PORTUGALIAE MATHEMATICA
Vol. 61 Fasc. 4 – 2004
Nova Série

ON THE VALIDITY OF CHAPMAN–ENSKOG EXPANSIONS FOR SHOCK WAVES WITH SMALL STRENGTH

Nabil Bedjaoui, Christian Klingenberg and Philippe G. LeFloch

Recommended by J.P. Dias

Abstract: We justify a Chapman–Enskog expansion for discontinuous solutions of hyperbolic conservation laws containing shock waves with *small* strength. Precisely, we establish pointwise uniform estimates for the difference between the traveling waves of a relaxation model and the traveling waves of the corresponding diffusive equations determined by a Chapman–Enskog expansion procedure to first- or second-order.

1 – Introduction

We consider scalar conservation laws of the form

$$\partial_t u + \partial_x f(u) = 0\,, \qquad u = u(x,t) \in \mathbb{R}, \quad t > 0\,, \tag{1.1}$$

where the flux-function $f : \mathbb{R} \to \mathbb{R}$ is a given, smooth mapping. It is well-known that initially smooth solutions of (1.1) develop singularities in finite time and that weak solutions satisfying (1.1) in the sense of distributions together with a suitable entropy condition must be sought. For instance, when the initial data have bounded variation, the Cauchy problem for (1.1) admits a unique entropy solution in the class of bounded functions with bounded variation. (See, for instance, [8].) In the present paper, we are primarily interested in shock waves of (1.1), i.e. step-functions propagating at constant speed.

Received: January 19, 2004.
AMS Subject Classification: 35L65, 76N10.
Keywords: conservation law; hyperbolic; shock wave; traveling wave; relaxation; diffusion; Chapman–Enskog expansion.

Entropy solutions of (1.1) can be obtained as limits of diffusion or relaxation models. For instance, under the sub-characteristic condition [9]

$$\sup |f'(u)| < a \ , \tag{1.2}$$

and when the relaxation parameter $\epsilon > 0$ tends to zero it is not difficult to check that solutions of

$$\begin{aligned} &\partial_t u_\epsilon + \partial_x v_\epsilon = 0 \ , \\ &\partial_t v_\epsilon + a^2 \, \partial_x u_\epsilon = \frac{1}{\epsilon} \Big(f(u_\epsilon) - v_\epsilon \Big) \ , \end{aligned} \tag{1.3}$$

converge toward entropy solutions of (1.1). More precisely, the first component $u := \lim_{\epsilon \to 0} u^\epsilon$ is an entropy solution of (1.1) and $f(u) := \lim_{\epsilon \to 0} v^\epsilon$ is the corresponding flux. See, for instance, Natalini [11] and the references therein for a review and references.

The Chapman–Enskog approach [2] allows one to approximate (to "first-order") the relaxation model (1.3) by a diffusion equation ((1.4) below). More generally, it provides a natural connection between the kinetic description of gas dynamics and the macroscopic description of continuum mechanics. The Chapman–Enskog expansion and its variants have received a lot of attention, from many different perspectives. For recent works on relaxation models like (1.3), Chapman–Enskog expansions, and related matters we refer to Liu [9], Caflisch and Liu [1], Szepessy [13], Natalini [11], Mascia and Natalini [10], Slemrod [12], Jin and Slemrod [6], Klingenberg and al. [7], and the many references therein.

Our goal in this paper is to initiate the investigation of the validity of the Chapman–Enskog expansion for discontinuous solutions containing shock waves. This expansion is described in the literature for solutions which are sufficiently smooth, and it is not a priori clear that such a formal procedure could still be valid for *discontinuous* solutions. This issue does not seem to have received the attention it deserves, however. Note first that, by the second equation in (1.3), we formally have

$$\begin{aligned} v_\epsilon &= f(u_\epsilon) - \epsilon \Big(\partial_t v_\epsilon + a^2 \, \partial_x u_\epsilon \Big) \\ &= f(u_\epsilon) - \epsilon \Big(\partial_t f(u_\epsilon) + a^2 \, \partial_x u_\epsilon \Big) + O(\epsilon^2) \\ &= f(u_\epsilon) - \epsilon \Big(-f'(u_\epsilon) \, \partial_x f(u_\epsilon) + a^2 \, \partial_x u_\epsilon \Big) + O(\epsilon^2) \ , \end{aligned}$$

as long as second-order derivatives of the solution remain uniformly bounded in ϵ. Keeping first-order terms only, we arrive at the diffusion equation

$$\partial_t u_\epsilon + \partial_x f(u_\epsilon) \; = \; \epsilon \, \partial_x \Big((a^2 - f'(u_\epsilon)^2) \, \partial_x u_\epsilon \Big) \ . \tag{1.4}$$

This expansion can be continued at higher-order to provide, for smooth solutions of (1.3), an approximation with higher accuracy. When solutions of (1.3) cease to be smooth and the gradient $\partial_x u_\epsilon$ becomes large, the terms collected in $O(\epsilon^2)$ above are clearly no longer negligible in a neighborhood of jumps. The validity of the first-order approximation (1.4), as well as higher-order expansions in powers of ϵ, becomes questionable.

The present paper is motivated by earlier results by Goodman and Majda [3] (validity of the equivalent equation associated with a difference scheme), Hou and LeFloch [5] (difference schemes in nonconservative form), and Hayes and LeFloch [4] (diffusive-dispersive schemes to compute nonclassical entropy solutions). In these three papers, the validity of an asymptotic method is investigated for *discontinuous* solutions, by restricting attention to *shock waves with sufficiently small strength.* This is the point of view we will adopt and, in the present paper, we provide a rigorous justification of the validity of the Chapman–Enskog expansion for solutions containing shocks with small strength.

Specifically, restricting attention to traveling wave solutions of the relaxation model (1.3), the first-order approximation (1.4), and the associated second-order approximation (see Section 2 below), we establish several pointwise, uniform estimates which show that the first- and the second-order approximations approach closely the shock wave solutions of (1.3) with sufficiently small strength. See Theorem 3.2 (for Burgers equation), Theorem 4.2 (general conservation laws), and Theorem 5.1 (generalization to second-order approximation). In the last section of the paper, we discuss whether our results are expected to generalize to higher-order approximations.

2 – Formal Chapman–Enskog expansions

2.1. Expanding v_ϵ only

In this section we will discuss two variants to derive a formal Chapman–Enskog expansion for (1.3), at any order. We begin by plugging the expansion $v = \sum_{k=0}^{\infty} \epsilon^k v_k$ into (1.3) while keeping u fixed. We obtain

$$\partial_t u + \sum_{k=0}^{\infty} \epsilon^k \partial_x v_k = 0 ,$$

$$\sum_{k=0}^{\infty} \epsilon^k \partial_t v_k + a^2 \partial_x u = \frac{f(u)}{\epsilon} - \sum_{k=0}^{\infty} \epsilon^{k-1} v_k .$$

The second identity above yields

$$f(u) = v_0 \ ,$$
$$\partial_t v_0 + a^2\,\partial_x u = -v_1 \ ,$$
$$\partial_t v_k = -v_{k+1}\,, \qquad k \geq 1 \ ,$$

which determines $v_0 = f(u)$ and, for $k \geq 1$, $v_k = (-1)^k\,\partial_t^{k-1}(\partial_t f(u) + a^2\,\partial_x u)$, while the function u is found to satisfy

$$\partial_t u + \partial_x f(u) \;=\; -\partial_x \sum_{k=1}^{\infty} (-\epsilon)^k\,\partial_t^{k-1}\Big(\partial_t f(u) + a^2\,\partial_x u\Big) \ . \tag{2.1}$$

For instance, to first order we find

$$\partial_t u + \partial_x f(u) \;=\; \epsilon\,\partial_x\Big(\partial_t f(u) + a^2\,\partial_x u\Big) \ , \tag{2.2}$$

and to second order

$$\partial_t u + \partial_x f(u) \;=\; \epsilon\,\partial_x\Big(\partial_t f(u) + a^2\,\partial_x u\Big) - \epsilon^2\,\partial_{xt}\Big(\partial_t f(u) + a^2\,\partial_x u\Big) \ . \tag{2.3}$$

The corresponding traveling wave equation satisfied by solutions of the form

$$u(x,t) = u(\xi)\,, \qquad \xi := (x - \lambda\,t)/\epsilon$$

read

$$-\lambda\,u' + f(u)' \;=\; \sum_{k=1}^{\infty} \lambda^{k-1}\Big((-\lambda\,f'(u) + a^2)\,u'\Big)^{(k)} \ . \tag{2.4}$$

To first order the traveling wave equation is

$$-\lambda\,u' + f(u)' \;=\; \Big((-\lambda\,f'(u) + a^2)\,u'\Big)' \tag{2.5}$$

and to second order

$$-\lambda\,u' + f(u)' \;=\; \Big((-\lambda\,f'(u) + a^2)\,u'\Big)' + \lambda\Big((-\lambda\,f'(u) + a^2)\,u'\Big)'' \ . \tag{2.6}$$

2.2 – Expanding both u_ϵ and v_ϵ

One can also expand both u^ϵ and v^ϵ, as follows:

$$u_\epsilon \;=\; u_0 + \epsilon\,u_1 + ... \;=\; u_0 + \sum_{k=1}^{\infty} \epsilon^k\,u_k \ ,$$

$$v_\epsilon \;=\; v_0 + \epsilon\,v_1 + ... \;=\; v_0 + \sum_{k=1}^{\infty} \epsilon^k\,v_k \ .$$

The solution at k^{th}-order is defined by

$$\widetilde{u}_k \;:=\; u_0 + \epsilon\, u_1 + ... + \epsilon^k\, u_k \;. \tag{2.7}$$

We also set

$$\widetilde{v}_k \;:=\; v_0 + \epsilon\, v_1 + ... + \epsilon^k\, v_k \;. \tag{2.8}$$

To first order, one can write (1.3) as

$$\begin{aligned}
&\partial_t u_0 + \epsilon\, \partial_t u_1 + \partial_x v_0 + \epsilon\, \partial_x v_1 + O(\epsilon^2) \;=\; 0 \;,\\
&\partial_t v_0 + \epsilon\, \partial_t v_1 + a^2\, (\partial_x u_0 + \epsilon\, \partial_x u_1) + O(\epsilon^2)\\
&\qquad\qquad = \; \frac{1}{\epsilon}\Big(f(u_0) + \epsilon\, f'(u_0)\, u_1 - v_0 - \epsilon\, v_1\Big) + O(\epsilon) \;,
\end{aligned}$$

which yields the following equations:

$$\begin{aligned}
&f(u_0) - v_0 = 0 \;,\\
&\partial_t u_0 + \partial_x v_0 = 0 \;,\\
&\partial_t u_1 + \partial_x v_1 = 0 \;,\\
&\partial_t v_0 + a^2\, \partial_x u_0 = f'(u_0)\, u_1 - v_1 \;.
\end{aligned}$$

Thus

$$\begin{aligned}
&\partial_t u_0 + \partial_x f(u_0) = 0 \;,\\
&\partial_t u_1 + \partial_x\Big(f'(u_0)\, u_1 - \partial_t v_0 - a^2\, \partial_x u_0\Big) \;=\; 0 \;.
\end{aligned}$$

Therefore, the first-order, Chapman–Enskog expansion leads us to

$$\begin{aligned}
\partial_t \widetilde{u}_1 + \partial_x f(\widetilde{u}_1) \;&=\; \partial_t(u_0 + \epsilon\, u_1) + \partial_x\Big(f(u_0) + \epsilon\, f'(u_0)\, u_1\Big)\\
&=\; \epsilon\, (\partial_{xt} v_0 + a^2 \partial_{xx} u_0)\\
&=\; \epsilon\, (a^2 \partial_{xx} u_0 - \partial_{tt} u_0) \;.
\end{aligned} \tag{2.9}$$

Using that $\partial_t u_0 = -\partial_x f(u_0)$ we get

$$\partial_t \widetilde{u}_1 + \partial_x f(\widetilde{u}_1) \;=\; \epsilon\, \partial_x\Big((-f'(u_0)^2 + a^2)\, \partial_x u_0\Big) + O(\epsilon^2) \;.$$

Neglecting the terms in $O(\epsilon^2)$ we may consider that $\widetilde{u}_1 = u_0 + \epsilon\, u_1$ is a solution of

$$\partial_t \widetilde{u}_1 + \partial_x f(\widetilde{u}_1) \;=\; \epsilon\, \partial_x\Big((-f'(\widetilde{u}_1)^2 + a^2)\, \partial_x \widetilde{u}_1\Big) \;. \tag{2.10}$$

By a similar, but more tedious calculation we can also derive the diffusive equation at second-order. Using (2.7) and (2.9), we have

$$\begin{aligned}\partial_t\widetilde{u}_2+\partial_x f(\widetilde{u}_2) &= \partial_t\widetilde{u}_1+\epsilon^2\,\partial_t u_2+\partial_x\Big(f(\widetilde{u}_1)+\epsilon^2\,f'(\widetilde{u}_1)\,u_2\Big)\\ &= \epsilon\Big(a^2\partial_{xx}u_0-\partial_{tt}u_0\Big)+\epsilon^2\Big(\partial_t u_2+\partial_x(f'(\widetilde{u}_1)\,u_2)\Big)\ .\end{aligned}$$

But, the second order expansion in (1.3) gives

$$\begin{aligned}&\partial_t u_2+\partial_x v_2=0\ ,\\ &\partial_t v_1+a^2\,\partial_x u_1=f'(\widetilde{u}_1)\,u_2-v_2\ ,\end{aligned}$$

and we get

$$\begin{aligned}\partial_t\widetilde{u}_2+\partial_x f(\widetilde{u}_2) &= \epsilon\Big(a^2\,\partial_{xx}u_0-\partial_{tt}u_0\Big)+\epsilon^2\,\partial_x\Big(f'(\widetilde{u}_1)\,u_2-v_2)\Big)\\ &= \epsilon\Big(a^2\,\partial_{xx}u_0-\partial_{tt}u_0\Big)+\epsilon^2\,\partial_x\Big(\partial_t v_1+a^2\,\partial_x u_1\Big)\\ &= \epsilon\Big(a^2\,\partial_{xx}u_0-\partial_{tt}u_0\Big)+\epsilon^2\Big(-\partial_{tt}u_1+a^2\,\partial_{xx}u_1\Big)\ .\end{aligned}$$

Finally, since $\widetilde{u}_1=u_0+\epsilon\,u_1$ we conclude that, to second order,

$$\partial_t\widetilde{u}_2+\partial_x f(\widetilde{u}_2)\ =\ \epsilon\Big(a^2\partial_{xx}\widetilde{u}_1-\partial_{tt}\widetilde{u}_1\Big)\ . \tag{2.11}$$

In exactly the same manner we have, for $n\geq 1$,

$$\partial_t\widetilde{u}_n+\partial_x f(\widetilde{u}_n)\ =\ \epsilon\Big(a^2\,\partial_{xx}\widetilde{u}_{n-1}-\partial_{tt}\widetilde{u}_{n-1}\Big)\ ,$$

so that

$$\partial_t\widetilde{u}_n+\partial_x f(\widetilde{u}_n)\ =\ \epsilon\ \Big(a^2\partial_{xx}\widetilde{u}_n-\partial_{tt}\widetilde{u}_n\Big)+O(\epsilon^{n+1})\ . \tag{2.12}$$

In general, the n^{th}-order equation is obtained by replacing $\partial_{tt}\widetilde{u}_{n-1}$ by derivatives with respect to x to obtain an equation of the form

$$\partial_t\widetilde{u}_n+\partial_x f(\widetilde{u}_n)\ =\sum_{k=1}^{n}\epsilon^k\,H_k(\widetilde{u}_n,\partial_x\widetilde{u}_n,...,\partial_x^{k+1}\widetilde{u}_n)\ . \tag{2.13}$$

We will refer to this expansion as *the Chapman–Enskog expansion to* n^{th} *order*.

So let us for instance derive in this fashion the second order equation satisfied by $\widetilde{u}_2$. We have first

$$\begin{aligned}\partial_{tt}\widetilde{u}_1 &= \partial_t(\partial_t\widetilde{u}_1)\\ &= \partial_t\Big(-\partial_x f(\widetilde{u}_1)+\epsilon\,\partial_x\Big((a^2-f'(\widetilde{u}_1)^2)\,\partial_x\widetilde{u}_1\Big)\Big)\\ &= -\partial_x\Big(f'(\widetilde{u}_1)\,\partial_t\widetilde{u}_1\Big)+\epsilon\,\partial_{xt}\Big((a^2-f'(\widetilde{u}_1))\,\partial_x\widetilde{u}_1\Big)\ .\end{aligned} \tag{2.14}$$

Then setting

$$g_1'(u) = a^2 - f'(u)^2 \quad \text{and} \quad g_2'(u) = (a^2 - f'(u)^2)\, f'(u) = g_1'(u)\, f'(u) \ ,$$

we get

$$\begin{aligned}
\partial_{tt}\widetilde{u}_1 \ &= \ -\partial_x\Big(f'(\widetilde{u}_1)\,\big(-f'(\widetilde{u}_1)\,\partial_x\widetilde{u}_1 + \epsilon\,\partial_{xx}g_1(\widetilde{u}_1)\big)\Big) + \epsilon\,\partial_{xxt}g_1(\widetilde{u}_1) + O(\epsilon^2)\\
&= \ \partial_x\Big(f'(\widetilde{u}_1)^2\,\partial_x\widetilde{u}_1\Big) - \epsilon\,\partial_x\Big(f'(\widetilde{u}_1)\,\partial_{xx}g_1(\widetilde{u}_1)\Big) + \epsilon\,\partial_{xx}\Big(g_1'(\widetilde{u}_1)\,\partial_t\widetilde{u}_1\Big) + O(\epsilon^2)\\
&= \ \partial_x\Big(f'(\widetilde{u}_1)^2\,\partial_x\widetilde{u}_1\Big) - \epsilon\,\partial_x\Big(f'(\widetilde{u}_1)\,\partial_{xx}g_1(\widetilde{u}_1)\Big)\\
&\quad\ + \epsilon\,\partial_{xx}\Big(g_1'(\widetilde{u}_1)\,(-f'(\widetilde{u}_1)\,\partial_x\widetilde{u}_1)\Big) + O(\epsilon^2)\\
&= \ \partial_x\Big(f'(\widetilde{u}_1)^2\,\partial_x\widetilde{u}_1\Big) - \epsilon\,\partial_x\Big(f'(\widetilde{u}_1)\,\partial_{xx}g_1(\widetilde{u}_1)\Big) - \epsilon\,\partial_{xxx}g_2(\widetilde{u}_1) + O(\epsilon^2) \ .
\end{aligned}$$

Finally, since $\widetilde{u}_1 = \widetilde{u}_2 + O(\epsilon^2)$, from (2.11) we obtain

$$\partial_t\widetilde{u}_2 + \partial_x f(\widetilde{u}_2) \ = \ \epsilon\,\partial_{xx}g_1(\widetilde{u}_2) + \epsilon^2\,\partial_x\Big(f'(\widetilde{u}_2)\,\partial_{xx}g_1(\widetilde{u}_2) + \partial_{xx}g_2(\widetilde{u}_2)\Big) \ . \tag{2.15}$$

Setting $u = \widetilde{u}_2$, we can rewrite the last equation in the form

$$\begin{aligned}
u_t + f(u)_x \ &= \ \epsilon\Big((a^2 - f'(u)^2)\,u_x\Big)_x\\
&\quad + \epsilon^2\left(f'(u)\,\Big((a^2 - f'(u)^2)\,u_x\Big)_x\right)_x\\
&\quad + \epsilon^2\left((a^2 - f'(u)^2)\,f'(u)\,u_x\right)_{xx} \ .
\end{aligned}\tag{2.16}$$

For later reference we record here the traveling wave equation associated with (2.16)

$$\begin{aligned}
-\lambda\,u' + f(u)' \ &= \ \Big((a^2 - f'(u)^2)\,u'\Big)'\\
&\quad + \left(f'(u)\,\Big((a^2 - f'(u)^2)\,u'\Big)'\right)'\\
&\quad + \Big((a^2 - f'(u)^2)\,f'(u)\,u'\Big)'' \ .
\end{aligned}\tag{2.17}$$

We arrive at the main issue in this paper: Does the solution $\widetilde{u}_n$ of (2.13) converge to some limit u when $n \to \infty$ and, if so, does this limit satisfy the equation

$$\partial_t u + \partial_x f(u) \ = \ \epsilon(a^2\,\partial_{xx}u - \partial_{tt}u) \ .$$

In other word, is this limit u a solution of the relaxation model (1.3)?

To make such a claim rigorous one would need to specify in which topology the limit is taken. As we are interested in the regime where shocks are present the convergence in the sense of distributions should be used. We will not address this problem at this level of general solutions, but will investigate the important situation of traveling wave solutions, at least as far as first- and second-order approximations are concerned.

3 – Burgers equation: validity of the first-order equations

We begin, in this section, with the simplest flux function $f(u) = u^2/2$. Modulo some rescaling $x \to x - \lambda t/\epsilon$, the traveling wave solutions $u = u(x)$, $v = v(x)$ of (1.3) are given by

$$\begin{aligned} &-\lambda\, u' + v' = 0\,, \\ &-\lambda\, v' + a^2\, u' = \frac{u^2}{2} - v\,, \end{aligned} \tag{3.1}$$

where λ represents the wave speed. Searching for solutions connecting left-hand states u_- and $v_- := f(u_-)$ to right-hand states u_+ and $v_+ := f(u_+)$ (so both at equilibrium), we see that

$$\lambda\,(u_+ - u_-) \;=\; v_+ - v_-\;,$$

so that the component u is a solution of the single first-order equation

$$(a^2 - \lambda^2)\, u' \;=\; \frac{1}{2}\,(u - u_-)\,(u - u_+)\;.$$

The shock speed is also given by $\lambda = (u_+ + u_-)/2$. Finally, an easy calculation based on (3.1) yields the following explicit formula for the solution, say $u = u_*(x)$ of (3.1) connecting u_- to u_+. It exists if and only if $u_- > u_+$ and then

$$u_*(x) \;:=\; u_- - \frac{(u_- - u_+)}{1 + \exp\left(-\frac{u_- - u_+}{2(a^2-\lambda^2)}\, x\right)}\;. \tag{3.2}$$

It will be useful to introduce the following one-parameter family of functions

$$\varphi_\mu(x) \;:=\; u_- - \frac{(u_- - u_+)}{1 + \exp\left(-\frac{x(u_- - u_+)}{2(a^2-\mu)}\right)}\,, \qquad \mu \in \mathbb{R}\backslash\{a^2\}\;, \tag{3.3}$$

in which μ is a parameter, not necessarily related to the speed λ. Clearly, we have

$$u_* = \varphi_{\lambda^2}\;.$$

Note that we have for all $\mu < a^2$, and $x \in \mathbb{R}$,

$$u_+ < \varphi_\mu(x) < u_- \ .$$

The following estimate in terms of the strength $\delta := (u_- - u_+)$ is easily derived from (3.3):

Lemma 3.1. *Given* $a > 0$ *and* $0 < h < a^2$ *there exist constants* $c, C > 0$ *such that for all* $\mu_1, \mu_2 \in (-a^2 + h, a^2 - h)$ *and for all* $x \in \mathbb{R}$ *we have*

$$|\varphi_{\mu_1}(x) - \varphi_{\mu_2}(x)| \le C\,\delta^2\,|x|\,|\mu_1 - \mu_2|\,e^{-c|x|\delta} \ . \tag{3.4}$$

Proof: We can write

$$\begin{aligned} |\varphi_{\mu_1}(x) - \varphi_{\mu_2}(x)| &= \delta \left| \frac{1}{1 + \exp\left(-\frac{x(u_- - u_+)}{2(a^2 - \mu_2)}\right)} - \frac{1}{1 + \exp\left(-\frac{x(u_- - u_+)}{2(a^2 - \mu_1)}\right)} \right| \\ &\le \frac{|x|}{2}\,\delta^2 \left| \frac{1}{a^2 - \mu_2} - \frac{1}{a^2 - \mu_1} \right| \sup_{x,k} \frac{\exp\left(-\frac{x\,\delta}{2(a^2-k)}\right)}{\left(1 + \exp\left(-\frac{x\,\delta}{2(a^2-k)}\right)\right)^2} \ . \end{aligned} \tag{3.5}$$

Here, the super bound is taken for $|k| < a^2 - h$ and $x \in \mathbb{R}$.

Then observe that for $y > 0$ we have

$$\begin{aligned} \frac{\exp\left(-\frac{y}{2(a^2-k)}\right)}{\left(1 + \exp\left(-\frac{y}{2(a^2-k)}\right)\right)^2} &\le \exp\left(-\frac{y}{2(a^2-k)}\right) \\ &\le \exp\left(-\frac{y}{2(a^2 + (a^2 - h))}\right) \ , \end{aligned}$$

while for $y < 0$ we have

$$\begin{aligned} \frac{\exp\left(-\frac{y}{2(a^2-k)}\right)}{\left(1 + \exp\left(-\frac{y}{2(a^2-k)}\right)\right)^2} &\le \frac{1}{1 + \exp\left(-\frac{y}{2(a^2-k)}\right)} \\ &\le \exp\left(\frac{y}{2(a^2-k)}\right) \\ &\le \exp\left(\frac{y}{2(a^2 + (a^2 - h))}\right) \ . \end{aligned}$$

This establishes the desired estimate. ∎

We are now in position to study the traveling waves of the first-order equations obtained by either the approaches in Subsections 2.1 and 2.2:

$$-\lambda\, u' + \left(\frac{u^2}{2}\right)' = \Big((a^2 - \lambda\, u)\, u'\Big)'$$

and

$$-\lambda\, u' + \left(\frac{u^2}{2}\right)' = \Big((a^2 - u^2)\, u'\Big)' ,$$

respectively. Note that they only differ by the diffusion coefficients in the right-hand sides. After integration, calling V_1 and W_1 the corresponding traveling wave solutions, we get

$$(a^2 - \lambda\, V_1)\, V_1' = \frac{1}{2}\,(V_1 - u_-)\,(V_1 - u_+) \tag{3.6}$$

and

$$(a^2 - W_1^2)\, W_1' = \frac{1}{2}\,(W_1 - u_-)\,(W_1 - u_+) , \tag{3.7}$$

respectively. For uniqueness, since the traveling waves are invariant by translation, we assume in addition that for example

$$u_*(0) = V_1(0) = W_1(0) = \frac{u_- + u_+}{2} . \tag{3.8}$$

To compare the first-order diffusive traveling waves W_1 and V_1 with the relaxation traveling wave u_*, we rely on monotonicity arguments. It is clear that the traveling waves are monotone, with $V_1', W_1' < 0$ and $u_- > V_1(x), W_1(x) > u_+$, so that setting

$$\Gamma_- = \min_{[u_+,u_-]} u^2 - b\,\delta , \qquad \Gamma_+ = \max_{[u_+,u_-]} u^2 + b\,\delta ,$$

where $b > 0$ is a sufficiently small constant such that $\Gamma_+ < a^2$, we find

$$\begin{aligned} (a^2 - \Gamma_-)\, W_1' &< \frac{1}{2}\,(W_1 - u_-)\,(W_1 - u_+) , \\ (a^2 - \Gamma_+)\, W_1' &> \frac{1}{2}\,(W_1 - u_-)\,(W_1 - u_+) . \end{aligned} \tag{3.9}$$

Therefore, setting

$$\tilde u = W_1 - \varphi_{\Gamma_-} ,$$

after some calculation we find

$$2\,(a^2 - \Gamma_-)\,\tilde u' - \tilde u^2 + \tilde u\,\delta\, \frac{1 - \exp\left(-\frac{x\delta}{2(a^2-\Gamma_-)}\right)}{1 + \exp\left(-\frac{x\delta}{2(a^2-\Gamma_-)}\right)} < 0 . \tag{3.10}$$

We have $\tilde{u}(\pm\infty) = 0$. As $x \to \pm\infty$ the last coefficient in (3.10) approaches ± 1 and the function $\tilde{u}$ satisfies

$$c\,\tilde{u}' \pm \tilde{u}\,\delta + \text{H.O.T.} < 0\,.$$

So, $\tilde{u}$ decreases exponentially at infinity while keeping a constant sign, and we deduce that $\tilde{u}(x) \neq 0$ for $|x| \geq M$, for some sufficiently large M.

Now, if $\tilde{u}$ vanishes at some point x_0 then, thanks to the inequality (3.10), we deduce that $\tilde{u}'(x_0) < 0$. This implies that there is at most one point, and thus exactly one point where $\tilde{u}$ vanishes, which is by (3.8) $x_0 = 0$. Therefore, we have $\operatorname{sgn}(x)\,\tilde{u}(x) < 0$.

A similar analysis applies to the function $W_1 - \varphi_{\Gamma_+}$ and we obtain

$$\operatorname{sgn}(x)\,\varphi_{\Gamma_+}(x) < \operatorname{sgn}(x)\,W_1(x) < \operatorname{sgn}(x)\,\varphi_{\Gamma_-}(x)\,, \qquad x \in \mathbb{R}\,. \tag{3.11}$$

Concerning the function V_1, by defining

$$\lambda_- := \min_{[u_+,u_-]} u\,, \qquad \lambda_+ := \max_{[u_+,u_-]} u\;,$$

and

$$\Lambda_- := \min(\lambda\,\lambda_-, \lambda\,\lambda_+)\; - b\,\delta\,, \qquad \Lambda_+ := \max(\lambda\,\lambda_-, \lambda\,\lambda_+)\; + b\,\delta\;,$$

where $b > 0$ is a sufficiently small constant such that $\Lambda_+ < a^2$, we obtain in the same manner as above

$$\operatorname{sgn}(x)\,\varphi_{\Lambda_+}(x) < \operatorname{sgn}(x)\,V_1(x) < \operatorname{sgn}(x)\,\varphi_{\Lambda_-}(x)\,, \qquad x \in \mathbb{R}\,. \tag{3.12}$$

Note that, for the same reasons, the function $u = u_*$ satisfies also (3.11) and (3.12). Finally, since $|\Gamma_+ - \Gamma_-|, |\Lambda_+ - \Lambda_-| \leq C\,\delta$, we can combine (3.11) and (3.12) with Lemma 3.1 and conclude:

Theorem 3.2. *Given two reals $a > M > 0$, there are constants c, $C > 0$ so that the following property holds for all $u_-, u_+ \in [-M, M]$. The uniform distance between the traveling wave of the relaxation model and the ones of the first-order diffusive equations derived in Section 2 is of cubic order, in the sense that*

$$|V_1(x) - u_*(x)|, \;\; |W_1(x) - u_*(x)| \leq C\,\delta^3\,|x|\,e^{-c\,\delta|x|}\,, \qquad x \in \mathbb{R}\,. \;\blacksquare \tag{3.13}$$

Note that the estimate is cubic on any compact set but is solely quadratic in the uniform norm on the real line:

$$\|V_1 - u_*\|_{L^\infty(\mathbb{R})}, \, \|W_1 - u_*\|_{L^\infty(\mathbb{R})} \;\leq\; C'\,\delta^2\;. \tag{3.14}$$

4 – Validity of the first-order expansions

We extend the result in Section 3 to general, strictly convex flux-functions. It is well-known that a traveling wave connecting u_- to u_+ must satisfy the condition $u_- > u_+$ which we assume from now on.

Set

$$P(u) = f(u) - f(u_-) - \lambda\,(u - u_-)\;, \tag{4.1}$$

and denote by u_* the solution of the relaxation equation and by V_1 and W_1 the first-order traveling waves corresponding to equation (2.2) (i.e., (2.5)) and to (2.10) respectively. We have

$$\begin{aligned}
&(a^2 - \lambda^2)\, u_*' = P(u_*)\;,\\
&(a^2 - \lambda\, f'(V_1))\, V_1' = P(V_1)\;,\\
&(a^2 - f'(W_1)^2)\, W_1' = P(W_1)\;,
\end{aligned} \tag{4.2}$$

together with the boundary conditions

$$\lim_{\pm\infty} u_*(x) = \lim_{\pm\infty} V_1(x) = \lim_{\pm\infty} W_1(x) = u_\pm\;.$$

The existence of solutions to these first-order O.D.E.'s can easily be checked, for instance using the following implicit formula:

$$F_k(u(x)) - F_k(u(0)) = x\;, \qquad x \in \mathbb{R}, \quad k = 0, 1, 2\;,$$

where

$$\begin{aligned}
F_0'(u) &:= \frac{(a^2 - \lambda^2)}{f(u) - f(u_-) - \lambda\,(u - u_-)}\;, \qquad u \in \mathbb{R}\;,\\
F_1'(u) &:= \frac{(a^2 - \lambda\, f'(u))}{f(u) - f(u_-) - \lambda\,(u - u_-)}\;, \qquad u \in \mathbb{R}\;,\\
F_2'(u) &:= \frac{(a^2 - f'(u)^2)}{f(u) - f(u_-) - \lambda\,(u - u_-)}\;, \qquad u \in \mathbb{R}\;.
\end{aligned} \tag{4.3}$$

To ensure uniqueness, we can impose, for example,

$$u_*(0) = V_1(0) = W_1(0) = \frac{u_- + u_+}{2}\;. \tag{4.4}$$

Now, as was done for Burgers' equation, let us define auxilliary functions φ_μ as the solutions of

$$(a^2 - \mu)\,\varphi'_\mu = P(\varphi_\mu)\ , \tag{4.5}$$

with the same boundary conditions as above. For $\mu < a^2$ we immediately have

$$u_+ < \varphi_\mu(x) < u_-\ , \qquad x \in \mathbb{R}\ .$$

Setting $\delta := (u_- - u_+)$ we get:

Lemma 4.1. *Suppose that f is a strictly convex flux-function and $u_- > u_+$. Given $a > 0$ and $0 < h < a^2$ there exist constants $c, C > 0$ such that, for all $\mu_1, \mu_2 \in (-a^2 + h, a^2 - h)$ and for all $x \in \mathbb{R}$,*

$$|\varphi_{\mu_1}(x) - \varphi_{\mu_2}(x)| \,\leq\, C\,\delta^2\,|x|\,|\mu_1 - \mu_2|\,e^{-c\,|x|\,\delta}\ . \tag{4.6}$$

Proof: Let ψ be the solution of

$$\psi' = P(\psi) = f(\psi) - f(u_-) - \lambda(\psi - u_-)\ .$$

We clearly have

$$\varphi_\mu(x) = \psi\left(\frac{x}{a^2 - \mu}\right)\ .$$

Now, we can write

$$\begin{aligned}
|\varphi_{\mu_1}(x) - \varphi_{\mu_2}(x)| \;&=\; \left|\psi\Big(\frac{x}{a^2 - \mu_1}\Big) - \psi\Big(\frac{x}{a^2 - \mu_2}\Big)\right| \\
&=\; \left|x\,\psi'(k(x)\,x)\left(\frac{1}{a^2 - \mu_1} - \frac{1}{a^2 - \mu_2}\right)\right| \\
&\leq\; C\,|\mu_1 - \mu_2|\,|x|\,|P(\psi(k(x)\,x))|\ .
\end{aligned}$$

Here, $k(x)$ is some real number lying in the interval $\left(\frac{1}{a^2-\mu_1}, \frac{1}{a^2-\mu_2}\right)$.

On the other hand we have

$$|P(\psi(x))| \,\leq\, C\,\delta\,|\psi(x) - u_-| \,\leq\, C\,\delta^2\ .$$

This implies that

$$|\varphi_{\mu_1}(x) - \varphi_{\mu_2}(x)| \,\leq\, C\,|\mu_1 - \mu_2|\,\delta^2\,|x|\ . \tag{4.7}$$

The behavior at $\pm\infty$ is described by

$$\psi(x) \sim k_+ \, e^{(f'(u_+)-\lambda)\, x} \,, \qquad x \to +\infty$$

and

$$\psi(x) \sim k_- \, e^{(f'(u_-)-\lambda)\, x} \,, \qquad x \to -\infty \,.$$

Since the coefficient $k(x)$ is bounded away from 0 and $f'(u_+) - \lambda = c_+ \, \delta$ and $f'(u_-) - \lambda = c_- \, \delta$ with $c_+ < 0$ and $c_- > 0$ (bounded away from zero since f is strictly convex), this completes the proof. ∎

Consider now the functions u_*, V_1 and W_1 the solutions of (4.2). Then, we have:

Theorem 4.2. *Let f be a strictly convex flux-function, $M > 0$ and $a > 0$ such that (1.2) holds in $[-M, M]$. Then there exist constants c, $C > 0$ so that the following inequality holds for all $u_-, u_+ \in [-M, M]$ with $u_- > u_+$: for all $x \in \mathbb{R}$*

$$|V_1(x) - u_*(x)|, \, |W_1(x) - u_*(x)| \, \leq \, C \, \delta^3 \, |x| \, e^{-c\,\delta |x|} \,. \tag{4.8}$$

The proof relies on the following lemma:

Lemma 4.3. *Suppose that f is a strictly convex flux-function and $u_- > u_+$. Assume that z_+ and z_- are the solutions of*

$$z'_+ = R_+(z_+) \,, \qquad z'_- = R_-(z_-) \,, \qquad z_+(0) = z_-(0) \,,$$

where $R_+ = R_+(u)$ and $R_- = R_-(u)$ are any smooth functions satisfying

$$R_+(u) < R_-(u) < 0 \qquad \textit{for all} \quad u \in (u_+, u_-) \,. \tag{4.9}$$

Then, the two corresponding curve solutions cross at $x = 0$ only, and

$$\begin{aligned} z_+ &> z_- \quad \textit{for} \;\; x < 0 \,, \\ z_+ &< z_- \quad \textit{for} \;\; x > 0 \,. \end{aligned} \tag{4.10}$$

Proof: If there is x_0 such that $z_+(x_0) = z_-(x_0)$ then thanks to (4.9),

$$z'_+(x_0) < z'_-(x_0) \,.$$

This implies that there cannot be more than one intersection point. So, $(0, z_+(0))$ is the only interaction point of the two trajectories, and (4.10) follows as well. ∎

Proof of Theorem 4.2: Setting

$$\lambda_- = \min_{[u_+,u_-]} f'(u)\,, \qquad \lambda_+ = \max_{[u_+,u_-]} f'(u)$$

and

$$\Lambda_- = \min(\lambda\,\lambda_-, \lambda\,\lambda_+) - b\,\delta\,, \qquad \Lambda_+ = \max(\lambda\,\lambda_-, \lambda\,\lambda_+) + b\,\delta\ ,$$

where, $b > 0$ is a sufficiently small constant such that $\Lambda_+ < a^2$, we have

$$\Lambda_- < \lambda\, f'(u) < \Lambda_+ \quad \text{and} \quad \Lambda_- < \lambda^2 < \Lambda_+\ ,$$

and thus

$$0 < a^2 - \Lambda_+ < a^2 - \lambda\, f'(u)\,, \qquad a^2 - \lambda^2 < a^2 - \Lambda_-\ . \tag{4.11}$$

Applying Lemma 4.3 we deduce that

$$\varphi_{\Lambda_-} < u_*,\, V_1, < \varphi_{\Lambda_+} \qquad x < 0\ ,$$
$$\varphi_{\Lambda_+} < u_*,\, V_1, < \varphi_{\Lambda_-} \qquad x > 0\ .$$

Now, concerning the third equation in (4.2), we set

$$\Gamma_- = \min_{[u_+,u_-]} f'(u)^2 - b\,\delta \quad \text{and} \quad \Gamma_+ = \max_{[u_+,u_-]} f'(u)^2 + b\,\delta\ ,$$

where $b > 0$ is sufficiently small such that $\Gamma_+ < a^2$. We obtain

$$0 < a^2 - \Gamma_+ < a^2 - f'(u)^2\,, \qquad a^2 - \lambda^2 < a^2 - \Gamma_- \tag{4.12}$$

and, by Lemma 4.3,

$$\varphi_{\Gamma_-} < u_*,\, W_1, < \varphi_{\Gamma_+} \qquad x < 0\ ,$$
$$\varphi_{\Gamma_+} < u_*,\, W_1, < \varphi_{\Gamma_-} \qquad x > 0\ .$$

Finally, since $|\Lambda_+ - \Lambda_-|,\, |\Gamma_+ - \Gamma_-| \le C\,\delta,$ by applying Lemma 4.1, we obtain (4.8). This completes the proof of Theorem 4.2. ∎

5 – Validity of a second-order expansion

Our next objective is to extend the estimate in Theorem 4.2 to the second-order equation obtained in Subsection 2.1.

We consider the equation (2.6) after integrating it once. The traveling wave connects u_- to u_+, with $u_- > u_+$, and is given by

$$P(u) := (-\lambda\, f'(u) + a^2)\, u' + \lambda \Big((-\lambda\, f'(u) + a^2)\, u'\Big)' . \tag{5.1}$$

Defining first- and second- order ODE operators:

$$Q_1 u \,=\, (a^2 - \lambda f'(u))\, u'$$

and

$$Q_2 u \,=\, (a^2 - \lambda f'(u))\, u' + \lambda\Big((a^2 - \lambda f'(u))\, u'\Big)' = Q_1 u + \lambda (Q_1 u)' .$$

The solution $u = V_2$ of (2.6) under consideration satisfies

$$Q_2 V_2 = P(V_2) . \tag{5.2}$$

Theorem 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex flux-function and $M > 0$. Then there exist constants $C, c, c_0 > 0$ so that the following property holds. For any $u_-, u_+ \in [-M, M]$ with $u_- > u_+$ and $0 < \delta = u_- - u_+ < c_0$, there exists a traveling wave $V_2 = V_2(y)$ of (5.2) connecting u_- to u_+. Moreover, this traveling wave approaches the relaxation traveling wave u_* to fourth-order in the shock strength, precisely:*

$$|V_2(x) - u_*(x)| \le C\, \delta^4\, |x|\, e^{-c\,|x|\,\delta} , \qquad x \in \mathbb{R} . \tag{5.3}$$

The estimate is only cubic in the uniform norm on the whole real line:

$$\|V_2 - u_*\|_{L^\infty(\mathbb{R})} \le C'\, \delta^3 . \tag{5.4}$$

Proof: Setting

$$d_\mu = \frac{\lambda}{a^2 - \mu} \qquad \text{and} \qquad \gamma_\lambda = d_{\lambda^2} = \frac{\lambda}{a^2 - \lambda^2} ,$$

then $u_* = \varphi_{\lambda^2}$ satisfies

$$Q_1 u_* \,=\, P(u_*)\, \Big(1 + \gamma_\lambda (\lambda - f'(u_*))\Big) \,=\, P(u_*)\, \Big(1 - \gamma_\lambda P'(u_*)\Big) ,$$

and a simple calculation gives

$$Q_2 u_* = P(u_*)\Big(1-\gamma_\lambda^2(f''(u_*)P(u_*)+(f'(u_*)-\lambda)^2)\Big) = P(u_*)\Big(1-\gamma_\lambda^2(P\,P')'(u_*)\Big)\,.$$

In the same manner, the function φ_μ, that is the solution of (4.5) satisfies the following equation

$$Q_1\varphi_\mu \;=\; P(\varphi_\mu)\,\Big(1+c_\mu+d_\mu(\lambda-f'(\varphi_\mu))\Big) = P(\varphi_\mu)\,\Big(1+c_\mu-d_\mu P'(\varphi_\mu)\Big)\,,$$

where

$$c_\mu := \frac{\mu-\lambda^2}{a^2-\mu}\,,$$

and

$$Q_2\varphi_\mu = P(\varphi_\mu)\,\Big(1+c_\mu\Big(1+d_\mu(f'(\varphi_\mu)-\lambda)\Big)-d_\mu^2\Big(f''(\varphi_\mu)\,P(\varphi_\mu)+(f'(\varphi_\mu)-\lambda)^2\Big)\Big)$$

or, equivalently,

$$Q_2\varphi_\mu = P(\varphi_\mu)\,\Big(1+c_\mu(1+d_\mu P'(\varphi_\mu))-d_\mu^2((PP')'(\varphi_\mu))\Big)\,.$$

Now, since $|f'(\varphi_\mu)-\lambda|\le C_0\,\delta$ and $|f''(\varphi_\mu)P(\varphi_\mu)+(f'(\varphi_\mu)-\lambda)^2|\le C_0\delta^2$, then for sufficiently small δ there exists a positive constant C such that the following property holds: by choosing μ_+ and μ_- in the form

$$\mu_+ = \lambda^2(1+C\delta^2)\,,\qquad \mu_- = \lambda^2(1-C\delta^2)\,,$$

we obtain

$$Q_2\varphi_{\mu_+} = P(\varphi_{\mu_+})\,(1+K_+(\varphi_{\mu_+})),\qquad \text{where}\ \ K_+(\varphi_{\mu_+})>0$$

and

$$Q_2\varphi_{\mu_-} = P(\varphi_{\mu_-})\,(1+K_-(\varphi_{\mu_-})),\qquad \text{where}\ \ K_-(\varphi_{\mu_-})<0\,.$$

Consider the corresponding functions φ_{μ_+} and φ_{μ_-} and let us use phase plane argument. The corresponding curves

$$\begin{aligned}&\mathcal{C}_+:\ \ \varphi_{\mu_+}\mapsto(\varphi_{\mu_+},\,w_{\mu_+}=Q_1\varphi_{\mu_+})\,,\\&\mathcal{C}_-:\ \ \varphi_{\mu_-}\mapsto(\varphi_{\mu_-},\,w_{\mu_-}=Q_1\varphi_{\mu_-})\end{aligned}\tag{5.5}$$

satisfy

$$\lambda\,l(\varphi_{\mu_+})\,w_{\mu_+}\frac{dw_{\mu_+}}{du}+w_{\mu_+} = P(\varphi_{\mu_+})\,(1+K_+(\varphi_{\mu_+}))\tag{5.6}$$

and

(5.7) $$\lambda\, l(\varphi_{\mu_-})\, w_{\mu_-} \frac{dw_{\mu_-}}{du} + w_{\mu_-} = P(\varphi_{\mu_-})\,(1 + K_-(\varphi_{\mu_-}))\ ,$$

where

$$l(u) := \frac{1}{a^2 - \lambda\, f'(u)}\ .$$

We claim that the curve $\mathcal{C}_+$ is "below" the curve $\mathcal{C}_-$.

This is true locally near the points $(u_-, 0)$ and $(u_+, 0)$, as it clear by comparing the tangents to the curves at these points (using (4.5)). Note that if $\lambda = 0$ we have $u = u_*$. We then distinguish between two cases:

Case 1: If $\lambda > 0$, suppose that the two curves issuing from $(u_-, 0)$, meet for the "first" time at some point (u_0, w_0) with $u_+ < u_0 < u_-$. Then, combining (5.6) and (5.7) at this point we get

$$\lambda\, l(u_0)\, w_0 \left(\frac{dw_{\mu_+}}{du}(u_0) - \frac{dw_{\mu_-}}{du}(u_0) \right) = P(u_0)\,(K_+(u_0) - K_-(u_0))\ .$$

This leads to a contradiction, since

$$w_0 < 0\,, \qquad \frac{dw_{\mu_+}}{du}(u_0) \le \frac{dw_{\mu_-}}{du}(u_0) \qquad \text{and} \qquad P(u_0)\,(K_+(u_0) - K_-(u_0)) < 0\ .$$

Consider now the equation (5.2) and let us study in the phase plane the trajectory issuing from $(u_-, 0)$ at $-\infty$. Comparing the eigenvalues we obtain that the tangent at this point lies between those of the reference curves $\mathcal{C}_+$ and $\mathcal{C}_-$.

In the same manner as before, we obtain that this curve cannot meet $\mathcal{C}_+$, nor $\mathcal{C}_-$, and necessarily converges to $(u_+, 0)$ as $y \to +\infty$.

Case 2: If $\lambda < 0$, we follow the same analysis by considering the trajectory of (2.5) arriving at $(u_+, 0)$ and the "last" intersection point.

In both cases, we obtain the existence (and uniqueness) of the solution of (5.2), denoted by $u = V_2$, and also that its trajectory called $\mathcal{C}$ is between $\mathcal{C}_+$ and $\mathcal{C}_-$.

Note that since our equations are autonomous, by choosing $u(0) = \varphi_{\mu_+}(0) = \varphi_{\mu_-}(0) = (u_- + u_+)/2$, we have

(5.8) $$\varphi_{\mu_+} < u < \varphi_{\mu_-}\,, \qquad x > 0$$

and

(5.9) $$\varphi_{\mu_-} < u < \varphi_{\mu_+}\,, \qquad x < 0\ .$$

Indeed, from the phase plane analysis, if for some $x_0 \in \mathbb{R}$, $u(x_0) = \varphi_{\mu_+}(x_0)$ then necessarily $w(x_0) > w_{\mu_+}(x_0)$ and then $u'(x_0) > \varphi'_{\mu_+}(x_0)$. This means that the curves $x \mapsto u(x) = V_2(x)$ and $x \mapsto \varphi_{\mu_+}(x)$ have only one intersection point, that is $(0, u(0))$, that satisfies in addition $u'(0) > \varphi'_{\mu_+}(0)$. We obtain in same manner that the two curves $x \mapsto u(x)$ and $x \mapsto \varphi_{\mu_-}(x)$ have only one intersection point, that is $(0, u(0))$, that satisfies in addition $u'(0) < \varphi'_{\mu_-}(0)$.

Now, using the inequalities (5.8) and (5.9) that are also satisfied by $u_* = \varphi_{\lambda^2}$ (since $\mu_- < \lambda^2 < \mu_+$), we can write

$$\begin{aligned} |u_*(x) - u(x)| &\le |\varphi_{\mu_+}(x) - \varphi_{\mu_-}(x)| \\ &\le |\mu_+ - \mu_-|\, \delta^2\, |x|\, e^{-c|x|\,\delta} \\ &\le C\, \delta^4\, |x|\, e^{-c|x|\,\delta} \;, \end{aligned}$$

which completes the proof of Theorem 5.1. ∎

6 – Conclusions

For the general expansion derived in Subsection 2.2 we now establish an identity which connects the relaxation equation with its Chapman–Enskog expansion at *any order* of accuracy. By defining the ODE operator

$$Q_n u := \sum_{k=1}^{n} \lambda^{k-1} \Big((-\lambda\, f'(u) + a^2)\, u' \Big)^{(k-1)} \;, \tag{6.1}$$

we have:

Theorem 6.1. *The traveling wave u_* of the relaxation model satisfies*

$$Q_n u_* = P(u_*)\, (1 - \gamma_\lambda^n\, R_n(u_*)) \;,$$

where $\gamma_\lambda := \lambda/(a^2 - \lambda^2)$, and the remainders R_n are defined by induction:

$$R_1 := P' \,, \qquad R_{n+1} := (P\, R_n)' \;\; \text{for} \;\; n \ge 1 \;.$$

Proof: Note that the ODE operators Q_n satisfy

$$Q_{n+1} u \;=\; Q_1 u + \lambda (Q_n u)' \;.$$

Now, assume that

$$Q_n u_* = P(u_*)\, (1 - \gamma_\lambda^n\, R_n(u_*)) \;,$$

then

$$Q_{n+1}u_* = P(u_*)\,(1-\gamma_\lambda\,P'(u_*)) + \lambda\Big(P'(1-\gamma_\lambda^n\,R_n(u_*)) - P(u_*)\,\gamma_\lambda^n\,R_n'(u_*)\Big)\,u_*'\ .$$

But since $u_*' = \frac{P(u_*)}{a^2-\lambda^2}$ it follows that

$$Q_{n+1}u_* \;=\; P(u_*)\,\Big(1-\gamma_\lambda^{n+1}\,(P\,R_n)'\,(u_*)\Big) \;=\; P(u_*)\,\Big(1-\gamma_\lambda^{n+1}\,R_{n+1}(u_*)\Big)\ ,$$

which completes the proof. ∎

Theorem 6.1 provides some indication that, by taking into account more and more terms in the Chapman–Enskog expansion, the approximating traveling wave should approach the traveling wave equation of the relaxation equation (1.3). For n large but *fixed* it is conceivable that, denoting V_n the solution of $Q_n u = P(u)$,

$$\|V_n - u_*\|_{L^\infty(\mathbb{R})} \;\le\; C_n\,\delta^{n+1}\ . \tag{6.2}$$

However, one may not be able to let $n \to \infty$ while keeping δ fixed. In fact, numerical experiments (with Burgers flux) have revealed that the remainders satisfy only

$$\|R_n(u_*)\|_{L^\infty} \;\le\; C_n'\,\delta^n\ ,$$

where the constants C_n' grow exponentially and cannot be compensated by the factor γ_λ^n. One can also easily check, directly from the definitions, that

$$\|R_n(u_*)\|_{L^\infty} \;\le\; C\delta^n\,n!\ .$$

In conclusion, although we successfully established *uniform* error estimates for first- and second-order models, it is an open problem whether such estimates should still be valid for higher-order approximations. Theorem 6.1 indicates that the convergence might hold but, probably, in a *weaker* topology.

ACKNOWLEDGEMENTS – The three authors gratefully acknowledge the support and hospitality of the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge, where this research was performed during the Semester Program “Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics” (Jan.–July 2003), organized by C.M. Dafermos, P.G. LeFloch, and E. Toro. PLF was also partially supported by the Centre National de la Recherche Scientifique (CNRS).

REFERENCES

[1] Caflisch, R.E. and Liu, T.P. – Stability of shock waves for the Broadwell equations, *Comm. Math. Phys.,* 114 (1988), 103–130.

[2] Chapman, S. and Cowling, T. – *Mathematical Theory of Non-Uniform Gases*, Cambridge University Press, Cambridge, 1970.

[3] Goodman, J.B. and Majda, A. – The validity of the modified equation for nonlinear shock waves, *J. Comput. Phys.,* 58 (1985), 336–348.

[4] Hayes, B.T. and LeFloch, P.G. – Nonclassical shocks and kinetic relations: finite difference schemes, *SIAM J. Numer. Anal.,* 35 (1998), 2169–2194.

[5] Hou, T.Y. and LeFloch, P.G. – Why nonconservative schemes converge to wrong solutions: Error analysis, *Math. of Comput.,* 62 (1994), 497–530.

[6] Jin, S. and Slemrod, M. – Regularization of the Burnett equations via relaxation, *J. Stat. Phys.,* 103 (2001), 1009–1033.

[7] Klingenberg, C.; Lu, Y. and Zhao, H. – L^1 singular limit for relaxation and viscosity approximation of extended traffic models, *Elect. J. Diff. Equa.,* 23 (2003), 1–11.

[8] LeFloch, P.G. – *Hyperbolic systems of conservation laws: The theory of classical and nonclassical shock waves*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 2002.

[9] Liu, T.P. – Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.,* 108 (1987), 153–175.

[10] Mascia, C. and Natalini, R. – L^1 nonlinear stability of traveling waves for a hyperbolic system with relaxation, *J. Differential Equations,* 132 (1996), 275–292.

[11] Natalini, R. – *Recent results on hyperbolic relaxation problems*, in "Analysis of Systems of Conservation Laws" (Aachen, 1997), Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Boca Raton, FL, 1999, pp. 128–198.

[12] Slemrod, M. – Constitutive relations for monatomic gases based on a generalized rational approximation to the sum of the Chapman–Enskog expansion, *Arch. Rational Mech. Anal.,* 150 (1999), 1–22.

[13] Szepessy, A. – *On the stability of Broadwell shocks*, in "Nonlinear Evolution Partial Differential Equations", (Beijing 1993), AMS/IP Stud. Adv. Math., 3, Amer. Math. Soc., Providence, RI, 1997, pp. 403–412.

Nabil Bedjaoui,
INSSET, Université de Picardie,
48 rue Raspail, 02109 Saint-Quentin – FRANCE
E-mail: bedjaoui@cmap.polytechnique.fr

and

Christian Klingenberg,
Applied Mathematics Department, Würzburg University,
Am Hubland, 97074 Würzburg – GERMANY
E-mail: klingen@mathematik.uni-wuerzburg.de

and

Philippe G. LeFloch,
Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, U.M.R. 7598,
Université Pierre et Marie Curie, 75252 Paris – FRANCE
E-mail: lefloch@ann.jussieu.fr