ strongly in $L^1_{loc}(Q)$.
If further $u_n \in W_0^{1,x}L_M(Q)$ then $u_n \to u$ strongly in $L^1(Q)$.

Proof: It is easily adapted from that given in [6] by using Theorem 2 and Remark 1 instead of lemma 8 of [21]. ∎

4 – Approximation and time mollification

In this section, Ω is an open subset of $\mathbb{R}^N$ with the segment property and I is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$.

Definition 1. We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha|\le 1} D_x^\alpha u_n^\alpha + u_n^0 \qquad \text{and} \qquad u = \sum_{|\alpha|\le 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence $\forall\, |\alpha| \le 1$ and $u_n^0 \to u^0$ strongly in $L^1(Q)$. □

This implies, in particular, that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ for the weak topology $\sigma(\Pi L_{\overline{M}} + L^1, \Pi L_M \cap L^\infty)$ in the sense that $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ where here and throughout the paper $\langle\;,\;\rangle$ means for either the pairing between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$, or between $W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$; indeed,

$$\langle u_n, v\rangle = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \; dx\, dt + \int_Q u_n^0 v \; dx\, dt$$

and since for all $|\alpha| \le 1$, $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q)$ for the modular convergence, and so for $\sigma(L_{\overline{M}}, L_M)$, we have

$$\sum_{|\alpha|\le 1} (-1)^{|\alpha|} \int_Q u_n^\alpha D_x^\alpha v \; dx\,dt + \int_Q u_n^0 v \; dx\,dt \;\to$$
$$\to \sum_{|\alpha|\le 1} (-1)^{|\alpha|} \int_Q u^\alpha D_x^\alpha v \; dx\,dt + \int_Q u^0 v \; dx\,dt \;=\; \langle u, v\rangle \;.$$

Moreover, if $v_n \to v$ in $W_0^{1,x} L_M(Q)$ for the modular convergence and weakly* in $L^\infty(Q)$, we have $\langle u_n, v_n\rangle \to \langle u, v\rangle$ as $n \to \infty$, see (3).

We shall prove the following approximation theorem which plays a fundamental role when proving the existence of solutions for parabolic problems.

Theorem 3. *If $u \in W^{1,x} L_M(Q) \cap L^1(Q)$ (resp. $W_0^{1,x} L_M(Q) \cap L^1(Q)$) and $\partial u/\partial t \in W^{-1,x} L_{\overline{M}}(Q) + L^1(Q)$ then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q})$ (resp. $\mathcal{D}(\overline{I}, \mathcal{D}(\Omega))$) such that*

$$v_j \to u \;\text{ in }\; W^{1,x} L_M(Q) \quad\text{ and }\quad \frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \;\text{ in }\; W^{-1,x} L_{\overline{M}}(Q) + L^1(Q)$$

for the modular convergence.

Proof: Let $u \in W^{1,x} L_M(Q) \cap L^1(Q)$ such that $\partial u/\partial t \in W^{-1,x} L_{\overline{M}}(Q) + L^1(Q)$ and let $\varepsilon > 0$ be given. Writing $\partial u/\partial t = \sum_{|\alpha|\le 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_{\overline{M}}(Q)$ for all $|\alpha| \le 1$ and $u^0 \in L^1(Q)$, we will show that there exists $\lambda > 0$ (depending only on u and N) and there exists $v \in \mathcal{D}(\overline{Q})$ for which we can write $\partial v/\partial t = \sum_{|\alpha|\le 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q})$ such that

$$\int_Q M\Big(\frac{D_x^\alpha v - D_x^\alpha u}{\lambda}\Big) dx\,dt \;\le\; \varepsilon\,, \qquad \int_Q \overline{M}\Big(\frac{v^\alpha - u^\alpha}{\lambda}\Big) dx\,dt \;\le\; \varepsilon \tag{7}$$
$$\forall\, |\alpha| \le 1 \quad\text{ and }\quad \|v^0 - u^0\|_{L^1(Q)} \le \varepsilon \;.$$

We will process as in [15] (see the proofs of Theorem 3 and Theorem 4). Since the approximation of u and $D_x^\alpha u$ can be obtained in the same way, we will only show that the approximation also holds for the time derivative. Thus, we consider $\varphi \in \mathcal{D}(\mathbb{R}^{N+1})$ with $0 \le \varphi \le 1$, $\varphi = 1$ for $|(x,t)| \le 1$ and $\varphi = 0$ for $|(x,t)| \ge 2$. Let $\varphi_r(x,t) = \varphi((x,t)/r)$ and let $u_r = \varphi_r u$.

On the one hand, we have

$$\begin{aligned}\frac{\partial u_r}{\partial t} &= \varphi_r\left(\sum_{|\alpha|\le 1} D_x^\alpha u^\alpha + u^0\right) + \frac{1}{r}\,\frac{\partial\varphi}{\partial t}\Big(\frac{(x,t)}{r}\Big)u \\ &= \sum_{|\alpha|\le 1} D_x^\alpha(\varphi_r u^\alpha) + \left[-\frac{1}{r}\sum_{|\alpha|=1} D_x^\alpha\varphi\Big(\frac{(x,t)}{r}\Big)u^\alpha\right] + \left[\frac{1}{r}\,\frac{\partial\varphi}{\partial t}\Big(\frac{(x,t)}{r}\Big)u + \varphi_r u^0\right] \\ &:= u_r^1 + u_r^2 + u_r^3\ .\end{aligned}$$

When $r \to \infty$, we have, by Lemma 5 of [15], $u_r^1 \to \sum_{|\alpha|\le 1} D_x^\alpha u^\alpha$ in $W^{-1,x}L_{\overline{M}}(Q)$ for the modular convergence and, by direct examination, $u_r^2 \to 0$ strongly in $L_{\overline{M}}(Q)$ and $u_r^3 \to u^0$ strongly in $L^1(Q)$. Hence, we can choose $\lambda > 0$ (namely such that $D_x^\alpha u/\lambda \in \mathcal{L}_M(Q)$ and $u^\alpha/\lambda \in \mathcal{L}_{\overline{M}}(Q)$ for all $|\alpha| \le 1$) and $r > 0$ such that

$$\int_Q M\Big((D_x^\alpha u_r - D_x^\alpha u)/\lambda\Big)\,dx\,dt \;\le\; \varepsilon \quad \forall\,|\alpha|\le 1\,, \qquad \int_Q \overline{M}(u_r^2/\lambda)\,dx\,dt \;\le\; \varepsilon$$

$$\|u_r^3 - u^0\|_{L^1(Q)} \le \varepsilon \quad \text{and} \quad \int_Q \overline{M}\Big((\varphi_r u^\alpha - u^\alpha)/\lambda\Big)\,dx\,dt \;\le\; \varepsilon \quad \forall\,|\alpha| \le 1\ . \tag{8}$$

On the other hand, let ψ_i be a C^∞ partition of unity on $\overline{Q}$ subordinate to a covering $\{U_i\}$ of $\overline{Q}$ satisfying the properties of lemma 7 of [15] and consider the translated function $(\psi_i v_r)_{t_i}$ defined by $(\psi_i v_r)_{t_i}(x,t) = (\psi_i v_r)((x,t)+t_i y_i)$ where y_i is the vector associated to U_i by the segment property. Let ρ_σ be a mollifier sequence in $\mathbb{R}^{N+1}$, that is, $\rho_\sigma \in \mathcal{D}(\mathbb{R}^{N+1})$, $\rho_\sigma(x,t) = 0$ for $|(x,t)| \ge \sigma$, $\rho_\sigma \ge 0$ and $\int_{\mathbb{R}^{N+1}} \rho_\sigma = 1$. Extending u_r outside Q by zero, we see that $\psi_i u_r$ vanishes identically for all $i \ge$ some i_r. As in [15], we define

$$v \;=\; \sum_{i=1}^{i_r} (\psi_i u_r)_{t_i} * \rho_{\sigma_i} \;\in\; \mathcal{D}(\overline{Q})\ .$$

Clearly, we have

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{i_r}(\psi_i u_r^1)_{t_i}*\rho_{\sigma_i} + \sum_{i=1}^{i_r}(\psi_i u_r^2)_{t_i}*\rho_{\sigma_i} + \sum_{i=1}^{i_r}(\psi_i u_r^3)_{t_i}*\rho_{\sigma_i} + \sum_{i=1}^{i_r}\Big(\frac{\partial\psi_i}{\partial t}u_r\Big)_{t_i}*\rho_{\sigma_i}$$

and since

$$\sum_{i=1}^{i_r}(\psi_i u_r^1)_{t_i}*\rho_{\sigma_i} \;=\; \sum_{i=1}^{i_r}\left(\psi_i\sum_{|\alpha|\le 1} D_x^\alpha(\varphi_r u^\alpha)\right)_{t_i} * \rho_{\sigma_i} \;=$$

$$= \sum_{i=1}^{i_r} \left(\sum_{|\alpha|\le 1} D_x^\alpha(\psi_i \varphi_r u^\alpha) \right)_{t_i} * \rho_{\sigma_i} - \sum_{i=1}^{i_r} \left(\sum_{|\alpha|=1} (D_x^\alpha \psi_i)\varphi_r u^\alpha \right)_{t_i} * \rho_{\sigma_i}$$

$$= \sum_{|\alpha|\le 1} \left(\sum_{i=1}^{i_r} \Big(D_x^\alpha(\psi_i \varphi_r u^\alpha) \Big)_{t_i} * \rho_{\sigma_i} \right) - \sum_{i=1}^{i_r} \left(\sum_{|\alpha|=1} (D_x^\alpha \psi_i)\varphi_r u^\alpha \right)_{t_i} * \rho_{\sigma_i}$$

we deduce that

$$\frac{\partial v}{\partial t} = \sum_{|\alpha|\le 1} D_x^\alpha v^\alpha + v^2 + v^3$$

where, as it can be easily seen

$$v^\alpha = \sum_{i=1}^{i_r} (\psi_i \varphi_r u^\alpha)_{t_i} * \rho_{\sigma_i} \quad \forall\, |\alpha| \le 1 ,$$

$$v^2 = \sum_{i=1}^{i_r} (\psi_i u_r^2)_{t_i} * \rho_{\sigma_i} - \sum_{i=1}^{i_r} \left(\sum_{|\alpha|=1} D_x^\alpha(\psi_i)\varphi_r u^\alpha \right)_{t_i} * \rho_{\sigma_i}$$

$$v^3 = \sum_{i=1}^{i_r} (\psi_i u_r^3)_{t_i} * \rho_{\sigma_i} + \sum_{i=1}^{i_r} \left(\frac{\partial \psi_i}{\partial t} u_r \right)_{t_i} * \rho_{\sigma_i} .$$

Now, for each $i = 1, ..., i_r$, we can choose (see lemma 5 of [15]) $0 < t_i < 1$ and $\rho_{\sigma_i} = \rho_i$ such that

$$\text{(9)} \qquad \begin{aligned} &\int_Q M\left(\left(\sum_{i=1}^{i_r} (\psi_i D_x^\alpha u_r)_{t_i} * \rho_i - D_x^\alpha u_r \right) / \lambda \right) dx\, dt \le \varepsilon \quad \forall\, |\alpha| \le 1 , \\ &\int_Q \overline{M}\Big((v^2 - u_r^2)/\lambda \Big)\, dx\, dt \le \varepsilon, \\ &\|v^3 - u_r^3\|_{L^1(Q)} \le \varepsilon , \\ &\int_Q \overline{M}\left(\left(\sum_{i=1}^{i_r} (\psi_i \varphi_r u^\alpha)_{t_i} * \rho_i - \varphi_r u^\alpha \right) / \lambda \right) dx\, dt \le \varepsilon \quad \forall\, |\alpha| \le 1 . \end{aligned}$$

Combining (8) and (9), we get the result.

The case where $u \in W_0^{1,x} L_M(Q) \cap L^1(Q)$ can be handled similarly without essential difficulty as it is mentioned in the proof of theorem 4 of [15]. ∎

Remark 2. The assumption $u \in L^1(Q)$ in theorem 3 is needed only when Q has infinite measure, since else, we have $L_M(Q) \subset L^1(Q)$ and so $W^{1,x} L_M(Q) \cap L^1(Q) = W^{1,x} L_M(Q)$. □

Remark 3. If, in the statement of theorem 3 above, one takes $I=\mathbb{R}$, we have that $\mathcal{D}(\Omega\times\mathbb{R})$ is dense in $\{u \in W_0^{1,x}L_M(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) : \partial u/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})\}$ for the modular convergence. This trivially follows from the fact that $\mathcal{D}(\mathbb{R},\mathcal{D}(\Omega)) \equiv \mathcal{D}(\Omega\times\mathbb{R})$. □

A first application of theorem 3 is the following trace result (see [19], Theorem 1.1, for the case of ordinary Sobolev spaces).

Lemma 5. *Let $a<b\in\mathbb{R}$ and Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. Then*

$$\Big\{u \in W_0^{1,x}L_M(\Omega\times(a,b))\colon\ \partial u/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b)) + L^1(\Omega\times(a,b))\Big\} \subset$$
$$\subset\ C([a,b],L^1(\Omega))\ .$$

Proof: Let $u \in W_0^{1,x}L_M(\Omega\times(a,b))$ such that $\partial u/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b)) + L^1(\Omega\times(a,b))$. After two consecutive reflections first with respect to $t=b$ and then with respect to $t=a$:

$$\hat{u}(x,t) \;=\; u(x,t)\chi_{(a,b)} + u(x,2b-t)\chi_{(b,2b-a)} \qquad \text{on}\ \ \Omega\times(a,2b-a)$$

and

$$\tilde{u}(x,t) \;=\; \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x,2a-t)\chi_{(3a-2b,a)} \qquad \text{on}\ \ \Omega\times(3a-2b,2b-a)\ ,$$

we get a function $\tilde{u} \in W_0^{1,x}L_M(\Omega\times(3a-2b,2b-a))$ with $\partial\tilde{u}/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times(3a-2b,2b-a)) + L^1(\Omega\times(3a-2b,2b-a))$. Now, by letting a function $\eta\in\mathcal{D}(\mathbb{R})$ with $\eta=1$ on $[a,b]$ and $\operatorname{supp}\eta \subset (3a-2b,2b-a)$, we set $\overline{u}=\eta\tilde{u}$; therefore, by standard arguments (see [7], Lemme IV and Remarque 10 p. 158), we have: $\overline{u}=u$ on $\Omega\times(a,b)$, $\overline{u}\in W_0^{1,x}L_M(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})$ and $\partial\overline{u}/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R})$.

Let now v_j the sequence given by theorem 3 corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \quad \text{in}\ \ W_0^{1,x}L_M(\Omega\times\mathbb{R})$$

and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \quad \text{in}\ \ W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})$$

for the modular convergence.

Throughout this paper, we denote T_k the usual truncation at height k defined on $\mathbb{R}$ by $T_k(s) = \min(k, \max(s, -k))$ and $S_k(s) = \int_0^s T_k(t)\,dt$ its primitive. We have,

$$\int_\Omega S_1(v_i - v_j)(\tau)\,dx = \int_\Omega \int_{-\infty}^\tau T_1(v_i - v_j)\Big(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t}\Big)\,dx\,dt \;\to\; 0 \quad \text{as } \; i, j \to \infty \;,$$

from which, by following [19], one deduces that v_j is a Cauchy sequence in $C(\mathbb{R}, L^1(\Omega))$ and hence $\overline{u} \in C(\mathbb{R}, L^1(\Omega))$. Consequently, $u \in C([a,b], L^1(\Omega))$. ∎

In order to deal with the time derivative, we introduce a time mollification of a function $u \in L_M(Q)$. Thus we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \mu \int_{-\infty}^t \tilde{u}(x,s) \exp\Big(\mu(s-t)\Big)\,ds \tag{10}$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of u.

Throughout the paper the index μ always indicates this mollification.

Proposition 1. *If $u \in L_M(Q)$ then u_μ is measurable in Q and $\partial u_\mu/\partial t = \mu(u - u_\mu)$ and if $u \in \mathcal{L}_M(Q)$ then*

$$\int_Q M(u_\mu)\,dx\,dt \le \int_Q M(u)\,dx\,dt \;.$$

Proof: Since $(x,t,s) \longmapsto u(x,s)\exp(\mu(s-t))$ is measurable in $\Omega\times[0,T]\times[0,T]$, we deduce that u_μ is measurable by Fubini's theorem. By Jensen's integral inequality we have, since $\int_{-\infty}^0 \mu \exp(\mu s)\,ds = 1$,

$$\begin{aligned} M\left(\int_{-\infty}^t \mu\,\tilde{u}(x,s)\exp\Big(\mu(s-t)\Big)ds\right) &= M\left(\int_{-\infty}^0 \mu\,\exp(\mu s)\,\tilde{u}(x,s+t)\,ds\right) \\ &\le \int_{-\infty}^0 \mu\,\exp(\mu s)\,M(\tilde{u}(x,s+t))\,ds \end{aligned}$$

which implies

$$\begin{aligned} \int_Q M(u_\mu(x,t))\,dx\,dt &\le \int_{\Omega\times\mathbb{R}} \left(\int_{-\infty}^0 \mu\,\exp(\mu s)\,M(\tilde{u}(x,s+t))\,ds\right)dx\,dt \\ &\le \int_{-\infty}^0 \mu\,\exp(\mu s)\left(\int_{\Omega\times\mathbb{R}} M(\tilde{u}(x,s+t))\,dx\,dt\right)ds \\ &\le \int_{-\infty}^0 \mu\,\exp(\mu s)\left(\int_Q M(u(x,t))\,dx\,dt\right)ds \\ &= \int_Q M(u)\,dx\,dt \;. \end{aligned}$$

Furthermore

$$\begin{aligned}\frac{\partial u_\mu}{\partial t} &= \lim_{\theta\to 0}\frac{1}{\theta}(e^{-\mu\theta}-1)\,u_\mu(x,t) + \lim_{\theta\to 0}\frac{1}{\theta}\int_t^{t+\theta} u(x,s)\,e^{\mu(s-(t+\theta))}\,ds \\ &= -\mu u_\mu + \mu u \,. \blacksquare\end{aligned}$$

Proposition 2.

1) *If* $u \in L_M(Q)$ *then* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $L_M(Q)$ *for the modular convergence.*

2) *If* $u \in W^{1,x}L_M(Q)$ *then* $u_\mu \to u$ *as* $\mu \to +\infty$ *in* $W^{1,x}L_M(Q)$ *for the modular convergence.*

Proof: 1) Let $(\varphi_k) \subset \mathcal{D}(Q)$ such that $\varphi_k \to u$ in $L_M(Q)$ for the modular convergence. Let $\lambda > 0$ large enough such that

$$\frac{u}{\lambda} \in \mathcal{L}_M(Q) \qquad \text{and} \qquad \int_Q M\Big(\frac{\varphi_k - u}{\lambda}\Big)dx\,dt \;\to\; 0 \quad \text{as } k\to\infty \,.$$

For a.e. $(x,t)\in Q$ we have

$$|(\varphi_k)_\mu(x,t) - \varphi_k(x,t)| \;=\; \frac{1}{\mu}\Big|\frac{\partial\varphi_k}{\partial t}(x,t)\Big| \;\le\; \frac{1}{\mu}\Big\|\frac{\partial\varphi_k}{\partial t}\Big\|_\infty \,.$$

On the other hand

$$\begin{aligned}\int_Q M\Big(\frac{u_\mu - u}{3\lambda}\Big)dx\,dt \;&\le\; \frac{1}{3}\int_Q M\Big(\frac{u_\mu - (\varphi_k)_\mu}{\lambda}\Big)dx\,dt + \frac{1}{3}\int_Q M\Big(\frac{(\varphi_k)_\mu - \varphi_k}{\lambda}\Big)dx\,dt \\ &\quad + \frac{1}{3}\int_Q M\Big(\frac{\varphi_k - u}{\lambda}\Big)dx\,dt \\ &\le\; \frac{1}{3}\int_Q M\Big(\frac{(\varphi_k - u)_\mu}{\lambda}\Big)dx\,dt + \frac{1}{3}\int_Q M\Big(\frac{(\varphi_k)_\mu - \varphi_k}{\lambda}\Big)dx\,dt \\ &\quad + \frac{1}{3}\int_Q M\Big(\frac{\varphi_k - u}{\lambda}\Big)dx\,dt \,.\end{aligned}$$

This implies that

$$\int_Q M\Big(\frac{u_\mu - u}{3\lambda}\Big)dx\,dt \;\le\; \frac{2}{3}\int_Q M\Big(\frac{\varphi_k - u}{\lambda}\Big)dx\,dt + \frac{1}{3}\,M\Big(\frac{1}{\mu\lambda}\Big\|\frac{\partial\varphi_k}{\partial t}\Big\|_\infty\Big)\,\mathrm{meas}(Q) \,.$$

Let $\varepsilon > 0$. There exists k such that

$$\int_Q M\Big(\frac{\varphi_k - u}{\lambda}\Big)dx\,dt \;\le\; \varepsilon$$

and there exists μ_0 such that

$$M\left(\frac{1}{\mu\lambda}\left\|\frac{\partial\varphi_k}{\partial t}\right\|_\infty\right)\mathrm{meas}(Q) \le \varepsilon \quad \text{for all } \ \mu \ge \mu_0 \ .$$

Hence

$$\int_Q M\left(\frac{u_\mu - u}{3\lambda}\right) dx\, dt \ \le \ \varepsilon \quad \text{for all } \ \mu \ge \mu_0 \ .$$

2) Since $\forall\,\alpha$, $|\alpha| \le 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result. ∎

Remark 4. If $u \in E_M(Q)$, we can choose λ arbitrary small since $\mathcal{D}(Q)$ is (norm) dense in $E_M(Q)$. Thus, for all $\lambda > 0$

$$\int_Q M\left(\frac{u_\mu - u}{\lambda}\right) dx\, dt \ \to \ 0 \quad \text{as } \ \mu \to +\infty$$

and $u_\mu \to u$ strongly in $E_M(Q)$. Idem for $W^{1,x}E_M(Q)$. □

Proposition 3. *If $u_n \to u$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence) then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence).*

Proof: For all $\lambda > 0$ (resp. for some $\lambda > 0$),

$$\int_Q M\left(\frac{D_x^\alpha((u_n)_\mu) - D_x^\alpha(u_\mu)}{\lambda}\right) dx\, dt \ \le \ \int_Q M\left(\frac{D_x^\alpha(u_n) - D_x^\alpha(u)}{\lambda}\right) dx\, dt \ \to \ 0$$

$$\text{as } \ n \to \infty \ ,$$

then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence). ∎

5 – Existence theorem

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \ge 2)$ with the segment property, $T > 0$ and set $Q = \Omega\times(0,T)$. Let M be an N-function.

Consider a second order partial differential operator $A\colon D(A) \subset W^{1,x}L_M(Q) \to W^{-1,x}L_{\overline{M}}(Q)$ in divergence form

$$A(u) \ = \ -\mathrm{div}\ a(x,t,u,\nabla u)$$

where $a : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $(x,t) \in \Omega\times[0,T]$ and all $s \in \mathbb{R}$, $\xi \neq \xi^* \in \mathbb{R}^N$:

$$|a(x,t,s,\xi)| \le \beta(|s|)\Big(c_1(x,t) + \overline{M}^{-1}M(\gamma|\xi|)\Big) \tag{11}$$

$$\Big[a(x,t,s,\xi) - a(x,t,s,\xi^*)\Big]\,[\xi - \xi^*] > 0 \tag{12}$$

$$a(x,t,s,\xi)\xi \ge \alpha\, M(|\xi|) \tag{13}$$

where $c_1(x,t) \in E_{\overline{M}}(Q)$, $c_1 \ge 0$; $\beta : [0,+\infty) \to [0,+\infty)$ a continuous and non-decreasing function; $\alpha, \gamma > 0$.

Note that, (13) written for $\xi = \varepsilon\zeta$, $\varepsilon > 0$, and the fact that a is a Carathéodory function, imply that

$$a(x,t,s,0) = 0 \quad \text{for almost every } (x,t) \in Q \text{ and every } s \in \mathbb{R}\,.$$

Let $g : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying for a.e. $(x,t) \in \Omega\times(0,T)$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \le b(|s|)\Big(c_2(x,t) + M(|\xi|)\Big) \tag{14}$$

$$g(x,t,s,\xi)s \ge 0 \tag{15}$$

where $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function. Furthermore let

$$f \in L^1(Q)\,. \tag{16}$$

Throughout this paper $\langle\ ,\ \rangle$ means for either the pairing between $W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ or between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$ and $Q_\tau = \Omega\times(0,\tau)$ for $\tau \in [0,T]$.

Consider, then, the following parabolic initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{17}$$

where u_0 is a given function in $L^1(\Omega)$.

Let us now precise in which sense the problem (17) will be solved. Thus, we state, as in [19], the following

Definition 2. A measurable function $u : \Omega\times(0,T) \to \mathbb{R}$ is called entropy solution of (17) if u belongs to $L^\infty(0,T;L^1(\Omega)), T_k(u)$ belongs to $D(A)\cap W_0^{1,x}L_M(Q)$ for every $k>0$, $S_k(u(.,t))$ belongs to $L^1(\Omega)$ for every $t\in[0,T]$ and every $k>0$, $g(x,t,u,\nabla u)$ is in $L^1(Q)$ and u satisfies:

$$\begin{aligned}
&\int_\Omega S_k(u-v)(\tau)\,dx + \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\,\nabla T_k(u-v)\;dx\,dt\;+\\
&\qquad\qquad + \int_{Q_\tau} g(x,t,u,\nabla u)\,T_k(u-v)\;dx\,dt\;\leq \qquad (18)\\
&\qquad \leq \int_{Q_\tau} f\,T_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0-v(0))\,dx
\end{aligned}$$

for every $\tau\in[0,T]$, $k>0$, and for all v in $W_0^{1,x}L_M(Q)\cap L^\infty(Q)$ such that $\partial v/\partial t$ belongs to $W^{-1,x}L_{\overline{M}}(Q)+L^1(Q)$ (recall that T_k is the usual truncation at height k defined on $\mathbb{R}$ by $T_k(s)=\min(k,\max(s,-k))$ and that $S_k(s)=\int_0^s T_k(t)dt$ is its primitive vanishing on 0).

Note that, all the terms in (18) make sense since $T_k(u-v)$ belongs to $W_0^{1,x}L_M(Q)$ $\cap L^\infty(Q)$. Moreover Lemma 5 implies that $v\in C([0,T],L^1(\Omega))$ and then the first and last terms are well defined. □

We shall prove the following existence theorem:

Theorem 4. *Assume that (11)–(16) hold true. Then the problem (17) admits at least one entropy solution $u\in C([0,T],L^1(\Omega))$ satisfying $u(x,0)=u_0(x)$ for a.e. $x\in\Omega$.*

Proof of Theorem 4: We divide the proof in four steps.

Step 1: A priori estimates.

Let (f_n) be a sequence of smooth functions such that $f_n\to f$ in $L^1(Q)$ and let (u_{0n}) be a sequence in $L^2(\Omega)$ such that $u_{0n}\to u_0$ in $L^1(\Omega)$.

Consider the sequence of approximate problems:

$$(19)\qquad \begin{cases} u_n \in D(A)\cap W_0^{1,x}L_M(Q)\cap C([0,T],L^2(\Omega)), \quad u_n(x,0)=u_{0n}\\ \partial u_n/\partial t - \operatorname{div}\Big(a(x,t,T_n(u_n),\nabla u_n)\Big) + g_n(x,t,u_n,\nabla u_n) = f_n \end{cases}$$

where $g_n(x,t,s,\xi)=T_n(g(x,t,s,\xi))$.

Note that $g_n(x,t,s,\xi)s \geq 0$, $|g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$. Since g_n is bounded for any fixed $n>0$, then, by Theorem 1 of [12], there exists at least one solution u_n of (19).

Note also that $\langle u_n', v\rangle$ is defined in the sense of distributions. Since $f_n - A(u_n) - g_n$ is in $W^{-1,x}L_{\overline{M}}(Q)$ we can extend $\langle u_n', v\rangle$ to all $v \in W_0^{1,x}L_M(Q)$.

Using in (19) the test function $T_k(u_n)\chi_{(0,\tau)}$, we get, for every $\tau \in (0,T)$

$$\int_\Omega S_k(u_n(\tau))\,dx + \int_{Q_\tau} a\Big(x,t,T_k(u_n),\nabla T_k(u_n)\Big)\,\nabla T_k(u_n)\,dx\,dt \;\leq\; c_1 k \tag{20}$$

where here and below c_i denote positive constants not depending on n and k.

On the other hand, thanks to Lemma 5.7 of [14], there exists two positive constants δ, λ such that

$$\int_Q M(v)\,dx\,dt \;\leq\; \delta \int_Q M(\lambda|\nabla v|)\,dx\,dt \qquad \text{for all} \;\; v \in W_0^{1,x}L_M(Q)\;. \tag{21}$$

Taking $v = T_k(u_n)/\lambda$ in (21) and using (20) with (13), give

$$\alpha \int_Q M\Big(\frac{T_k(u_n)}{\lambda}\Big)dx\,dt \;\leq\; c_2\,k$$

which implies that

$$\operatorname{meas}\Big\{(x,t) \in Q\colon\; |u_n| > k\Big\} \;\leq\; \frac{c_3\,k}{M(k/\lambda)}$$

so that

$$\lim_{k\to\infty}\Big(\operatorname{meas}\Big\{(x,t) \in Q\colon\; |u_n| > k\Big\}\Big) \;=\; 0 \qquad \text{uniformly with respect to} \;\; n\;. \tag{22}$$

Consider now for $\theta, \varepsilon > 0$ a function $\rho_\theta^\varepsilon \in C^1(\mathbb{R})$ such that

$$\begin{aligned} &\rho_\theta^\varepsilon(s) = 0 && \text{if } |s| \leq \theta\,,\\ &\rho_\theta^\varepsilon(s) = \operatorname{sign}(s) && \text{if } |s| \geq \theta + \varepsilon\,,\\ &(\rho_\theta^\varepsilon)'(s) \geq 0 && \forall\, s \in \mathbb{R} \end{aligned}$$

then, by using $\rho_\theta^\varepsilon(u_n)$ as a test function in (19) and following [19], we can see that

$$\int_{\{|u_n|>\theta\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \;\leq \int_{\{|u_n|>\theta\}} |f_n|\,dx\,dt + \int_{\{|u_{0n}|>\theta\}} |u_{0n}|\,dx\,dt \tag{23}$$

and so by letting $\theta \to 0$ and using Fatou's lemma, we deduce that $g_n(x,t,u_n,\nabla u_n)$ is a bounded sequence in $L^1(Q)$.

Moreover, we have from (20) that $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$ for every $k > 0$. Take a $C^2(\mathbb{R})$, and nondecreasing function ζ_k such that $\zeta_k(s) = s$ for $|s| \le k/2$ and $\zeta_k(s) = k\,\mathrm{sign}(s)$ for $|s| \ge k$. Multiplying the approximating equation by $\zeta_k'(u_n)$, we get

$$\begin{aligned}\frac{\partial}{\partial t}(\zeta_k(u_n)) - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\,\zeta_k'(u_n)\Big) &+ a(x,t,u_n,\nabla u_n)\,\zeta_k''(u_n)\\ &+ g_n(x,t,u_n,\nabla u_n)\,\zeta_k'(u_n) \;=\; f_n\,\zeta_k'(u_n)\ ,\end{aligned}$$

in the sense of distributions. This implies, thanks to (20) and the fact that ζ_k' has compact support, that $\zeta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$ while its time derivative $\frac{\partial}{\partial t}(\zeta_k(u_n))$ is bounded in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$, hence Corollary 1 allows us to conclude that $\zeta_k(u_n)$ is compact in $L^1(Q)$. Therefore, following [19], we can see that there exists a measurable function u in $L^\infty(0,T;L^1(\Omega))$ such that for every $k > 0$ and a subsequence, not relabeled,

$$(24)\qquad T_k(u_n) \to T_k(u) \qquad \begin{aligned}&\text{weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}})\ ,\\ &\text{strongly in } L^1(Q) \text{ and a.e. in } Q\ .\end{aligned}$$

To prove that $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q))^N$. Let $\varphi \in (E_M(Q))^N$ with $\|\varphi\|_{M,Q} = 1$. In view of (12), we have

$$\int_Q \Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\varphi)\Big]\,[\nabla T_k(u_n) - \varphi]\,dx\,dt \;\ge\; 0$$

which gives

$$\begin{aligned}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\,\varphi\;dx\,dt \;\le\; &\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,dx\,dt\\ &- \int_Q a(x,t,T_k(u_n),\varphi)\,[\nabla T_k(u_n) - \varphi]\,dx\,dt\ .\end{aligned}$$

On the one hand, by (20), we have $\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt \le C$, where here and below C denotes a positive constant not depending on n. On the other hand, using (11), we see that

$$\overline{M}\left(\frac{|\,a(x,t,T_k(u_n),\varphi)\,|}{2\,\beta(k)}\right) \;\le\; \overline{M}(c_1(x,t)) + M(\gamma|\varphi|)$$

and hence $a(x,t,T_k(u_n),\varphi)$ is bounded in $(L_{\overline{M}}(Q))^N$, implying that, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$

$$\left|\int_Q a(x,t,T_k(u_n),\varphi)\,[\nabla T_k(u_n)-\varphi]\,dx\,dt\right| \;\le\; C$$

and so, by using the dual norm, $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q))^N$.

Thus, up to subsequences

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{in } (L_{\overline{M}}(Q))^N \;\text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M)\;, \tag{25}$$

for some $h_k \in (L_{\overline{M}}(Q))^N$.

Step 2: Almost everywhere convergence of the gradients.

Fix $k>0$ and let $\varphi(s)=s\,e^{\delta\,s^2}$, $\delta>0$. It is well known that when $\delta \ge \left(\frac{b(k)}{2\alpha}\right)^2$ one has

$$\varphi'(s)-\frac{b(k)}{\alpha}\,|\varphi(s)| \;\ge\; \frac{1}{2} \qquad \text{for all }\; s\in\mathbb{R}\;. \tag{26}$$

Let $v_j\in\mathcal{D}(Q)$ be a sequence such that

$$v_j \to T_k(u) \quad \text{in } W_0^{1,x}L_M(Q) \qquad \text{for the modular convergence} \tag{27}$$

and let $\psi_i\in\mathcal{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Set $\omega^i_{\mu,j}=T_k(v_j)_\mu+e^{-\mu t}T_k(\psi_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, see (10). Note that $\omega^i_{\mu,j}$ is a smooth function having the following properties:

$$\begin{cases} \dfrac{\partial}{\partial t}(\omega^i_{\mu,j})=\mu(T_k(v_j)-\omega^i_{\mu,j})\,, \quad \omega^i_{\mu,j}(0)=T_k(\psi_i)\,, \quad |\omega^i_{\mu,j}|\le k\;, \\[2mm] \omega^i_{\mu,j}\to T_k(u)_\mu+e^{-\mu t}\,T_k(\psi_i) \quad \text{in } W_0^{1,x}L_M(Q) \\ \qquad\qquad\qquad \text{for the modular convergence as } j\to\infty\;, \\[2mm] T_k(u)_\mu+e^{-\mu t}T_k(\psi_i)\to T_k(u) \quad \text{in } W_0^{1,x}L_M(Q) \\ \qquad\qquad\qquad \text{for the modular convergence as } \mu\to\infty\;. \end{cases}$$

Let now the function ρ_m defined on $\mathbb{R}$ by

$$\rho_m(s) = \begin{cases} 1 & \text{if } |s| \le m\,, \\ m+1-|s| & \text{if } m \le |s| \le m+1\,, \\ 0 & \text{if } |s| \ge m+1\,, \end{cases}$$

where $m > k$. Let $\theta^{\mu,i}_{n,j} = T_k(u_n) - \omega^i_{\mu,j}$ and $z^{\mu,i}_{n,j,m} = \varphi(\theta^{\mu,i}_{n,j})\rho_m(u_n)$.

Using in (19) the test function $z^{\mu,i}_{n,j,m}$, we get (u'_n denotes the distributional time derivative of u_n),

$$\begin{aligned} \langle u'_n, z^{\mu,i}_{n,j,m}\rangle \;&+\; \int_Q a(x,t,u_n,\nabla u_n)\,[\nabla T_k(u_n) - \nabla\omega^i_{\mu,j}]\,\varphi'(\theta^{\mu,i}_{n,j})\,\rho_m(u_n)\;dx\,dt \\ &+\; \int_Q a(x,t,u_n,\nabla u_n)\,\varphi(\theta^{\mu,i}_{n,j})\,\rho'_m(u_n)\;dx\,dt \\ &+\; \int_Q g_n(x,t,u_n,\nabla u_n)\,z^{\mu,i}_{n,j,m}\;dx\,dt \;=\; \int_Q f_n\,z^{\mu,i}_{n,j,m}\;dx\,dt \end{aligned}$$

which implies, since $g_n(x,t,u_n,\nabla u_n)\,\varphi(T_k(u_n) - \omega^i_{\mu,j})\,\rho_m(u_n) \ge 0$ on $\{|u_n| > k\}$:

$$\begin{aligned} \langle u'_n, z^{\mu,i}_{n,j,m}\rangle \;&+\; \int_Q a(x,t,u_n,\nabla u_n)\,[\nabla T_k(u_n) - \nabla\omega^i_{\mu,j}]\,\varphi'(\theta^{\mu,i}_{n,j})\,\rho_m(u_n)\;dx\,dt \\ &+\; \int_Q a(x,t,u_n,\nabla u_n)\,\varphi(\theta^{\mu,i}_{n,j})\,\rho'_m(u_n)\;dx\,dt \\ &+\; \int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\,\varphi(T_k(u_n) - \omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt \;\le \\ &\le\; \int_Q f_n\,z^{\mu,i}_{n,j,m}\;dx\,dt\,. \end{aligned} \tag{28}$$

In the sequel and throughout the paper, we will omit for simplicity the dependence on x and t in the function $a(x,t,s,\xi)$ and denote $\varepsilon(n,j,\mu,i,s,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{s\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} \varepsilon(n,j,\mu,i,s,m) \;=\; 0$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then j, μ, i, s and finally m. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,j)$, ... to mean that the limits are made only on the specified parameters.

We will deal with each term of (28). First of all, observe that

$$\int_Q f_n\,\varphi(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\,dx\,dt \;=\; \varepsilon(n,j,\mu) \tag{29}$$

since $\varphi(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n) \rightharpoonup \varphi(T_k(u)-\omega^i_{\mu,j})\,\rho_m(u)$ weakly * in $L^\infty(Q)$ as $n\to\infty$, $\varphi(T_k(u)-\omega^i_{\mu,j})\,\rho_m(u) \rightharpoonup \varphi(T_k(u)-T_k(u)_\mu+e^{-\mu t}T_k(\psi_i))\,\rho_m(u)$ weakly * in $L^\infty(Q)$ as $j\to\infty$ and finally $\varphi(T_k(u)-T_k(u)_\mu+e^{-\mu t}T_k(\psi_i))\rho_m(u)\rightharpoonup 0$ weakly * in $L^\infty(Q)$ as $\mu\to\infty$.

On the one hand, from (19) one deduces that $u_n\in W^{1,x}_0L_M(Q)$ and $\partial u_n/\partial t\in W^{-1,x}(Q)+L^1(Q)$ and then by theorem 3 there exists a smooth function $u_{n\sigma}$ such that, as $\sigma\to 0^+$, $u_{n\sigma}\to u_n$ in $W^{1,x}_0L_M(Q)$ and $\partial u_{n\sigma}/\partial t\to\partial u_n/\partial t$ in $W^{-1,x}(Q)+L^1(Q)$ for the modular convergence, so that, $\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\rho_m(u_{n\sigma})\to z^{\mu,i}_{n,j,m}$ in $W^{1,x}_0L_M(Q)$ for the modular convergence and weakly * in $L^\infty(Q)$. This implies

$$\begin{aligned}\langle u_n', z^{\mu,i}_{n,j,m}\rangle &= \lim_{\sigma\to0^+}\int_Q u'_{n\sigma}\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\,\rho_m(u_{n\sigma})\;dx\,dt\\ &= \lim_{\sigma\to0^+}\int_Q[(R_m(u_{n\sigma}))']\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\;dx\,dt\end{aligned}$$

where $R_m(s)=\int_0^s\rho_m(\eta)\,d\eta$. Hence

$$\begin{aligned}\langle u_n', z^{\mu,i}_{n,j,m}\rangle &= \lim_{\sigma\to0^+}\left[\int_Q(R_m(u_{n\sigma})-T_k(u_{n\sigma}))'\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\;dx\,dt\right.\\ &\qquad\left.+\int_Q(T_k(u_{n\sigma}))'\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\;dx\,dt\right]\\ &= \lim_{\sigma\to0^+}\left\{\left[\int_\Omega(R_m(u_{n\sigma})-T_k(u_{n\sigma}))\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\;dx\right]_0^T\right.\\ &\qquad-\int_Q(R_m(u_{n\sigma})-T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma})-\omega^i_{\mu,j})\,(T_k(u_{n\sigma})-\omega^i_{\mu,j})'\;dx\,dt\\ &\qquad\left.+\int_Q(T_k(u_{n\sigma}))'\,\varphi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\;dx\,dt\right\}\\ &= \lim_{\sigma\to0^+}\Big\{I_1(\sigma)+I_2(\sigma)+I_3(\sigma)\Big\}\,.\end{aligned}$$

Observe that for $|s|\le k$ we have $R_m(s)=T_k(s)=s$ and for $|s|>k$ we have $|R_m(s)|\ge|T_k(s)|$ and, since both $R_m(s)$ and $T_k(s)$ have the same sign of s, we deduce that $\operatorname{sign}(s)\,(R_m(s)-T_k(s)) \geq 0$. Consequently

$$I_1(\sigma) \;=\; \left[\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\; dx\right]_0^T$$
$$\geq\; -\int_{\{|u_{n\sigma}(0)|>k\}} (R_m(u_{n\sigma}(0)) - T_k(u_{n\sigma}(0)))\,\varphi(T_k(u_{n\sigma})(0) - \omega^i_{\mu,j}(0))\; dx$$

and so, by letting $\sigma\to 0^+$ in the last integral, we get

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \;\geq\; -\int_{\{|u_{0n}|\geq k\}} (R_m(u_{0n}) - T_k(u_{0n}))\,\varphi(T_k(u_{0n}) - T_k(\psi_i))\; dx\;.$$

Letting $n\to\infty$, the right hand side of the above inequality clearly tends to

$$-\int_{\{|u_0|\geq k\}} (R_m(u_0) - T_k(u_0))\,\varphi(T_k(u_0) - T_k(\psi_i))\; dx$$

which obviously goes to 0 as $i\to\infty$. We deduce then that

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \;\geq\; \varepsilon(n,i)\;.$$

About $I_2(\sigma)$, we have, since $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,(T_k(u_{n\sigma}))' = 0$

$$I_2(\sigma) \;=\; \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\,(\omega^i_{\mu,j})'\; dx\,dt$$
$$=\; \mu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\,(T_k(v_j) - \omega^i_{\mu,j})\; dx\,dt$$
$$\geq\; \mu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\,(T_k(v_j) - T_k(u_{n\sigma}))\; dx\,dt$$

by using the fact that $\varphi'\geq 0$ and that $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\chi_{\{|u_{n\sigma}|>k\}} \geq 0$ and so, by letting $\sigma\to 0^+$ in the last integral

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \;\geq$$
$$\geq\; \mu\int_{\{|u_n|\geq k\}} (R_m(u_n) - T_k(u_n))\,\varphi'(T_k(u_n) - \omega^i_{\mu,j})\,(T_k(v_j) - T_k(u_n))\; dx\,dt$$

and since, as it can be easily seen, the last integral is of the form $\varepsilon(n,j)$ we deduce that

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \;\geq\; \varepsilon(n,j) \;.$$

For what concerns $I_3(\sigma)$, one has

$$\begin{aligned} I_3(\sigma) \;=\; & \int_Q (T_k(u_{n\sigma}) - \omega^i_{\mu,j})' \,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\; dx\,dt \\ & + \int_Q (\omega^i_{\mu,j})' \,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\; dx\,dt \end{aligned}$$

and then, by setting $\Phi(s) = \int_0^s \varphi(\eta)\,d\eta$ and integrating by parts

$$\begin{aligned} I_3(\sigma) \;=\; & \Big[\int_\Omega \Phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\, dx \Big]_0^T \\ & + \mu \int_Q (T_k(v_j) - \omega^i_{\mu,j})\,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\; dx\,dt \end{aligned}$$

which implies, since $\Phi \geq 0$ and $(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j}) \geq 0$

$$\begin{aligned} I_3(\sigma) \;\geq\; & -\int_\Omega \Phi\Big(T_k(u_{n\sigma}(0)) - T_k(\psi_i)\Big)\, dx \\ & + \mu \int_Q (T_k(v_j) - T_k(u_{n\sigma}))\,\varphi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\; dx\,dt \;, \end{aligned}$$

so that

$$\begin{aligned} \limsup_{\sigma\to 0^+} I_3(\sigma) \;\geq\; & -\int_\Omega \Phi\Big(T_k(u_{0n}) - T_k(\psi_i)\Big)\, dx \\ & + \mu \int_Q (T_k(v_j) - T_k(u_n))\,\varphi(T_k(u_n) - \omega^i_{\mu,j})\; dx\,dt \end{aligned}$$

and by letting $n \to \infty$ in the last side, we obtain

$$\begin{aligned} \limsup_{\sigma\to 0^+} I_3(\sigma) \;\geq\; & -\int_\Omega \Phi(T_k(u_0) - T_k(\psi_i)\Big)\, dx \\ & + \mu \int_Q (T_k(v_j) - T_k(u))\,\varphi(T_k(u) - \omega^i_{\mu,j})\; dx\,dt \;+\; \varepsilon(n) \;. \end{aligned}$$

Since the first integral of the last side is of the form $\varepsilon(i)$ while the second one is of the form $\varepsilon(j)$ we deduce that

$$\limsup_{\sigma\to 0^+} I_3(\sigma) \;\geq\; \varepsilon(n,j,i) \;.$$

Combining these estimates, we conclude that

$$\Big\langle u_n',\, \varphi(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\Big\rangle \,\geq\, \varepsilon(n,j,i)\ . \tag{30}$$

On the other hand, the second term of the left hand side of (28) reads as

$$\begin{aligned}
&\int_Q a(u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt \;=\\
&\quad= \int_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt\\
&\quad= \int_Q a(T_k(u_n),\nabla T_k(u_n))\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\;dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt
\end{aligned}$$

where we have used the fact that, since $m>k$, $\rho_m(u_n)=1$ on $\{|u_n|\le k\}$.

Setting for $s>0$, $Q^s=\{(x,t)\in Q\colon |\nabla T_k(u)|\le s\}$ and $Q^s_j=\{(x,t)\in Q\colon |\nabla T_k(v_j)|\le s\}$ and denoting by χ^s and χ^s_j the characteristic functions of Q^s and Q^s_j respectively, we deduce that

$$\begin{aligned}
&\int_Q a(u_n,\nabla u_n)\,[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt \;=\\
&\quad= \int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]\,[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\\
&\qquad\quad\times\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\;dx\,dt\\
&\qquad+\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\,[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\;dx\,dt\\
&\qquad+\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\,\chi^s_j\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\;dx\,dt\\
&\qquad-\int_Q a(u_n,\nabla u_n)\,\nabla\omega^i_{\mu,j}\,\varphi'(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt\\
&\quad:=\;J_1+J_2+J_3+J_4\ .
\end{aligned}$$

We shall go to the limit as n,j,μ and $s\to\infty$ in the last three integrals of the last side. Starting with J_2, we have by letting $n\to\infty$

$$\begin{aligned}
J_2 \;=\; &\int_Q a(T_k(u),\nabla T_k(v_j)\chi^s_j)\,[\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j]\,\varphi'(T_k(u)-\omega^i_{\mu,j})\,\rho_m(u)\;dx\,dt\\
&+\varepsilon(n)
\end{aligned}$$

since $a(T_k(u_n),\nabla T_k(v_j)\chi_j^s) \to a(T_k(u),\nabla T_k(v_j)\chi_j^s)$ strongly in $(E_{\overline{M}}(Q))^N$ by using (11) and Lebesgue theorem while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(Q))^N$ by (25).

Letting $j\to\infty$ in the first term of the right hand side of the above equality, one has, since $a(T_k(u),\nabla T_k(v_j)\chi_j^s) \to a(T_k(u),\nabla T_k(u)\chi^s)$ strongly in $(E_{\overline{M}}(Q))^N$ by using (11), (27) and Lebesgue theorem while $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(L_M(Q))^N$,

$$J_2 = \varepsilon(n,j)\ .$$

About $J_3(n,j,\mu,s)$, we have by letting $n\to\infty$ and using (25)

$$J_3 \ = \int_Q h_k\,\nabla T_k(v_j)\,\chi_j^s\,\varphi'(T_k(u)-\omega_{\mu,j}^i)\,\rho_m(u)\ dx\,dt\ +\ \varepsilon(n)$$

which gives by letting $j\to\infty$, thanks to (27) (recall that $\rho_m(u)=1$ on $\{|u|\le k\}$)

$$J_3 \ = \int_Q h_k\,\nabla T_k(u)\,\chi^s\,\varphi'\Big(T_k(u)-T_k(u)_\mu+e^{-\mu t}\,T_k(\psi_i)\Big)\ dx\,dt\ +\ \varepsilon(n,j)$$

implying that, by letting $\mu\to\infty$, $J_3=\int_Q h_k\nabla T_k(u)\chi^s\,dx\,dt+\varepsilon(n,j,\mu)$, and thus

$$J_3 \ = \int_Q h_k\nabla T_k(u)\,dx\,dt\ +\ \varepsilon(n,j,\mu,s)\ .$$

For what concerns J_4 we can write, since $\rho_m(u_n)=0$ on $\{|u_n|>m{+}1\}$

$$\begin{aligned}
J_4 &= -\int_Q a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\,\nabla\omega_{\mu,j}^i\,\varphi'(T_k(u_n)-\omega_{\mu,j}^i)\,\rho_m(u_n)\ dx\,dt\\
&= -\int_{\{|u_n|\le k\}} a(T_k(u_n),\nabla T_k(u_n))\,\nabla\omega_{\mu,j}^i\,\varphi'(T_k(u_n)-\omega_{\mu,j}^i)\,\rho_m(u_n)\ dx\,dt\\
&\quad -\int_{\{k<|u_n|\le m+1\}} a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\,\nabla\omega_{\mu,j}^i\,\varphi'(T_k(u_n){-}\omega_{\mu,j}^i)\,\rho_m(u_n)\ dx\,dt
\end{aligned}$$

and, as above, by letting $n\to\infty$

$$\begin{aligned}
J_4 \ =& \ -\int_{\{|u|\le k\}} h_k\nabla\omega_{\mu,j}^i\,\varphi'(T_k(u)-\omega_{\mu,j}^i)\ dx\,dt\\
&\qquad -\int_{\{k\le|u|\le m+1\}} h_{m+1}\nabla\omega_{\mu,j}^i\,\varphi'(T_k(u)-\omega_{\mu,j}^i)\,\rho_m(u)\ dx\,dt\ +\ \varepsilon(n)
\end{aligned}$$

which implies that, by letting $j \to \infty$

$$\begin{aligned} J_4 \;=\; &-\int_{\{|u|\le k\}} h_k\Big[\nabla T_k(u)_\mu - e^{-\mu t}\nabla T_k(\psi_i)\Big]\,\varphi'\Big(T_k(u) - T_k(u)_\mu - e^{-\mu t}T_k(\psi_i)\Big)\,dx\,dt \\ &+\varepsilon(n,j) \\ &-\int_{\{k\le|u|\le m+1\}} h_{m+1}\Big[\nabla T_k(u)_\mu - e^{-\mu t}\nabla T_k(\psi_i)\Big] \\ &\qquad\times\,\varphi'\Big(T_k(u) - T_k(u)_\mu - e^{-\mu t}T_k(\psi_i)\Big)\,\rho_m(u)\;dx\,dt \end{aligned}$$

so that, by letting $\mu \to \infty$

$$J_4 \;=\; -\int_Q h_k \nabla T_k(u)\,dx\,dt \;+\; \varepsilon(n,j,\mu)\;.$$

We conclude then that

$$\begin{aligned} &\int_Q a(u_n,\nabla u_n)\,[\nabla T_k(u_n) - \nabla\omega^i_{\mu,j}]\,\varphi'(T_k(u_n) - \omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt \;= \\ &\quad= \int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]\,[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j] \\ &\qquad\times\,\varphi'(T_k(u_n) - \omega^i_{\mu,j})\;dx\,dt \;+\; \varepsilon(n,j,\mu,s)\;. \end{aligned} \tag{31}$$

To deal with the third term of the left hand side of (28), observe that

$$\begin{aligned} &\left|\int_Q a(x,t,u_n,\nabla u_n)\,\varphi(\theta^{\mu,i}_{n,j})\,\rho'_m(u_n)\;dx\,dt\right| \;\le \\ &\qquad\le\; \varphi(2k)\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\,\nabla u_n\;dx\,dt\;. \end{aligned}$$

On the other hand, using $\theta_m(u_n)$ as a test function in (19) where $\theta_m(s) = T_1(s - T_m(s))$, we get

$$\begin{aligned} &\langle u'_n, \theta_m(u_n)\rangle \;+\int_Q a(u_n,\nabla u_n)\,\nabla u_n\theta'_m(u_n)\;dx\,dt \;+\int_Q g(u_n,\nabla u_n)\,\theta_m(u_n)\;dx\,dt \;= \\ &\qquad= \int_Q f_n\,\theta_m(u_n)\,dx\,dt \end{aligned}$$

which gives, by setting $\Theta_m(s) = \int_0^s \theta_m(\eta)\,d\eta$ (observe that $\theta_m(s)s \ge 0$)

$$\left[\int_\Omega \Theta_m(u_n(t))\,dx\right]_0^T + \int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\,\nabla u_n\,dx\,dt \;\le\; \int_{\{|u_n|\ge m\}} |f_n|\,dx\,dt$$

and since $\Theta_m \ge 0$, we deduce that

$$\int_{\{m\le|u_n|\le m+1\}} a(u_n,\nabla u_n)\,\nabla u_n\;dx\,dt \;\le\; \int_\Omega \Theta_m(u_{0n})\,dx \;+\int_{\{|u_n|\ge m\}} |f_n|\,dx\,dt\;.$$

Since, as it can be easily seen, each integral of the right hand side is of the form $\varepsilon(n,m)$ we obtain

$$\left|\int_Q a(x,t,u_n,\nabla u_n)\,\varphi(\theta^{\mu,i}_{n,j})\,\rho'_m(u_n)\;dx\,dt\right| \le \varepsilon(n,m)\;. \tag{32}$$

We now turn to the fourth term of the left hand side of (28). We can write

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\,\varphi(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt\right| \le\\
&\quad\le b(k)\int_Q c_2(x,t)\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt\\
&\qquad+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt\;.
\end{aligned}\tag{33}$$

Since $c_2(x,t)$ belongs to $L^1(Q)$ it is easy to see that

$$b(k)\int_Q c_2(x,t)\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt \;=\; \varepsilon(n,j,\mu)\;.$$

On the other hand, the second term of the right hand side of (33) reads as

$$\begin{aligned}
&\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt \;=\\
&=\frac{b(k)}{\alpha}\int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]\,[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\\
&\qquad\times\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt\\
&\quad+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\,[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\;|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt\\
&\quad+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(v_j)\,\chi^s_j\;|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt
\end{aligned}$$

and, as above, by letting first n then j,μ and finally s go to infinity, we can easily see that each one of last two integrals is of the form $\varepsilon(n,j,\mu)$. This implies that

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\,\varphi(T_k(u_n)-\omega^i_{\mu,j})\,\rho_m(u_n)\;dx\,dt\right| \le\\
&\le\frac{b(k)}{\alpha}\int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\,|\varphi(T_k(u_n)-\omega^i_{\mu,j})|\;dx\,dt\;+\;\varepsilon(n,j,\mu)\;.
\end{aligned}\tag{34}$$

Combining (28), (29), (30), (31), (32) and (34) we get

$$\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big] \times [\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s] \left[\varphi'(T_k(u_n)-\omega_{\mu,j}^i) - \frac{b(k)}{\alpha}\,|\varphi(T_k(u_n)-\omega_{\mu,j}^i)|\right] dx\,dt \;\le\; \le\; \varepsilon(n,j,\mu,i,s,m)$$

and so, thanks to (26)

$$\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\; dx\,dt \;\le\; \le\; \varepsilon(n,j,\mu,i,s,m)\;. \tag{35}$$

On the other hand, we have

$$\begin{aligned}
&\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)\Big] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\; dx\,dt \;- \\
&- \int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big] [\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\; dx\,dt \;= \\
&\qquad = \int_Q a(T_k(u_n),\nabla T_k(u_n))\,[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s]\; dx\,dt \\
&\qquad\quad - \int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)\,[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\; dx\,dt \\
&\qquad\quad + \int_Q a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\,[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\; dx\,dt
\end{aligned}$$

and, as it can be easily seen, each integral of the right hand side is of the form $\varepsilon(n,j,s)$ implying that

$$\begin{aligned}
&\int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)\Big] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\; dx\,dt \;= \\
&\qquad = \int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\; dx\,dt \;+\; \varepsilon(n,j,s)\;.
\end{aligned} \tag{36}$$

For $r \le s$, we have

$$\begin{aligned}
0 \;&\le\; \int_{Q^r} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big] [\nabla T_k(u_n)-\nabla T_k(u)]\; dx\,dt \\
&\le\; \int_{Q^s} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big] [\nabla T_k(u_n)-\nabla T_k(u)]\; dx\,dt \;=
\end{aligned}$$

$$
\begin{aligned}
&= \int_{Q^s} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)\Big]\,[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\;dx\,dt \\
&\le \int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s)\Big]\,[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\;dx\,dt \\
&= \int_Q \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big]\,[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\;dx\,dt \\
&\quad + \varepsilon(n,j,s) \\
&\le \varepsilon(n,j,\mu,i,s,m)
\end{aligned}
$$

hence by passing to the limit sup over n, we get

$$
\begin{aligned}
0 &\le \limsup_{n\to\infty} \int_{Q^r} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\,[\nabla T_k(u_n) - \nabla T_k(u)]\;dx\,dt \\
&\le \lim_{n\to\infty} \varepsilon(n,j,\mu,i,s,m)
\end{aligned}
$$

in which we can let successively j,μ,i,s and m go to infinity, to obtain

$$
\int_{Q^r} \Big[a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\Big]\,[\nabla T_k(u_n) - \nabla T_k(u)]\;dx\,dt \;\to\; 0 \qquad \text{as } n\to\infty
$$

and thus, as in the elliptic case (see [3]), there exists a subsequence also denoted by u_n such that

$$
\nabla u_n \to \nabla u \quad \text{a.e. in } \; Q\,. \tag{37}
$$

We deduce then that,

$$
\begin{aligned}
&a(x,t,T_k(u_n),\nabla T_k(u_n)) \;\rightharpoonup\; a(x,t,T_k(u),\nabla T_k(u)) \\
&\text{weakly in } \; (L_{\overline{M}}(Q))^N \;\text{ for }\; \sigma(\Pi L_{\overline{M}},\Pi E_M) \;\text{ for every } k>0\,.
\end{aligned} \tag{38}
$$

Step 3: Modular convergence of the truncations and equi-integrability of the nonlinearities.

Thanks to (35) and (36), we can write

$$
\begin{aligned}
&\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\;dx\,dt \;\le \\
&\qquad\le \int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u)\,\chi^s\;dx\,dt \\
&\qquad\quad + \int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)\,[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\;dx\,dt \\
&\qquad\quad + \varepsilon(n,j,\mu,i,s,m)
\end{aligned}
$$

and then

$$\limsup_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\;dx\,dt\;\le$$
$$\le\;\int_Q a(T_k(u),\nabla T_k(u))\,\nabla T_k(u)\,\chi^s\;dx\,dt\;+\;\lim_{n\to\infty}\varepsilon(n,j,\mu,i,s,m)$$

in which we can pass to the limit as $j,\mu,i,s,m\to\infty$ to obtain

$$\limsup_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,dx\,dt\;\le\;\int_Q a(T_k(u),\nabla T_k(u))\,\nabla T_k(u)\,dx\,dt\,.$$

On the other hand, Fatou's lemma implies

$$\int_Q a(T_k(u),\nabla T_k(u))\,\nabla T_k(u)\;dx\,dt\;\le\;\liminf_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\;dx\,dt$$

and thus, as $n\to\infty$

$$a(T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\;\to\;a(T_k(u),\nabla T_k(u))\,\nabla T_k(u)\quad\text{in}\;\;L^1(Q)\tag{39}$$

implying by using (13) and Vitali's theorem that

$$\nabla T_k(u_n)\to\nabla T_k(u)\;\;\text{in}\;\;(L_M(Q))^N\quad\text{for the modular convergence}\,.$$

We shall now prove that $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ a.e. in Q, thanks to (24) and (37), it suffices to prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q. Let $E\subset Q$ be a measurable subset of Q. We have for any $m>0$:

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt\;=\\
&\qquad=\;\int_{E\cap\{|u_n|\le m\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt\;+\int_{\{|u_n|>m\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt\\
&\qquad\le\;\frac{b(m)}{\alpha}\int_E a(T_m(u_n),\nabla T_m(u_n))\,\nabla T_m(u_n)\,dx\,dt\;+\;b(m)\int_E c_2(x,t)\,dx\,dt\\
&\qquad\quad+\int_{\{|u_n|>m\}}|f_n|\,dx\,dt\;+\int_{\{|u_{0n}|>m\}}|u_{0n}|\,dx\,dt\;,
\end{aligned}$$

where we have used (14) and (23). Therefore, it is easy to see that there exists $\eta > 0$ such that

$$|E| < \eta \implies \int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \varepsilon, \quad \forall\, n$$

which shows that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q as required.

Step 4: Passage to the limit.

Let $v \in W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ such that $\partial v/\partial t \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$. There exists a prolongation $\overline{v}$ of v such that (see the proof of lemma 5)

$$\overline{v} = v \quad \text{on } Q\ , \qquad \overline{v} \in W_0^{1,x}L_M(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})\ ,$$
$$\text{and} \qquad \partial\overline{v}/\partial t \in W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})\ . \tag{40}$$

By Theorem 3 (see also Remark 3, Section 4), there exists a sequence $(w_j) \subset \mathcal{D}(\Omega\times\mathbb{R})$ such that

$$\text{and} \quad \begin{aligned} &w_j \to \overline{v} \quad \text{in } W_0^{1,x}L_M(\Omega\times\mathbb{R}) \\ &\frac{\partial w_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \quad \text{in } W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}) \end{aligned} \tag{41}$$

for the modular convergence and $\|w_j\|_{\infty,Q} \le (N+2)\,\|v\|_{\infty,Q}$.

Go back to approximate equations (19) and use $T_k(u_n - w_j)\chi_{(0,\tau)}$ (which belongs to $W_0^{1,x}L_M(Q)$) as a test function, one has, for every $\tau \in [0,T]$:

$$\begin{aligned} \Big\langle u_n', T_k(u_n - w_j)\Big\rangle_{Q_\tau} &+ \int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\,\nabla T_k(u_n - w_j)\;dx\,dt \\ &+ \int_{Q_\tau} g_n(u_n,\nabla u_n)\,T_k(u_n - w_j)\;dx\,dt \;= \\ &= \int_{Q_\tau} f_n\,T_k(u_n - w_j)\;dx\,dt\ , \end{aligned} \tag{42}$$

where $\overline{k} = k + C\|v\|_{\infty,Q}$.

The second term of the left hand side of (42) reads as

$$\begin{aligned} \int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\,\nabla T_k(u_n - w_j)\;dx\,dt \;&= \\ = \int_{Q_\tau\cap\{|u_n-w_j|\le k\}} &a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\,\nabla u_n\;dx\,dt \\ - \int_{Q_\tau\cap\{|u_n-w_j|\le k\}} &a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\,\nabla w_j\;dx\,dt \end{aligned}$$

and, by using Fatou's lemma in the first integral of the last side and (38) in the second one, we deduce that

$$\int_{Q_\tau} a(T_{\overline{k}}(u),\nabla T_{\overline{k}}(u))\,\nabla T_k(u-w_j)\;dx\,dt \;\le$$
$$\le\; \liminf_{n\to\infty} \int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\,\nabla T_k(u_n-w_j)\;dx\,dt\;.$$

Since $T_k(u_n-w_j)\to T_k(u-w_j)$ weakly* in $L^\infty(Q)$ as $n\to\infty$, we have (as $n\to\infty$)

$$\int_{Q_\tau} g_n(u_n,\nabla u_n)\,T_k(u_n-w_j)\;dx\,dt \;\to\; \int_{Q_\tau} g(u,\nabla u)\,T_k(u-w_j)\;dx\,dt$$

and

$$\int_{Q_\tau} f_n\,T_k(u_n-w_j)\;dx\,dt \;\to\; \int_{Q_\tau} f\,T_k(u-w_j)\;dx\,dt\;.$$

For what concerns the first term of (42), we have, by setting $S_k(s)=\int_0^s T_k(\eta)\,d\eta$

$$\begin{aligned}
\Big\langle u_n', T_k(u_n-w_j)\Big\rangle_{Q_\tau} &= \Big\langle u_n'-w_j', T_k(u_n-w_j)\Big\rangle_{Q_\tau} + \Big\langle w_j', T_k(u_n-w_j)\Big\rangle_{Q_\tau}\\
&= \int_\Omega S_k(u_n-w_j)(\tau)\,dx \;-\int_\Omega S_k(u_{0n}-w_j(0))\,dx\\
&\quad+\int_{Q_\tau}\frac{\partial w_j}{\partial t}\,T_k(u_n-w_j)\;dx\,dt\;,
\end{aligned}\tag{43}$$

and, in order to pass to the limit (as $n\to\infty$) in (43), we will first prove that $u_n\to u$ in $C([0,T],L^1(\Omega))$ (implying, in particular, that $u\in C([0,T],L^1(\Omega))$).

Let now, for every $l\ge k$, $\omega^{i,l}_{j,\mu}=T_l(v^l_j)_\mu+e^{-\mu t}T_l(\psi_i)$ and $\omega^{i,l}_\mu=T_l(u)_\mu+e^{-\mu t}T_l(\psi_i)$, where $v^l_j\in\mathcal{D}(Q)$ is a sequence such that: $v^l_j\to T_l(u)$ in $W^{1,x}_0L_M(Q)$ for the modular convergence as $j\to+\infty$. We have for every $\tau\in(0,T]$

$$\begin{aligned}
\Big\langle (\omega^{i,l}_{j,\mu})', T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q_\tau} &= \mu\int_{Q_\tau}(T_l(v^l_j)-\omega^{i,l}_{j,\mu})\,T_k(u_n-\omega^{i,l}_{j,\mu})\;dx\,dt\\
&\to \mu\int_{Q_\tau}(T_l(v^l_j)-\omega^{i,l}_{j,\mu})\,T_k(u-\omega^{i,l}_{j,\mu})\;dx\,dt\\
&\to \mu\int_{Q_\tau}(T_l(u)-\omega^{i,l}_{\mu})\,T_k(u-\omega^{i,l}_{\mu})\;dx\,dt\;\ge\;0
\end{aligned}\tag{44}$$

as first n and then j go to infinity, where we have used the fact that $|\omega^{i,l}_\mu|\le l$ to get the positiveness of the last integral.

On the other hand, by using (19)

$$\begin{aligned}\Big\langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q_\tau} &= \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\,[\nabla\omega_{j,\mu}^{i,l} - \nabla u_n]\,\chi_{\{|u_n-\omega_{j,\mu}^{i,l}|\le k\}}\,dx\,dt \\ &\quad + \int_{Q_\tau} g(x,t,u_n,\nabla u_n)\,T_k(\omega_{j,\mu}^{i,l} - u_n)\,dx\,dt \\ &\quad + \int_{Q_\tau} f\,T_k(u_n - \omega_{j,\mu}^{i,l})\,dx\,dt\end{aligned}$$

in which we can use Fatou's lemma and Lebesgue theorem to pass to the limit sup first over n and then over j, μ, l, to get, for every fixed $k > 0$,

$$\Big\langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q_\tau} \le \varepsilon(n,j,\mu,l) \quad \text{not depending on } \tau\,. \tag{45}$$

Therefore, by writing

$$\begin{aligned}\int_\Omega S_k\Big(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau)\Big)\,dx &= \\ &= \Big\langle u_n' - (\omega_{j,\mu}^{i,l})', T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q_\tau} + \int_\Omega S_k(u_0 - T_l(\psi_i))\,dx \\ &= \Big\langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q_\tau} - \Big\langle (\omega_{j,\mu}^{i,l})', T_k(u_n - \omega_{j,\mu}^{i,l})\Big\rangle_{Q_\tau} + \int_\Omega S_k(u_0 - T_l(\psi_i))\,dx\end{aligned}$$

and using (44) and (45), we see that, for every fixed $k > 0$,

$$\int_\Omega S_k\Big(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau)\Big)\,dx \le \varepsilon(n,j,\mu,l,i) \quad \text{not depending on } \tau$$

which implies, by writing (recall that S_k is a convex function)

$$\begin{aligned}\int_\Omega S_k\Big[\frac{1}{2}(u_n(\tau) - u_m(\tau))\Big]\,dx &\le \\ &\le \int_\Omega S_k\Big(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau)\Big)\,dx + \int_\Omega S_k\Big(u_m(\tau) - \omega_{j,\mu}^{i,l}(\tau)\Big)\,dx\,,\end{aligned}$$

that

$$\int_\Omega S_k\Big[\frac{1}{2}(u_n(\tau) - u_m(\tau))\Big]\,dx \le \varepsilon_1(n,m)$$

where $\varepsilon_i(n,m)$ $(i = 1,2)$ is a term not depending on τ and which tends to 0 as n and m go to infinity.

We deduce then that (see for instance, the proof of Theorem 1.1 of [19]),

$$\int_\Omega |u_n(\tau) - u_m(\tau)|\,dx \le \varepsilon_2(n,m) \quad \text{not depending on } \tau$$

and thus (u_n) is a Cauchy sequence in $C([0,T],L^1(\Omega))$ (the space of continuous functions from $[0,T]$ into $L^1(\Omega)$) equipped with the topology of uniform convergence). Since the limit of u_n in $L^1(Q)$ is u, we have

$$u_n \to u \quad \text{in } \ C([0,T],L^1(\Omega))\ .$$

Moreover, since $S_k(u_n - w_j)(\tau) \le k|u_n(\tau)| + k|w_j(\tau)|$, we have by using Lebesgue theorem

$$\int_\Omega S_k(u_n - w_j)(\tau)\,dx \ \to \int_\Omega S_k(u - w_j)(\tau)\,dx \qquad \text{as } \ n \to \infty$$

therefore we can pass to the limit in n in each term of the right hand side of (43) to get

$$\lim_{n\to\infty} \Big\langle u_n', T_k(u_n - w_j)\Big\rangle_{Q_\tau} = \\ = \int_\Omega S_k(u-w_j)(\tau)\,dx - \int_\Omega S_k(u_0-w_j(0))\,dx + \int_{Q_\tau} \frac{\partial w_j}{\partial t}\,T_k(u-w_j)\,dx\,dt\ ,$$

and thus, by passing to the limit inf over n in (42), we have

$$\begin{aligned}&\int_\Omega S_k(u - w_j)(\tau)\,dx + \int_{Q_\tau} \frac{\partial w_j}{\partial t}\,T_k(u - w_j)\,dx\,dt\ + \\ &\qquad + \int_{Q_\tau} a(u,\nabla u)\,\nabla T_k(u-w_j)\,dx\,dt + \int_{Q_\tau} g(u,\nabla u)\,T_k(u-w_j)\,dx\,dt \ \le \\ &\qquad\qquad \le \int_{Q_\tau} f\,T_k(u-w_j)\,dx\,dt + \int_\Omega S_k(u_0 - w_j(0))\,dx\ .\end{aligned} \tag{46}$$

To go to the limit in (46) as $j \to \infty$, observe that, thanks to (41), we have

$$\int_{Q_\tau} \frac{\partial w_j}{\partial t}\,T_k(u - w_j)\,dx\,dt \ \to \ \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle_{Q_\tau}\ .$$

Moreover, for every $\tau \in [0,T]$

$$\int_\Omega S_1(w_i-w_j)(\tau)\,dx \ = \int_\Omega \int_{-\infty}^0 T_1(w_i - w_j)\left(\frac{\partial w_i}{\partial t} - \frac{\partial w_j}{\partial t}\right) dx\,dt \ \to \ 0 \\ \text{as } \ i,j \to \infty\ ,$$

implying, as above, that $\|w_i(\tau) - w_j(\tau)\|_{L^1(\Omega)} \to 0$ as $i,j \to \infty$ and so $\|w_j(\tau) - v(\tau)\|_{L^1(\Omega)} \to 0$ as $j \to \infty$.

Therefore, we can go to the limit, as $j \to \infty$, in each integral of (46), to get

$$\int_\Omega S_k(u-v)(\tau)\,dx \;+\; \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle_{Q_\tau} \;+$$
$$+\int_{Q_\tau} a(u,\nabla u)\,\nabla T_k(u-v)\,dx\,dt \;+\int_{Q_\tau} g(u,\nabla u)\,T_k(u-v)\,dx\,dt \;\le$$
$$\le \int_{Q_\tau} f\,T_k(u-v)\,dx\,dt \;+\int_\Omega S_k(u_0-v(0))\,dx\;,$$

where for the first and last integrals, we have used the fact that $S_k(u-w_j)(\tau) \le S_k(u(\tau)) + k|w_j(\tau)|$, and thus, u is an entropy solution of (17). This completes the proof of theorem 4. ∎

Remark 5. Assume that a satisfies (11)–(13) with β bounded from above (i.e. $\beta(s) \le$ some β_0), and let g satisfying, in addition to (14) and (15), the following coercivity condition:

$$|g(x,t,s,\xi)| \;\ge\; \delta M(|\xi|/\lambda)\;,$$

for all $|s| \ge \theta > 0$, $\xi \in \mathbb{R}^N$ and for a.e. $(x,t) \in Q$ with $\delta, \lambda > 0$. If f is in $L^1(Q)$ then there exists a solution of

$$\text{(47)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) + g(x,t,u,\nabla u) = f & \text{in } Q\\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T)\\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

such that $u \in W_0^{1,x}L_M(Q)$, $g(x,t,u,\nabla u) \in L^1(Q)$ and the equation is satisfied in distributional sense, if and only if u_0 belongs to $L^1(\Omega)$.

Indeed, if there exists a distributional solution u in $W_0^{1,x}L_M(Q)$ with $g(x,t,u,\nabla u)$ in $L^1(Q)$, then $\partial u/\partial t \in W^{-1,x}L_{\overline{M}}(Q)+L^1(Q)$ and hence by Lemma 5, $u \in C([0,T],L^1(\Omega))$. So that u_0 must be in $L^1(\Omega)$.

Conversely, the existence of one distributional solution of (47) can be obtained by adapting the above proof to the approximate equations,

$$\text{(48)}\qquad \begin{cases} u_n \in D(A)\cap W_0^{1,x}L_M(Q)\cap C([0,T],L^2(\Omega))\,, \qquad u_n(x,0) = u_{0n}\\ \dfrac{\partial u_n}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big) + g(x,t,u_n,\nabla u_n) = f_n \end{cases}$$

where, further, the sequence of the approximating solutions u_n is now bounded in $W_0^{1,x}L_M(Q)$ (it suffices to use $T_\theta(u_n)$ as a test function), which allow to pass to the limit in (48) in distributional sense. □

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