PORTUGALIAE MATHEMATICA
Vol. 60 Fasc. 4 – 2003
Nova Série

# UNIFORM STABILIZATION AND EXACT CONTROL OF A MULTILAYERED PIEZOELECTRIC BODY

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*Recommended by E. Zuazua*

**Abstract:** A transmission problem for a class of dynamic coupled system of hyperbolic equations having piecewise constant coefficients in a bounded three-dimensional domain is considered. Assuming that in the entire boundary, dissipative mechanisms are present and that suitable geometric conditions on the domain and the interfaces are satisfied, we prove that the total energy associated with the model decays exponentially as $t \to +\infty$. Exact boundary controllability is then obtained through Russell's "controllability via stabilizability" principle.

## 1 – Introduction

This paper is devoted to study the uniform stabilization as $t \to +\infty$ of the solutions of a transmission problem for a class of dynamic coupled system of hyperbolic equations from which a distinguish example is the coupled system of electromagneto-elasticity governed by Maxwell equations and the system of elastic waves. Let $\Omega$ be a bounded region of $\mathbb{R}^3$ with smooth boundary $\partial\Omega = S$.

*Received*: January 1, 2002; *Revised*: July 8, 2002.
*AMS Subject Classification*: 35Q99, 35Q60, 35L99.
*Keywords*: uniform stabilization; transmission problem; piezoelectric body.

We will assume that $\Omega$ is occupied by a multilayered piezoelectric body whose motion is governed by the system (see [4] and [7]):

$$
(1.1)\qquad
\begin{cases}
\rho\, u_{tt} - \displaystyle\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\,\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\,(A_i^* E) \;=\; 0 \\[2ex]
\dfrac{\partial}{\partial t}\left\{DE + \displaystyle\sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\right\} - \operatorname{curl} H \;=\; 0 \\[2ex]
\beta H_t + \operatorname{curl} E \;=\; 0 \\[2ex]
\operatorname{div}\left\{DE + \displaystyle\sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\right\} \;=\; 0 \\[2ex]
\operatorname{div} H \;=\; 0
\end{cases}
$$

in $\Omega \times (0, +\infty)$. Here $x = (x_1, x_2, x_3) \in \Omega$ and $t$ denotes the time variable. In (1.1) we denote by

$$
\begin{aligned}
&u = (u_1, u_2, u_3) \;=\; \text{the displacement vector}\\
&E = (E_1, E_2, E_3) \;=\; \text{the electric field}\\
&H = (H_1, H_2, H_3) \;=\; \text{the magnetic field}\\
&\beta(x) \;=\; \text{the electric permeability}\\
&\rho \;=\; \text{the density}
\end{aligned}
$$

and the $3 \times 3$ matrices $A_{ij}(x)$, $A_i$ and $D(x)$ will satisfy suitable assumptions given below. In the simplest case, when we consider an isotropic medium, then, we will have that

$$
\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\,\frac{\partial}{\partial x_j}\right) = \mu\Delta + (\lambda + \mu)\nabla \operatorname{div}
$$

where $\lambda$ and $\mu$ are the Lame's constants ($\mu > 0$, $\lambda + \mu > 0$), $D$ will be the identity matrix, $\nabla$ the gradient operator, $\Delta$ the (vector) Laplacian,

$$
\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\,(A_i^* E) \;=\; \alpha \operatorname{curl} E
$$

and

$$
\frac{\partial}{\partial t} \sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i} \;=\; -\alpha \operatorname{curl} u_t
$$

where $\alpha$ is a coupling constant. Here $A_i^*$ denotes the adjoint of $A_i$. The coupled system (1.1) is complemented with initial conditions

$$\begin{cases} u(x,0) = f_1(x)\,, \quad u_t(x,0) = f_2(x) \\ E(x,0) = f_3(x)\,, \quad H(x,0) = f_4(x) \end{cases} \quad \text{in } \Omega \tag{1.2}$$

and boundary conditions

$$\begin{cases} \displaystyle\sum_{i,j=1}^{3} A_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E\,\eta_i \;=\; -\,a(x)\,u_t - b(x)\,u \\ \eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta) \;=\; \alpha(x)\,H\mathbf{x}\,\eta \;+\; \gamma(x)\displaystyle\int_0^t \Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(x)\,(t-\tau)\Big)\,d\tau \end{cases} \tag{1.3}$$

on $\partial\Omega \times (0,+\infty)$ where "$\mathbf{x}$" denotes the usual vector product and $\eta = \eta(x)$ denotes the unit outward normal to $\partial\Omega = S$ at $x$. The functions $a(x)$, $b(x)$, $\alpha(x)$, $\gamma(x)$ and $\sigma(x)$ will satisfy suitable conditions given below. In the simplest case they are just positive constants.

Finding uniform rates of decay of the solution of problem (1.1), (1.2) and (1.3) as $t \to +\infty$ is of interest to understand the evolution of the model and consequently for the phenomenon described by it. Even more interesting is the so called transmission problem associated with model (1.1)–(1.3). Let us describe our main result of this article: Let $\Omega \subseteq \mathbb{R}^3$ be as above and consider a finite number of subsets of $\Omega$, $\{B_k\}_{k=1}^n$ which are open, connected, with smooth boundary $\partial B_k = S_k$ and such that $\overline{B}_k \subset B_{k+1}$ for $1 \le k \le n-1$. We denote by $\Omega_0 = B_1$, $\Omega_k = B_{k+1}\backslash\overline{B}_k$ for $k = 1,2,...,n-1$ and $\Omega_n = \Omega\backslash\overline{B}_n$. Now, we consider system (1.1) restricted to each set $\Omega_k \times (0,T)$, $k = 0,1,2,...,n$. We complement (1.1) with the initial data (1.2) also restricted to $\Omega_k$, $k = 0,1,2,...,n$. The boundary conditions on $S \times (0,+\infty)$ are given by (1.3). Furthermore, we will require the following interface conditions to be satisfied

$$\begin{cases} u^{(k-1)} = u^{(k)} \\ \displaystyle\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial u^{(k-1)}}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E^{(k-1)}\eta_i \;=\; \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E^{(k)}\eta_i \\ \eta\,\mathbf{x}\,E^{(k-1)} = \eta\,\mathbf{x}\,E^{(k)} \\ \eta\,\mathbf{x}\,H^{(k-1)} = \eta\,\mathbf{x}\,H^{(k)} \end{cases} \tag{1.4}$$

for any $(x,t) \in S_k \times (0,+\infty)$, $k = 1,2,...,n$. Here, $\eta = \eta(x) = (\eta_1,\eta_2,\eta_3)$ is the unit normal vector pointing the exterior of $B_k$ and $A_{ij}^{(k)}$, $u^{(k)}$, $E^{(k)}$ and $H^{(k)}$ are the restrictions of $A_{ij}$, $u$, $E$ and $H$ to $\Omega_k$ respectively.

We will assume to be valid the following conditions

**HYPOTHESIS I.**

**1)** $A_{ij} = A_{ij}(x)$ are $3 \times 3$ matrices given by $A_{ij}(x) = \left[C^{ij}_{kh}(x)\right]_{3\times 3}$ where

$$C^{ij}_{kh}(x) = (1 - \delta_{ih}\delta_{ik})\, a_{ikjh}(x) + \delta_{ik}\delta_{jh}\, a_{ihjk}(x)$$

with $\delta_{\ell k} = \begin{cases} 1 & \text{if } \ell = k \\ 0 & \text{if } \ell \neq k \end{cases}$ and $a_{ijkh}$ are Cartesian components of the elastic tensor with the symmetric properties

$$a_{ijkh} = a_{jikh} = a_{khij} \ .$$

**2)** $A_i$ and $D = D(x)$ are $3 \times 3$ matrices given by

$$A_i = [e_{khi}] \quad \text{and} \quad D(x) = [d_{ij}(x)]$$

where $e_{khi}$ and $d_{ij}(x)$ are Cartesian components of the piezoelectric and electric permittivity tensors respectively and satisfy the following conditions:

$$d_{kh} = d_{hk}\,, \qquad \sum_{k,h=1}^{3} d_{kh}\, \xi_k \xi_h \ \geq\ d_0\, |\xi|^2$$

for some $d_0 > 0$ and any vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$.

**3)** The matrices $A_{ij}(x)$ satisfy the condition

$$\sum_{i,j=1}^{3} A_{ij}(x)\, v_j \cdot v_i \ \geq\ c_0 \sum_{i=1}^{3} |v_i|^2$$

for some $c_o > 0$ and any vector $v_i = (v_i^1, v_i^2, v_i^3) \in \mathbb{R}^3$. Here the dot $\cdot$ denotes the inner product in $\mathbb{R}^3$.

**4)** We assume that $a_{ijkh}(x)$, $d_{ij}(x)$ and $\beta(x) > 0$ are piecewise constant functions which lose continuity only on $S_1, S_2, ..., S_n$ .

**5)** $\rho$ and $e_{khi}$ are real constants, $\rho > 0$.

**6)** The functions $a = a(x)$, $b = b(x)$, $\alpha(x)$, $\gamma(x)$ and $\sigma(x)$ are real-valued and continuously differentiable functions on $S = \partial\Omega$. Furthermore, $a > 0$, $b > 0$, $\alpha > 0$, $\gamma \geq 0$ and $\sigma > 0$ for all $x \in S$. $\square$

Observe that from the symmetry of the $a_{ijkh}$ it follows that $A^*_{ij} = A_{ji}$. Also, for an isotropic medium, the constants $a_{ijkh}$ are given by

$$a_{ijkh} \;=\; \lambda\,\delta_{ij}\delta_{kh} + \mu\,(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$$

where $\lambda$ and $\mu$ are the Lame's constants. Furthermore, assumption 3) in Hypothesis I holds for an isotropic medium with the constant $c_o = \mu > 0$. In fact, in that case, direct calculation shows that

$$\sum_{i,j=1}^{3} A_{ij}\, v_j \cdot v_i \;=\; (\lambda+\mu)\left(\sum_{i=1}^{3} v_i^i\right)^2 + \mu \sum_{i,j=1}^{3} (v_i^j)^2 \;\geq\; \mu \sum_{i=1}^{3} |v_i|^2 \,.$$

Let $\{u, E, H\}$ be the global solution of problem (1.1) satisfying the initial conditions (1.2), the boundary conditions (1.3) and the interface conditions (1.4). We consider the (total) energy $\mathcal{E}(t)$ given by

$$\begin{aligned} \mathcal{E}(t) \;=\; & \sum_{k=0}^{n} \int_{\Omega_k} \Big\{ \rho\,|u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)}\, \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \\ & + D^{(k)} E^{(k)} \cdot E^{(k)} + \beta^{(k)} |H^{(k)}|^2 \Big\}\, dx \\ & + \int_S \Big\{ b\,|u^{(n)}|^2 + \gamma\, \Big| \int_0^t \Big[ H(x,\tau)\,\mathbf{x}\,\eta \Big] \exp\Big(-\sigma(t-\tau)\Big)\, d\tau \Big|^2 \Big\}\, dS \end{aligned} \tag{1.5}$$

where $\beta^{(n)} = \beta$ and $u^{(n)} = u$.

We (formally) calculate the derivative of $\mathcal{E}(t)$, use the equations together with the boundary conditions as well as the interface conditions to obtain that

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} \;=\; & -2 \int_S \Big\{ a\,|u_t|^2 + \alpha\,|H\,\mathbf{x}\,\eta|^2 \\ & + \sigma\,\gamma\, \Big| \int_0^t \Big[ H(x,\tau)\,\mathbf{x}\,\eta \Big] \exp\Big(-\sigma(t-\tau)\Big)\, d\tau \Big|^2 \Big\}\, dS \,. \end{aligned} \tag{1.6}$$

Thus

$$\frac{d\mathcal{E}(t)}{dt} \;\leq\; 0 \,.$$

Assuming suitable geometric conditions on $\Omega$ (and $S_k$) as well as monotonicity assumptions on the coefficients of the system, we are able to prove that

$$\mathcal{E}(t) \;\leq\; c\, \exp(-wt)\, \mathcal{E}(0)$$

for any $t \geq 0$ where $c$ and $w$ are positive constants.

As an application of the above result, we study the following exact controllability problem: Assume that $\gamma \equiv 0$. Given a time $T > 0$, the initial distribution $F(x) = (f_1(x), f_2(x), f_3(x), f_4(x))$ and a desired terminal state $G(x) = (g_1(x), g_2(x), g_3(x), g_4(x))$ where $F$ and $G$ belong to an appropriate function space, to find vector-valued functions $\vec{p}(x,t)$ and $\vec{q}(x,t)$ such that the solution of (1.1), (1.2) and (1.4) with boundary conditions

$$\left\{ \begin{array}{l} \displaystyle\sum_{i,j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* E \, \eta_i + b \, u = \vec{p}(x,t) \\ \eta \, \mathbf{x} \, E = \vec{q}(x,t) \end{array} \right. \tag{1.7}$$

on $S \times (0,T)$, satisfy

$$u(x,T) = g_1(x), \quad u_t(x,T) = g_2(x), \quad E(x,T) = g_3(x), \quad H(x,T) = g_4(x) \; . \tag{1.8}$$

Let us mention some bibliographical comments: Boundary controllability in transmission problems for the wave equation has been considered by J.-L. Lions [22] and S. Nicaise in [24] and [25]. Uniform stabilization and exact control for the Maxwell system in multilayered media were studied by B. Kapitonov in [10]. Boundary controllability in transmission problems for a class of second order hyperbolic systems has been studied by J. Lagnese [18]. Stabilization and exact boundary controllability for the system of elasticity were considered by J. Lagnese [17], [18], F. Alabau and V. Komornik [1] and M. Horn [6] among others. The exact controllability problem for the Maxwell system has been studied by D. Russell [27] for a circular cylindrical region, by K. Kime [14] for a spherical region and by J. Lagnese for a general region. In [9] and [19] the exact controllability problem has been studied by means of the Hilbert Uniqueness Method introduced by J.-L. Lions [20], [21]. Uniform exponential decay of solutions of Maxwell's equation with boundary dissipation was proved by B. Kapitonov in [10] and [11], including the uniform "simultaneous" stabilization for a pair of Maxwell's equations.

The results obtained in this article generalize previous work of the authors [12], [13] where a transmission problem was considered either for Maxwell system with boundary conditions with memory and for the system of electromagneto-elasticity.

Let us describe the sections of this paper: Solvability of (1.1)–(1.4) in the appropriate class of functions is shown in Section 2. This is done via semigroup theory and the main technical difficulty comes from the memory term (see (1.3)) on $\partial\Omega \times (0, +\infty)$. In Section 3 we prove the uniform exponential decay of the energy $\mathcal{E}(t)$ via the multiplier method. At this point, we needed to modified "slightly" the usual multipliers in order to take care of the additional boundary terms which appear after integration (in space) of the fundamental identity. We also needed to assume suitable geometric conditions on $\Omega$ and $S_k$ as well as some monotonicity assumptions on $A_{ij}^{(k)}$, $D^{(k)}$ and $\beta^{(k)}$. In the last section, the controllability problem (1.1), (1.2), (1.4), (1.7)–(1.8) when $\gamma \equiv 0$ is solved.

Since $\rho$ is a positive constant we may assume without lost of generality that $\rho \equiv 1$. When studying system (1.1) the restrictions of $u$, $E$, $H$, $\beta$, $A_{ij}$, $D$, to $\Omega_k$ $(k = 0, 1, 2, ..., n-1)$ will be denoted by $u^{(k)}$, $E^{(k)}$, $H^{(k)}$, etc. When $k = n$ we will write $u^{(n)} = u$, $E^{(n)} = E$, etc. At each point $x$ belonging to one of the boundaries $S = \partial\Omega, S_1, S_2, ..., S_n$ , the unit normal vector pointing the exterior will be denote by $\eta = \eta(x)$ and its components by $\eta_i$ . We use the standard notations, for example $H^m(\Omega)$ and $H^r(\partial\Omega)$ will denote the Sobolev spaces of order $m$ and $r$ on $\Omega$ and $\partial\Omega$ respectively. The norm of a vector $v \in \mathbb{R}^3$ will be denote by $|v|$. Due to the techniques we use in this article (the multiplier method) in order to achieve the result on the exponential decay we needed to assume that $b = b(x)$ is bounded above by a suitable constant (see (3.15) in Theorem 3.5). This is, apparently, a limitation of the method.

We conclude this introduction with some comments on the boundary conditions (1.3). The second line in (1.3) combines the so-called Leontovich's boundary condition (when $\gamma \equiv 0$) and a dissipative term of memory type with an exponentially decaying kernel. A boundary condition in electromagnetism with memory was introduced by M. Fabrizio and A. Morro in [5]. Later V. Berti [2] studied the asymptotic stability of such models. When $\gamma \equiv 0$ and $\alpha(x) > 0$ then, Leontovich's boundary condition is also of dissipative type. In V. Komornik's book [16] (pg. 120) a nice geometrical meaning of such boundary condition is given in case $\alpha \equiv 1$: The tangential component of the magnetic field $H$ is obtained from the tangential component of the electrical field $E$ by a rotation of angle 90° in the positive direction in the tangent plane. The term $-a(x)u_t$ in the first line of (1.3) is also a dissipative mechanism and the left hand side (of the first line of (1.3)) could be interpreted as an stress tensor for the system at the boundary $S$.

## 2 – Well-posedness

In this section we will prove the well-posedness of problem (1.1)–(1.4) using semigroup theory. The main (technical) difficulty arises from the memory term appearing on the boundary condition (1.3).

Let us consider the Hilbert space $X$ consisting on triples $v = (v_1, v_2, v_3)$ of three-component vector-value functions $v_j(x)$ such that

$$v_1, v_2 \in [L^2(\Omega_k)]^3\,, \qquad k = 0, 1, 2, ..., n\;,$$

$$\operatorname{curl} v_1, \operatorname{curl} v_2 \in [L^2(\Omega_k)]^3\,, \qquad k = 0, 1, 2, ..., n$$

and

$$v_3 \in [L^2(S)]^3\,, \qquad v_3 \cdot \eta = 0 \;\text{ on } S\;.$$

We define the inner product in $X$ as follows. If $v, w \in X$, then

$$\begin{aligned}(v,w)_X \;=\; \sum_{k=0}^{n} \int_{\Omega_k} \Big\{ &\operatorname{curl} v_1^{(k)} \cdot \operatorname{curl} w_1^{(k)} + \operatorname{curl} v_2^{(k)} \cdot \operatorname{curl} w_2^{(k)} \\ &+ D^{(k)} v_1^{(k)} \cdot w_1^{(k)} + \beta^{(k)} v_2^{(k)} \cdot w_2^{(k)} \Big\}\, dx + \int_S \gamma\, v_3 \cdot w_3 \, dS\;.\end{aligned}$$

The following lemma was proved in [12] (see also B.V. Kapitonov [8]):

**Lemma 2.1.** *Assume that $\alpha(x)$ and $\gamma(x)$ belong to $C^1(S)$. Then, the mapping*

$$u = (u_1, u_2, u_3) \;\mapsto\; u_1 - \eta(u_1 \cdot \eta) - \alpha\, u_2 \times \eta - \gamma\, u_3$$

*from $[\widetilde{C}^1(\overline{\Omega})]^3 = \{u^{(k)} \in [C^1(\overline{\Omega}_k)]^3,\; k = 0, 1, 2, ..., n\}$ into $[C^1(S)]^3$ extends by continuity to a continuous linear mapping from $X$ into $[H^{-1/2}(S)]^3$ which we also denote by*

$$u \;\mapsto\; u_1 - \eta(u_1 \cdot \eta) - \alpha\, u_2 \times \eta - \gamma\, u_3 \;\equiv\; w(u; \alpha, \gamma)\;.$$

**Remark 2.2.** Well known results (see for instance the book of G. Duvaut and J.-L. Lions [3]) imply that for any $u \in X$, the expressions $\eta \times u_1$ and $\eta \times u_2$ where $\eta = \eta(x)$ is the unit normal vector pointing the exterior of $S_k$, are well defined on $S_k$ and belong to $[H^{-1/2}(S_k)]^3$. □

Lemma 2.1 and Remark 2.2 make it possible to introduce in $X$, the closed subspace

$$\begin{aligned}V = \Big\{ &u = (u_1, u_2, u_3) \in X \;\text{ such that} \\ &\eta \times u_1^{(k-1)} = \eta \times u_1^{(k)},\;\; \eta \times u_2^{(k-1)} = \eta \times u_2^{(k)} \;\text{ on } S_k\,,\;\; k = 1, 2, ..., n \\ &\text{and }\; u_1 - \eta(u_1 \cdot \eta) - \alpha\, u_2 \times \eta - \gamma\, u_3 = 0 \;\text{ on } S \Big\}\;.\end{aligned}$$

Let us denote by $Z$ the (real) Hilbert space which consists of all elements $w = (w_1, w_2, w_3, w_4, w_5)$ of three-component vector-valued functions $w_j(x)$ such that $w_1^{(k)} \in [H^1(\Omega_k)]^3$, $w_2^{(k)}, w_3^{(k)}, w_4^{(k)} \in [L^2(\Omega_k)]^3$, $k = 0, 1, ..., n$, $w_5 \in [L^2(S)]^3$, $w_1^{(k)} = w_1^{(k-1)}$ on $S_k$, $k = 1, 2, ..., n$. The inner product in $Z$ is given by: If $w, v \in Z$, then

$$
\begin{aligned}
(w, v)_Z \;=\; & \sum_{k=0}^{n} \int_{\Omega_k} \Bigg\{ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \cdot \frac{\partial v_1^{(k)}}{\partial x_i} + w_2^{(k)} \cdot v_2^{(k)} \\
& + D^{(k)} w_3^{(k)} \cdot v_3^{(k)} + \beta^{(k)} w_4^{(k)} \cdot v_4^{(k)} \Bigg\} dx \\
& + \int_S (b\, w_1 \cdot v_1 + \gamma\, w_5 \cdot v_5)\, dS \;.
\end{aligned}
$$

The norm in the space $Z$ will be denote by $\|\cdot\|_Z = (\,\cdot\,,\,\cdot\,)_Z^{1/2}$. In $Z$ we define the unbounded operator $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ which consists of all elements $w = (w_1, w_2, w_3, w_4, w_5) \in Z$ such that, for $k = 1, 2, ..., n$

$$
\begin{aligned}
& \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} - \sum_{i=1}^{3} A_i^* w_3^{(k)} \in [H^1(\Omega_k)]^3 \\
& \qquad w_2^{(k)} \in [H^1(\Omega_k)]^3, \quad (w_3, w_4, w_5) \in V, \quad w_4 \mathbf{x}\, \eta \in [L^2(S)]^3 \\
& \sum_{i,j=1}^{3} A_{ij} \frac{\partial w_1}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3\, \eta_i + a\, w_2 + b\, w_1 = 0 \quad \text{on } S \\
& \qquad w_2^{(k-1)} = w_2^{(k)} \quad \text{on } S
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i,j=1}^{3} A_{ij}^{(k-1)} \frac{\partial w_1^{(k-1)}}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3^{(k-1)} \eta_i \;= \\
& \qquad = \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3^{(k)} \eta_i \quad \text{on } S_k\,, \quad k = 1, 2, ..., n\,,
\end{aligned}
$$

then $\mathcal{A}$: $\mathcal{D}(\mathcal{A}) \subseteq Z \mapsto Z$ is defined as

$$
\begin{aligned}
\mathcal{A} w \;=\; & \Bigg( w_2\,,\; \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \Big( A_{ij} \frac{\partial w_1}{\partial x_j} \Big) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} A_i^* w_3\,,\; D^{-1} \Big( \operatorname{curl} w_4 - \sum_{i=1}^{3} A_i \frac{\partial w_2}{\partial x_i} \Big)\,, \\
& - \beta^{-1} \operatorname{curl} w_3\,,\;\; w_4 \mathbf{x}\, \eta - \sigma\, w_5 \Bigg)
\end{aligned}
$$

whenever $w = (w_1, w_2, w_3, w_4, w_5) \in \mathcal{D}(\mathcal{A})$.

Next, we consider the adjoint operator $\mathcal{A}^*$. We can verified in a similar manner as in [12] that the domain of $\mathcal{A}^*$ coincides with the following subspace

$$\mathcal{D}(\mathcal{A}^*) = \Big\{ v=(v_1,v_2,v_3,v_4,v_5)\in Z \ \text{ such that } \ v_2^{(k)}\in[H^1(\Omega_k)]^3, \ \ (v_3,v_4,v_5)\in\widetilde{V},$$

$$\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_j} - \sum_{i=1}^{3} A_i^* v_3^{(k)} \in [H^1(\Omega_k)]^3, \quad v_4\,\mathbf{x}\,\eta\in[L^2(S)]^3,$$

$$\sum_{i,j=1}^{3} A_{ij}\frac{\partial v_1}{\partial x_j}\eta_i - \sum_{i=1}^{3} A_i^* v_3\,\eta_i - a\,v_2 + b\,v_1 = 0 \quad \text{on } S\,,$$

$$v_2^{(k-1)} = v_2^{(k)} \quad \text{on } S_k\,, \quad k=1,2,...,n\,,$$

$$\sum_{i,j=1}^{3} A_{ij}^{(k-1)}\frac{\partial v_1^{(n-1)}}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* v_3^{(k-1)}\eta_i = \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* v_3^{(k)}\eta_i$$

$$\text{on } S_k\,, \quad k=1,2,...,n \Big\}$$

where $\widetilde{V}$ is as in the definition of $V$ with $-\alpha(x)$ instead of $\alpha(x)$. Given $v = (v_1,v_2,v_3,v_4,v_5)\in\mathcal{D}(\mathcal{A}^*)$ then, we have that

$$\mathcal{A}^*v = -\Bigg( v_2,\ \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\Big(A_{ij}\frac{\partial v_1}{\partial x_j}\Big) - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}A_i^* v_3\,,\ \ D^{-1}\Big(\operatorname{curl} v_4 - \sum_{i=1}^{3} A_i\frac{\partial v_2}{\partial x_i}\Big),$$
$$-\,\beta^{-1}\operatorname{curl} v_3\,,\ \ v_4\,\mathbf{x}\,\eta+\sigma\,v_5\Bigg)\ .$$

We note that the operator $\mathcal{A}$ is closed since it coincides with the adjoint operator of $\mathcal{A}^*$. Clearly $\mathcal{A}$ is densely defined. Furthermore, we have

**Lemma 2.3.** *Assuming Hypothesis I given in the introduction (with $\rho=1$), then, the operators $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, that is,*

$$(\mathcal{A}w,w)_Z \le 0 \quad \text{for any } \ w\in\mathcal{D}(\mathcal{A}) \tag{2.1}$$

*and*

$$(\mathcal{A}^*v,v)_Z \le 0 \quad \text{for any } \ v\in\mathcal{D}(\mathcal{A}^*)\ . \tag{2.2}$$

**Proof:** It is enough to prove (2.1) for a dense subset of $\mathcal{D}(\mathcal{A})$. In fact, the set of piecewise smooth vector-valued functions $w = (w_1, w_2, w_3, w_4, w_5) \in \mathcal{D}(\mathcal{A})$ such that $w_1 \in [C^2(\Omega_k)]^3$, $w_2, w_3, w_4 \in [C^1(\Omega_k)]^3$, $w_5 \in [C(S)]^3$, $k = 0, 1, 2, ..., n$ is dense in $\mathcal{D}(\mathcal{A})$. Let $w$ be an element of such dense subset. Taking the inner product of $\mathcal{A}w$ with $w$ in $Z$ and using the divergence theorem we obtain that

$$
\begin{aligned}
(\mathcal{A}w, w)_Z \;=\; & \sum_{k=0}^{n} \int_{\Omega_k} \Bigg\{ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_2^{(k)}}{\partial x_j} \cdot \frac{\partial w_1^{(k)}}{\partial x_i} \\
& \quad + \left[ \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left( A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \right) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (A_i^* w_3^{(k)}) \right] \cdot w_2^{(k)} \\
& \quad + \left[ \operatorname{curl} w_4^{(k)} - \sum_{i=1}^{3} A_i \frac{\partial w_2^{(k)}}{\partial x_i} \right] \cdot w_3^{(k)} - \operatorname{curl} w_3^{(k)} \cdot w_4^{(k)} \Bigg\} \, dx \\
& + \int_S \Big\{ b\, w_2 \cdot w_1 \,+\, \gamma (w_4 \mathbf{x}\, \eta - \sigma\, w_5) \cdot w_5 \Big\} \, dS \\
=\; & \sum_{i=1}^{3} \int_{S_k} \Bigg\{ \left[ \sum_{i,j=1}^{3} A_{ij}^{(k-1)} \frac{\partial w_1^{(k-1)}}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3^{(k-1)} \eta_i \right] \cdot w_2^{(k-1)} \\
& \quad - \left[ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3^{(k)} \eta_i \right] \cdot w_2^{(k)} \\
& \quad + \; w_4^{(k-1)} \cdot (w_3^{(k-1)} \mathbf{x}\, \eta) \;-\; w_4^{(k)} \cdot (w_3^{(k)} \mathbf{x}\, \eta) \Bigg\} \, dS_k \\
& + \int_S \Bigg\{ \left[ \sum_{i,j=1}^{3} A_{ij} \frac{\partial w_1}{\partial x_j} \eta_i - \sum_{i=1}^{3} A_i^* w_3\, \eta_i \right] \cdot w_2 \\
& \quad + \; w_4 \cdot (w_3 \mathbf{x}\, \eta) \,+\, b\, w_2 \cdot w_1 + \gamma (w_4 \mathbf{x}\, \eta - \sigma\, w_5) \cdot w_5 \Bigg\} \, dS \\
=\; & \int_S \Big\{ (-\, a\, w_2 - b\, w_1) \cdot w_2 \,+\, w_4 \cdot (w_3 \mathbf{x}\, \eta) \,+\, b\, w_2 \cdot w_1 \\
& \quad + \; \gamma (w_4 \mathbf{x}\, \eta - \sigma\, w_5) \cdot w_5 \Big\} \, dS \; .
\end{aligned} \tag{2.3}
$$

Now, we use the fact that $(w_3, w_4, w_5) \in V$. Therefore

$$
w_3 - \eta (w_3 \cdot \eta) - \alpha\, w_4 \mathbf{x}\, \eta - \gamma\, w_5 \;=\; 0 \qquad \text{on} \;\; S
$$

which together with the fact that $w_5 \cdot \eta = 0$ on $S$ give us that

$$
\begin{aligned}
&(-aw_2 - bw_1) \cdot w_2 + w_4 \cdot (w_3 \,\mathbf{x}\, \eta) + bw_2 \cdot w_1 + \gamma(w_4 \,\mathbf{x}\, \eta - \sigma w_5) \cdot w_5 = \\
&\quad = -a|w_2|^2 - bw_1 \cdot w_2 + w_3 \cdot (\eta \,\mathbf{x}\, w_4) + bw_2 \cdot w_1 + \gamma(w_4 \,\mathbf{x}\, \eta) \cdot w_5 - \gamma\sigma|w_5|^2 \\
&\quad = -a|w_2|^2 - \gamma\sigma|w_5|^2 + (w_4 \,\mathbf{x}\, \eta) \cdot (\gamma w_5 - w_3) \\
&\quad = -a|w_2|^2 - \gamma\sigma|w_5|^2 + w_4 \,\mathbf{x}\, \eta \cdot \Big[-\eta(w_3 \cdot \eta) - \alpha w_4 \,\mathbf{x}\, \eta\Big] \\
&\quad = -a|w_2|^2 - \gamma\sigma|w_5 \,\mathbf{x}\, \eta|^2 - \alpha|w_4 \,\mathbf{x}\, \eta|^2 \, .
\end{aligned}
\tag{2.4}
$$

Therefore, from (2.3) and (2.4) we obtain that

$$
(\mathcal{A}w, w)_Z = -\int_S \Big[a|w_2|^2 + \gamma\sigma|w_5 \,\mathbf{x}\, \eta|^2 + \alpha|w_4 \,\mathbf{x}\, \eta|^2\Big]\, dS \le 0 \, .
$$

The proof that (2.2) also holds for $\mathcal{A}^*$ can be done in a similar way.

Therefore, $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operators and clearly $\mathcal{A}$ is a densely defined closed operator. We use a classical result (see [26], Corollary I.4.4, which says "Let $A$ be a densely defined closed linear operator. If both $A$ and $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on the Hilbert space $Z$")) to conclude that $\mathcal{A}$ is a generator of a $C^0$ semigroup of contractions $\{U(t)\}_{t\ge 0}$ on $Z$. ∎

**Lemma 2.4.** *Let $M_1$ be the orthogonal complement of the subspace $M = \{v \in \mathcal{D}(\mathcal{A}^*)$ such that $\mathcal{A}v^* = 0\}$ in $Z$. Assume Hypothesis I given in the Introduction (with $\rho = 1$). Then, the following properties are valid:*

**1)** *$U(t)$ takes $M_1 \cap \mathcal{D}(\mathcal{A})$ into itself.*

**2)** *Any element $w = (w_1, w_2, w_3, w_4, w_5) \in M_1 \cap \mathcal{D}(\mathcal{A})$ has the following property*

$$
\operatorname{div}\left\{D^{(k)} w_3^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k)}}{\partial x_i}\right\} = 0\,, \quad \operatorname{div} w_4^{(k)} = 0\,, \qquad k = 0, 1, ..., n
$$

*in the sense of distributions.*

**3)** *Any element $w = (w_1, w_2, w_3, w_4, w_5) \in M_1 \cap \mathcal{D}(\mathcal{A})$ satisfies the additional interface conditions*

$$
\beta^{(k-1)}\, w_4^{(k-1)} \cdot \eta = \beta^{(k)}\, w_4^{(k)} \cdot \eta
$$

$$
\left(D^{(k-1)} w_3^{(k-1)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k-1)}}{\partial x_i}\right) \cdot \eta = \left(D^{(k)} w_3^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k)}}{\partial x_i}\right) \cdot \eta
$$

*for any $x \in S_k\,,\ k = 1, 2, ..., n$.*

**Proof:** First, we observe that the kernel of $\mathcal{A}^*$ is nonempty. In fact, it contains elements of the form $v=(v_1,0,\nabla\varphi_1,\nabla\varphi_2,0)$ where $\varphi_1$ and $\varphi_2$ belong to $H^2(\Omega)\cap H_0^1(\Omega)$ and $v_1$ is a solution of the following problem

$$\text{(2.5)}\quad \begin{cases} \displaystyle\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(A_{ij}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_j}\right)=\sum_{i=1}^{3}\frac{\partial}{\partial x_i}A_i^*\,\nabla\varphi_1 \quad \text{in } \Omega_k\,,\quad k=0,1,...,n\\ \qquad v_1^{(k-1)}=v_1^{(k)}\,,\\ \displaystyle\sum_{i,j=1}^{3}A_{ij}^{(k-1)}\frac{\partial v_1^{(k-1)}}{\partial x_j}\,\eta_i=\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_j}\,\eta_i \quad \text{on } S_k\,,\quad k=1,2,...,n\,,\\ \displaystyle\sum_{i,j=1}^{3}A_{ij}\frac{\partial v_1}{\partial x_j}\,\eta_i+bv_1=\sum_{i=1}^{3}A_i^*\eta_i\nabla\varphi_1 \quad \text{on } S\,. \end{cases}$$

The proof of 1) is simple. Indeed, if $v\in\mathrm{Ker}(\mathcal{A}^*)$ and $w\in M_1\cap\mathcal{D}(\mathcal{A})$, then

$$\frac{d}{dt}(U(t)w,v)_Z=(\mathcal{A}U(t)w,v)_Z=(U(t)w,\mathcal{A}^*v)_Z=0$$

which proves 1). Now, let us prove 2): We will prove that

$$\text{(2.6)}\qquad \int_{\Omega_k}\left[D^{(k)}w_3^{(k)}+\sum_{i=1}^{3}A_i\,\frac{\partial w_1^{(k)}}{\partial x_i}\right]\cdot\nabla\varphi_1\,dx=0$$

for an arbitrary $\varphi_1\in H^2(\Omega)$ with support contained in $\Omega_k$. Clearly (2.6) implies that

$$\mathrm{div}\left\{D^{(k)}w_3^{(k)}+\sum_{i=1}^{3}A_i\,\frac{\partial w_1^{(k)}}{\partial x_i}\right\}=0$$

in the sense of distributions. Let us take any such $\varphi_1$ and $v_1$ a solution of problem (2.5). We consider the element

$$\tilde v=(v_1,0,\nabla\varphi_1,0,0)$$

which belongs to the kernel of $\mathcal{A}^*$. Then, for any

$$w=(w_1,w_2,w_3,w_4,w_5)\in M_1\cap\mathcal{D}(\mathcal{A})$$

we have that

$$\text{(2.7)}\qquad \begin{aligned} 0=(w,\tilde v)_Z&=\sum_{k=0}^{n}\int_{\Omega_k}\left\{\sum_{i,j=1}^{3}A_{ij}^{(k)}\,\frac{\partial w_1^{(k)}}{\partial x_j}\cdot\frac{\partial v_1^{(k)}}{\partial x_i}+D^{(k)}w_3^{(k)}\cdot\nabla\varphi_1\right\}dx\\ &\quad+\int_S b\,w_1\cdot v_1\,dS\,. \end{aligned}$$

However, using the divergence theorem and (2.5) we deduce that

$$\sum_{k=0}^{n}\int_{\Omega_k}\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial w_1^{(k)}}{\partial x_j}\cdot\frac{\partial v_1^{(k)}}{\partial x_i}\,dx\;=$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\int_{\Omega_k}-\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\left(A_{ji}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_i}\right)\cdot w_1^{(k)}\,dx\\
&\quad+\sum_{k=1}^{n}\int_{S_k}\left\{\sum_{i,j=1}^{3}A_{ji}^{(k-1)}\frac{\partial v_1^{(k-1)}}{\partial x_i}\eta_j\cdot w_1^{(k-1)}-\sum_{i,j=1}^{3}A_{ji}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_i}\eta_j\cdot w_1^{(k)}\right\}dS_k\\
&\quad+\int_S\sum_{i,j=1}^{3}A_{ji}\frac{\partial v_1}{\partial x_i}\eta_j\cdot w_1\,dS\\
&= \sum_{k=0}^{n}\int_{\Omega_k}-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}A_i^*\nabla\varphi_1\cdot w_1^{(k)}\,dx\\
&\quad+\sum_{k=1}^{3}\int_{S_k}\left\{\sum_{i,j=1}^{3}A_{ij}^{(k-1)}\frac{\partial v_1^{(k-1)}}{\partial x_j}\eta_i-\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial v_1^{(k)}}{\partial x_j}\eta_i\right\}\cdot w_1^{(k)}\,dS_k\\
&\quad+\int_S\left\{\sum_{i=1}^{3}A_i^*\eta_i\nabla\varphi_1-bv_1\right\}\cdot w_1\,dS\\
&= \int_{\Omega_k}\sum_{i=1}^{3}A_i\frac{\partial w_1^{(k)}}{\partial x_j}\cdot\nabla\varphi_1\,dx-\int_S b\,w_1\cdot v_1\,dS\,.
\end{aligned}\tag{2.8}$$

Substitution of (2.8) into (2.7) completes the proof of (2.6). It can be shown in a similar way that $\operatorname{div} w_4^{(k)}=0$ by taking in this case $\tilde v=(0,0,0,\nabla\varphi_2,0)$ where $\varphi_2$ is an arbitrary element of $H^2(\Omega)$ with support in $\Omega_k$.

Finally, let us prove 3): Since $\tilde v=(0,0,0,\nabla\varphi_2,0)$ belongs to the kernel of $\mathcal{A}^*$ for an arbitrary $\varphi_2\in H^2(\Omega)\cap H_0^1(\Omega)$ it follows that for $w\in M_1\cap\mathcal{D}(\mathcal{A})$ we have that

$$\begin{aligned}
0\;=\;(w,\tilde v)_Z\;&=\;\sum_{k=0}^{n}\int_{\Omega_k}\beta^{(k)}\,w_4^{(k)}\cdot\nabla\varphi_2\,dx\\
&=\int_{S_1}\beta^{(0)}\,w_4^{(0)}\cdot\eta\,\varphi_2\,dS_1-\int_{S_1}\beta^{(1)}\,w_4^{(1)}\cdot\eta\,\varphi_2\,dS_1+\cdots\\
&\quad+\int_{S_n}\beta^{(n-1)}\,w_4^{(n-1)}\cdot\eta\,\varphi_2\,dS_n-\int_{S_n}\beta^{(n)}\,w_4^{(n)}\cdot\eta\,\varphi_2\,dS_n\,.
\end{aligned}$$

Now, we choose $\varphi_2$ such that $\varphi_2\equiv0$ on $S_1,...,S_{k-1},S_{k+1}$. Then

$$\int_{S_k}\left\{\beta^{(k-1)}w_4^{(k-1)}\cdot\eta-\beta^{(k)}w_4^{(k)}\cdot\eta\right\}\varphi_2\,dS_k\;=\;0$$

which implies that

$$\beta^{(k-1)}\, w_4^{(k-1)} \cdot \eta = \beta^{(k)} w_4^{(k)} \cdot \eta \quad \text{on } \ S_k\,, \quad k = 1, 2, ..., n\ .$$

Now, elements of the form $\tilde{v} = (v_1, 0, \nabla\varphi_1, 0, 0)$ belong to the kernel of $\mathcal{A}^*$ for an arbitrary $\varphi_1 \in H_0^2(\Omega)$ with $v_1$ being a solution of (2.5). Thus, for any $w \in M_1 \cap \mathcal{D}(\mathcal{A})$ we have that

$$\begin{aligned} 0 \ = \ (w, \tilde{v})_Z \ &= \sum_{k=0}^{n} \int_{\Omega_k} \left\{ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \cdot \frac{\partial v_1^{(k)}}{\partial x_i} + D^{(k)} w_3^{(k)} \cdot \nabla\varphi_1 \right\} dx \\ &\quad + \int_S b\, w_1 \cdot v_1\, dS\ . \end{aligned} \tag{2.9}$$

Using the divergence theorem and (2.5) we deduce that

$$\begin{aligned} &\sum_{k=0}^{n} \int_{\Omega_k} \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial w_1^{(k)}}{\partial x_j} \cdot \frac{\partial v_1^{(k)}}{\partial x_i}\, dx \ = \\ &\quad = \sum_{k=0}^{n} \int_{\Omega_k} -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \left( A_{ji}^{(k)} \frac{\partial v_1^{(k)}}{\partial x_i} \right) \cdot w_1^{(k)}\, dx \\ &\quad + \sum_{k=1}^{n} \int_{S_k} \left\{ \sum_{i,j=1}^{3} A_{ji}^{(k-1)} \frac{\partial v_1^{(k-1)}}{\partial x_i} \eta_j \cdot w_1^{(k-1)} - \sum_{i,j=1}^{3} A_{ji}^{(k)} \frac{\partial v_1^{(k)}}{\partial x_i} \eta_j \cdot w_1^{(k)} \right\} dS_k \\ &\quad + \int_S \sum_{i,j=1}^{3} A_{ji} \frac{\partial v_1}{\partial x_i} \eta_j \cdot w_1\, dS \\ &\quad = \sum_{k=0}^{n} \int_{\Omega_k} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} A_i^* \nabla\varphi_1 \cdot w_1^{(k)}\, dx + \int_S -b v_1 \cdot w_1\, dS \\ &\quad = \sum_{k=0}^{n} \int_{\Omega_k} \sum_{i=1}^{3} A_i \frac{\partial}{\partial x_i} w_1^{(k)} \cdot \nabla\varphi_1\, dx - \int_S b\, v_1 \cdot w_1\, dS\ . \end{aligned} \tag{2.10}$$

Substitution of (2.10) into (2.9) implies that

$$\begin{aligned} 0 \ = \ (w, \tilde{v})_Z \ &= \sum_{k=0}^{n} \int_{\Omega_k} \left\{ D^{(k)} w_3^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k)}}{\partial x_i} \right\} \cdot \nabla\varphi_1\, dx \\ &= \sum_{k=1}^{n} \int_{S_k} \left[ D^{(k-1)} w_3^{(k-1)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k-1)}}{\partial x_i} \right] \cdot \eta\, \varphi_1\, dS_k \\ &\quad - \sum_{k=1}^{n} \int_{S_k} \left[ D^{(k)} w_3^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k)}}{\partial x_i} \right] \cdot \eta\, \varphi_1\, dS_k\ . \end{aligned} \tag{2.11}$$

Now, we choose $\varphi_1$ such that $\varphi_1 \equiv 0$ on $S_1, ..., S_{k-1}, S_{k+1}, ..., S_n$ and obtain from (2.11) that

$$\left[D^{(k-1)} w_3^{(k-1)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k-1)}}{\partial x_i}\right] \cdot \eta = \left[D^{(k)} w_3^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial w_1^{(k)}}{\partial x_i}\right] \cdot \eta$$

on $S_k$, $k = 1, 2, ..., n$, which completes the proof of Lemma 2.4. ∎

**Theorem 2.5.** *Let $M_1$ be the orthogonal complement of the subspace $\{w \in \mathcal{D}(\mathcal{A}^*)$ such that $\mathcal{A}^* w = 0\}$ in $Z$. Assume Hypothesis I given in the Introduction (with $\rho = 1$) and let $f = (f_1, f_2, f_3, f_4, 0) \in M_1 \cap \mathcal{D}(\mathcal{A})$ then, there exists a unique solution $\{u, E, H\}$ of problem (1.1)–(1.4) such that*

$$\beta^{(k-1)} H^{(k-1)} \cdot \eta = \beta^{(k)} H^{(k)} \cdot \eta$$

$$\left[D^{(k-1)} E^{(k-1)} + \sum_{i=1}^{3} A_i \frac{\partial u^{(k-1)}}{\partial x_i}\right] \cdot \eta = \left[D^{(k)} E^{(k)} + \sum_{i=1}^{3} A_i \frac{\partial u^{(k)}}{\partial x_i}\right] \cdot \eta \tag{2.12}$$

*for any $x \in S_k$, $k = 1, 2, ..., n$ and $t \geq 0$. Furthermore*

$$\left(u,\, u_t,\, E,\, H,\, \int_0^t \Big[H(x,\tau) \,\mathbf{x}\, \eta\Big] \exp\Big(-\sigma(x)\,(t-\tau)\Big)\, d\tau\right) \in M_1 \cap \mathcal{D}(\mathcal{A})$$

*for any $t \geq 0$ and (1.6) is valid for any $t \geq 0$ where $\mathcal{E}(t)$ is given by (1.5).*

**Proof:** Let $w = (w_1, w_2, w_3, w_4, w_5) = U(t) f \in M_1 \cap \mathcal{D}(\mathcal{A})$ then $(w_1, w_3, w_4) = (u, E, H)$. The relation

$$\frac{d}{dt} w = \frac{d}{dt} U(t) f = \mathcal{A} w$$

give us that

$$\begin{cases} \dfrac{\partial w_1}{\partial t} = w_2 \\ \dfrac{\partial w_2}{\partial t} = \displaystyle\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial w_1}{\partial x_j}\right) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} A_i^* w_3 \\ \dfrac{\partial w_3}{\partial t} = D^{-1}\left(\operatorname{curl} w_4 - \displaystyle\sum_{i=1}^{3} A_i \frac{\partial w_2}{\partial x_i}\right) \\ \dfrac{\partial w_4}{\partial t} = \beta^{-1} \operatorname{curl} w_3 \\ \dfrac{\partial w_5}{\partial t} = w_4 \,\mathbf{x}\, \eta - \sigma w_5 \,. \end{cases} \tag{2.13}$$

From the last equation in (2.13) we obtain the identity

$$(w_4 \,\mathbf{x}\, \eta)\, \exp(\sigma(x)\, t) \;=\; \frac{\partial}{\partial t}\Big(\exp(\sigma(x)\, t)\, w_5\Big)$$

which implies that

$$w_5(x,t) \;=\; \int_0^t \Big[w_4(x,\tau)\,\mathbf{x}\,\eta\Big]\, \exp\Big(-\sigma(t-\tau)\Big)\, d\tau \tag{2.14}$$

because $w_5(x,0) = 0$. Since $w \in M_1 \cap \mathcal{D}(\mathcal{A})$ then $(w_3, w_4, w_5) \in V$ (see the definition of $V$ after Remark 2.2). Consequently

$$w_3 - \eta(w_3 \,\centerdot\, \eta) - \alpha\, w_4 \,\mathbf{x}\, \eta - \gamma\, w_5 \;=\; 0 \quad \text{ on } \; S\;. \tag{2.15}$$

Substitution of (2.14) into (2.15) and writting $w_3 = E$, $w_4 = H$ implies that

$$\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta) \;-\; \alpha\, H \mathbf{x}\, \eta \;-\; \gamma \int_0^t \Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\, \exp\Big(-\sigma(t-\tau)\Big)\, d\tau \;=\; 0$$

on $S$ because $\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta) = E - \eta(E \,\centerdot\, \eta)$, $(|\eta| = 1)$. The first boundary condition in (1.3) is also satisfy because $w \in M_1 \cap \mathcal{D}(\mathcal{A})$ and the interface conditions (1.4) for the same reason.

Lemma 2.4 implies the validity of (2.12) as well as the last equation in (1.1). Finally, let us prove (1.6) for a dense subset of $M_1 \cap \mathcal{D}(\mathcal{A})$, namely, the set of piecewise smooth vector-valued functions $w = (w_1, w_2, w_3, w_4, 0)$ belonging to $M_1 \cap \mathcal{D}(\mathcal{A})$ such that $w_1 \in [C^2(\Omega_k)]^3$, $w_j \in [C^1(\Omega_k)]^3$, $j = 2,3,4$ and $k = 0,1,2,\ldots,n$. Let us take the inner product of $2\,u_t$, $2E$ and $2H$ by the first, second and third equation of (1.1) respectively. We obtain the identity

$$\begin{aligned} 0 \;=\; & 2\,u_t \,\centerdot \left\{ u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\, \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\, A_i^* E \right\} \\ & + \; 2E \,\centerdot \left\{ \frac{\partial}{\partial t}\left( DE + \sum_{i=1}^{3} A_i\, \frac{\partial u}{\partial x_i} \right) - \operatorname{curl} H \right\} \;+\; 2H \,\centerdot\, \Big\{\beta H_t + \operatorname{curl} E\Big\}\,. \end{aligned} \tag{2.16}$$

Using the identity $\operatorname{div}(U \times V) = V \centerdot \operatorname{curl} U - U \centerdot \operatorname{curl} V$ valid for any pair of vectors $U$ and $V$ in $\mathbb{R}^3$, we obtain from (2.16) that

$$\begin{aligned} 0 \;=\; & \frac{\partial}{\partial t}\Big\{|u_t|^2 + DE \,\centerdot\, E + \beta |H|^2\Big\} \;-\; 2\, \operatorname{div}(H \times E) \\ & + \; 2\,u_t \,\centerdot \left\{ \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\, A_i^* E \right\} \;+\; 2E \,\centerdot \sum_{i=1}^{3} A_i\, \frac{\partial u_t}{\partial x_i} \\ & - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ 2\,u_t \,\centerdot \sum_{j=1}^{3} A_{ij}\, \frac{\partial u}{\partial x_j} \right\} \;+\; 2 \sum_{i,j=1}^{3} A_{ij}\, \frac{\partial u}{\partial x_j} \,\centerdot\, \frac{\partial u_t}{\partial x_i}\;. \end{aligned} \tag{2.17}$$

Since $A_{ij}^* = A_{ji}$ and

$$2\sum_{i=1}^{3} A_i \frac{\partial u_t}{\partial x_i} \cdot E = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\{2\,u_t \cdot A_i^*\,E\}$$

$$2\,u_t \cdot \sum_{i=1}^{3} \frac{\partial}{\partial x_i} A_i^* E + 2\sum_{i=1}^{3} A_i \frac{\partial u_t}{\partial x_i} \cdot E = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\{2\,u_t \cdot A_i^* E\} - 2\sum_{i=1}^{3} \frac{\partial u_t}{\partial x_i} \cdot A_i^* E\ .$$

Then, we can rewritte (2.17) as follows

$$\begin{aligned} 0 \;=\; & \frac{\partial}{\partial t}\left\{|u_t|^2 + \sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} + DE \cdot E + \beta|H|^2\right\} \\ & - 2\,\mathrm{div}(H \,\mathbf{x}\, E) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left\{2\,u_t \cdot \left(\sum_{j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j} - A_i^* E\right)\right\}. \end{aligned} \tag{2.18}$$

Integration of identity (2.18) over $\Omega_k$ and summation in $k$ from zero up to $n$ give us

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{k=0}^{n}\int_{\Omega_k}&\left\{|u_t^{(k)}|^2 + \sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} + D^{(k)}E^{(k)} \cdot E^{(k)} + \beta|H^{(k)}|^2\right\} dx \;= \\ &= \sum_{k=1}^{n}\int_{S_k}\left(F^{(k-1)} - F^{(k)}\right) dS + \int_S F\,dS \end{aligned} \tag{2.19}$$

where

$$F^{(k)} \;=\; 2\left(H^{(k)}\,\mathbf{x}\,E^{(k)}\right) \cdot \eta \;-\; 2\,u_t^{(k)} \cdot \left\{\sum_{i,j=1}^{3} A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E^{(k)}\eta_i\right\}$$

and

$$F \;=\; 2\,(H \,\mathbf{x}\, E) + 2\,u_t \cdot \left\{\sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E\,\eta_i\right\}. \tag{2.20}$$

Due to the interface conditions (1.4), the integrals over $S_k$ on the right hand side of (2.20) are equal to zero for $k = 1, 2, ..., n$. Now, we use the boundary conditions (1.3) to get the identities

$$\begin{aligned} 2\,(H \,\mathbf{x}\, E) \cdot \eta \;&=\; 2\,E \cdot (\eta \,\mathbf{x}\, H) \\ &=\; 2\left\{\eta(E \cdot \eta) + \eta \,\mathbf{x}\,(E \,\mathbf{x}\, \eta)\right\} \cdot \{\eta \,\mathbf{x}\, H\}\,, \qquad |\eta| = 1\ , \\ &=\; 2\,\{\eta \,\mathbf{x}\,(E \,\mathbf{x}\, \eta)\} \cdot \{\eta \,\mathbf{x}\, H\} \\ &=\; 2\,\{\eta \,\mathbf{x}\, H\} \cdot \left\{\alpha(H \,\mathbf{x}\, \eta) + \gamma\int_0^t (H \,\mathbf{x}\, \eta)\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right\} \;= \end{aligned} \tag{2.21}$$

$$= \;-2\,\alpha\,|H\,\mathbf{x}\,\eta|^2 - \frac{\partial}{\partial t}\left\{\gamma\left|\int_0^t (H\,\mathbf{x}\,\eta)\exp\Bigl(-\sigma(t-\tau)\Bigr)\,d\tau\right|^2\right\}$$
$$-\,2\,\gamma\,\sigma\,\left|\int_0^t (H\,\mathbf{x}\,\eta)\exp\Bigl(-\sigma(t-\tau)\Bigr)\,d\tau\right|^2$$

and

$$2\,u_t\,\boldsymbol{\cdot}\left\{\sum_{i,j=1}^3 A_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i - \sum_{i=1}^3 A_i^*\,E\,\eta_i\right\} = 2\,u_t\,\boldsymbol{\cdot}\,\{-a u_t - b u\} = -2\,a\,|u_t|^2 - \frac{\partial}{\partial t}\,(b\,|u|^2)\;. \tag{2.22}$$

Using (2.21) and (2.22) together with (2.19) where $F$ is given by (2.20), we obtain that

$$\frac{\partial}{\partial t}\left\{\sum_{k=0}^n \int_{\Omega_k}\left\{|u_t^{(k)}|^2 + \sum_{i,j=1}^3 A_{ij}^{(k)}\,\frac{\partial u^{(k)}}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u^{(k)}}{\partial x_i} + D^{(k)}E^{(k)}\boldsymbol{\cdot}E^{(k)} + \beta^{(k)}|H^{(k)}|^2\right\}dx\right\} =$$
$$= -\int_S \left\{2\,\alpha\,|H\,\mathbf{x}\,\eta|^2 + 2\,a\,|u_t|^2 - 2\,\gamma\,\sigma\left|\int_0^t (H\,\mathbf{x}\,\eta)\exp\Bigl(-\sigma(t-\tau)\Bigr)\,d\tau\right|^2\right.$$
$$\left. + \frac{\partial}{\partial t}\left(\gamma\,\left|\int_0^t (H\,\mathbf{x}\,\eta)\exp\Bigl(-\sigma(t-\tau)\Bigr)\,d\tau\right|^2 + b\,|u|^2\right)\right\}dS$$

which implies (1.6). This concludes the proof of Theorem 2.5. ∎

**Corollary 2.6.** *Under the assumptions of Theorem 2.5, let* $f=(f_1,f_2,f_3,f_4,0) \in Z$, *then* $U(t)f$ *is the weak solution of the problem*

$$\frac{dw}{dt} = \mathcal{A}w\,, \qquad w(0) = f\;.$$

**Proof:** Let $f^{(m)} = \Bigl(f_1^{(m)}, f_2^{(m)}, f_3^{(m)}, f_4^{(m)}, 0\Bigr) \in \mathcal{D}(\mathcal{A})$ such that $f^{(m)} \to f$ in $Z$ as $m \to \infty$. Then, $U(t)f^{(m)}$ satisfies the following identity

$$\int_0^T \left\{\left(U(t)f^{(m)}, \frac{d\psi}{dt}\right)_Z + \Bigl(U(t)f^{(m)}, \mathcal{A}^*\psi\Bigr)_Z\right\}dt \;=\; -\Bigl(f^{(m)}, \psi(0)\Bigr)_Z \tag{2.23}$$

for any $\psi \in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$ such that $\psi_t \in L^2(0,T;Z)$ and $\psi(T) = 0$. Passing to the limit in (2.23) as $n \to +\infty$, we obtain

$$\int_0^T \left\{\left(U(t)f, \frac{d\psi}{dt}\right)_Z + \Bigl(U(t)f, \mathcal{A}^*\psi\Bigr)_Z\right\}dt \;=\; -\Bigl(f, \psi(0)\Bigr)_Z \tag{2.24}$$

which proves Corollary 2.6. ∎

**Remark 2.7.** We note that $U(t)$ takes $M_1$ into itself. Indeed, if $g \in \operatorname{Ker}(\mathcal{A}^*)$ and take $\psi(t) = (T-t)g$, then from (2.17) it follows that

$$\int_0^T (U(t)f, g)_Z \;=\; T(f, g)_Z$$

which implies that $(U(t)f, g)_Z = (f, g)_Z$, $\forall\, t \geq 0$ whenever $f \in M_1$ . □

## 3 – Stabilization

In this section we will prove the main result of this article, that is, the exponential stabilization of the solution of problem (1.1)–(1.4). The proof is based on the theory of multipliers and it is motivated by the invariance of system (1.1) (with constant coefficients) relative to the one-parameter group of dilations in all variables. The multipliers have to be conveniently modified in such a way that the extra boundary terms appearing in the identities can be estimated by appropriate bounds. Let $\varphi = \varphi(x)$ be an auxiliary (scalar) smooth function on $\overline{\Omega}$ which we will choose later. Let us fix $t_0 > 0$ and consider the multiplier

$$L_1 u \;=\; (t+t_0)u_t + (\nabla\varphi \,.\, \nabla)u + u \tag{3.1}$$

where $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\right)$,

$$\nabla\varphi \,.\, \nabla \;=\; \frac{\partial\varphi}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial\varphi}{\partial x_3}\frac{\partial}{\partial x_3}$$

and $u = u(x,t) = (u_1, u_2, u_3)$.

We also consider the multipliers

$$L_2 \;=\; L_2(E,H) \;=\; (t+t_0)E + \beta\,\nabla\varphi \,\mathbf{x}\, H \tag{3.2}$$

and

$$L_3 \;=\; L_3(H,E,u) \;=\; (t+t_0)H - \nabla\varphi \;\mathbf{x} \left[DE + \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right] . \tag{3.3}$$

We take the inner product (in $\mathbb{R}^3$) of $L_1 u$, $L_2$ and $L_3$ with

$$u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(A_i^* E) \;,$$

$$\frac{\partial}{\partial t}\left(DE + \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) - \operatorname{curl} H$$

and

$$\beta H_t + \operatorname{curl} E \;,$$

respectively. Finally, we multiply $\operatorname{div}\left\{DE + \sum_{i=1}^{3} A_i \dfrac{\partial u}{\partial x_i}\right\}$ by $E \cdot \nabla\varphi$ and $\operatorname{div} H$ by $\beta H \cdot \nabla\varphi$. Since $\{u, E, H\}$ is a solution of (1.1) then, adding the identities we obtain that

$$\frac{\partial F}{\partial t} - \operatorname{div}_x G - \sum_{i=1}^{3} \frac{\partial I_i}{\partial x_i} - J = 0 \tag{3.4}$$

where

$$\begin{aligned} F \;=\; & (t+t_0)\left\{|u_t|^2 + \sum_{i,j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} + DE \cdot E + \beta\,|H|^2 \right. \\ & + 2\,u_t \cdot (\nabla\varphi \cdot \nabla)u \;+\; 2\,u_t \cdot u \;+\; 2\,\beta(\nabla\varphi\,\mathbf{x}\,H) \cdot DE \\ & \left. + \;2\,\beta(\nabla\,\underline{\mathbf{x}}\,H) \cdot \left(\sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right)\right\}\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} G \;=\; & 2\,(t+t_0)\,H\,\mathbf{x}\,E \;+ \nabla\varphi\,(DE \cdot E) \;+\; (\nabla\varphi)\,\beta\,|H|^2 - \;2\,DE\,(E \cdot \nabla\varphi) \\ & -\;2\,\beta H(H \cdot \nabla\varphi) \;+\; 2\,E\;\mathbf{x}\left(\nabla\varphi\;\mathbf{x}\;\sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right)\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} I_i \;=\; & 2\left[(t+t_0)\,u_t + (\nabla\varphi \cdot \nabla)\,u + u\right] \cdot \left[\sum_{j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} - A_i^* E\right] \\ & + \frac{\partial\varphi}{\partial x_i}\left\{|u_t|^2 - \sum_{p,q=1}^{3} A_{pq} \frac{\partial u}{\partial x_q} \cdot \frac{\partial u}{\partial x_p}\right\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} J \;=\; & (\Delta\varphi - 1) \sum_{i,j=1}^{3} A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \;-\; 2 \sum_{i,j,p=1} \frac{\partial^2\varphi}{\partial x_i\,\partial x_p} A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_p} \\ & + \;(3 - \Delta\varphi)\,|u_t|^2 + \;2 \sum_{i,j,k=1}^{3} \frac{\partial^2\varphi}{\partial x_i\,\partial x_k} d_{ij}\,E_j\,E_k \;-\; (\Delta\varphi - 1)\,DE \cdot E \\ & + \;2 \sum_{i,j=1}^{3} \frac{\partial^2\varphi}{\partial x_i\,\partial x_j} \beta\,H_i\,H_j \;-\; (\Delta\varphi - 1)\,\beta |H|^2 \\ & + \;2\,E \cdot \left\{\sum_{i,k=1}^{3} \frac{\partial^2\varphi}{\partial x_i\,\partial x_k} A_k \frac{\partial u}{\partial x_i} \;+\; \left(\sum_{k=1}^{3} A_k \frac{\partial u}{\partial x_k} \cdot \nabla\right) \nabla\varphi \right. \\ & \left. -\;(\Delta\varphi - 1) \sum_{k=1}^{3} A_k \frac{\partial u}{\partial x_k}\right\}\,. \end{aligned} \tag{3.8}$$

Observe that if we consider $\varphi(x) = \frac{1}{2}|x-x_0|^2$ for some fixed $x_0 \in \Omega$, then $J \equiv 0$. In this case (3.4) will be a conservation law. However, due to the expressions of $G$ and $I_i$ we can see that we will need (after integration in $\Omega_k$ of identity (3.4)) a definite sign for $\frac{\partial\varphi}{\partial\eta}\cdot$ We will choose $\varphi(x)$ as a "little" perturbation of $\frac{1}{2}|x-x_0|^2$ for some $x_0 \in \Omega$. Let $f = (f_1, f_2, f_3, f_4, 0) \in M_1 \cap \mathcal{D}(\mathcal{A})$ and $\{u, E, H\}$ be the corresponding solution of problem (1.1)–(1.4) obtained in Theorem 2.5. Integration over $\Omega_k \times (0,t)$ of the identity (3.4) and summation over $k$ implies that

$$
\begin{aligned}
&(t+t_0)\sum_{k=0}^{n}\int_{\Omega_k}\left\{|u_t^{(k)}|^2+\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}+D^{(k)}E^{(k)}\cdot E^{(k)}+\beta^{(k)}|H^{(k)}|^2\right\}dx\,\Bigg|_{t=0}^{t=T}+\\
&\quad+2\sum_{k=0}^{n}\int_{\Omega_k}\Bigg\{u_t^{(k)}\cdot(\nabla\varphi\cdot\nabla)u^{(k)}+u_t^{(k)}\cdot u^{(k)}+\beta^{(k)}(\nabla\varphi\,\mathbf{x}\,H^{(k)})\cdot D^{(k)}E^{(k)}\\
&\quad+\beta^{(k)}(\nabla\varphi\,\mathbf{x}\,H^{(k)})\cdot\left(\sum_{i=1}^{3}A_i\frac{\partial u^{(k)}}{\partial x_i}\right)\Bigg\}dx\,\Bigg|_{t=0}^{t=T}=\\
&\quad=\sum_{k=1}^{n}\int_0^T\!\!\int_{S_k}(V_{k-1}-V_k)\,dS_k\,dt+\int_0^T\!\!\int_S V_n\,dS\,dt+\sum_{k=0}^{n}\int_0^T\!\!\int_{\Omega_k}J_k(x,t)\,dx\,dt
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
V_k &= 2\left\{(t+t_0)u_t^{(k)}+(\nabla\varphi\cdot\nabla)u^{(k)}+u^{(k)}\right\}\cdot\left\{\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\eta_i-\sum_{i=1}^{3}A_i^{*}E^{(k)}\eta_i\right\}\\
&\quad+\frac{\partial\varphi}{\partial\eta}\left\{|u_t^{(k)}|^2-\sum_{i,j=1}^{3}A_{ij}^{(k)}\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}\right\}+2\,(t+t_0)\,\eta\cdot(H^{(k)}\,\mathbf{x}\,E^{(k)})\\
&\quad+\frac{\partial\varphi}{\partial\eta}D^{(k)}E^{(k)}\cdot E^{(k)}+\frac{\partial\varphi}{\partial\eta}\beta^{(k)}|H^{(k)}|^2-2\,(D^{(k)}E^{(k)}\cdot\eta)\,(E^{(k)}\cdot\nabla\varphi)\\
&\quad-2\,\beta^{(k)}\,(H^{(k)}\cdot\eta)\,(H^{(k)}\cdot\nabla\varphi)+2\left\{\nabla\varphi\,\mathbf{x}\left(\sum_{i=1}^{3}A_i\frac{\partial u^{(k)}}{\partial x_i}\right)\right\}\cdot\left\{\eta\,\mathbf{x}\,E^{(k)}\right\}
\end{aligned}
\tag{3.10}
$$

and $J_k(x,t)$ is the restriction of $J(x,t)$ (given in (3.8)) to the subset $\Omega_k$. Here $\frac{\partial\varphi}{\partial\eta}$ denotes the normal derivative of $\varphi$ at $x \in S_k$.

The proof of the main result will follow as long as we can get appropriate estimates for all terms on the right hand side of identity (3.9). The following three Lemmas will take care of such estimates. Since their proofs are quite long and technical we prefer to give the precise statement postponing their proofs to the end of the section. The first Lemma tell us that the differences $V_{k-1} - V_k$ will have a "good" sign if we choose $\varphi$ conveniently together with a monotonicity condition on $\{A_{ij}^{(k)}\}$, $\{D^{(k)}\}$ and $\{\beta^{(k)}\}$.

**Lemma 3.1.** *Let $f = (f_1, f_2, f_3, f_4, 0) \in M_1 \cap \mathcal{D}(\mathcal{A})$ and $\{u, E, H\}$ be the corresponding solution of problem (1.1)–(1.4) obtained in Theorem 2.5. Then, the identity*

$$
\begin{aligned}
V_{k-1} - V_k \;=\; -\frac{\partial \varphi}{\partial \eta} \Bigg\{ & \sum_{i,j=1}^{3} (A_{ij}^{(k-1)} - A_{ij}^{(k)}) \frac{\partial u^{(k-1)}}{\partial x_j} \cdot \frac{\partial u^{(k-1)}}{\partial x_i} \\
& + \sum_{i,j=1}^{3} A_{ij}^{(k)} \Big( \frac{\partial u^{(k)}}{\partial x_j} - \frac{\partial u^{(k-1)}}{\partial x_j} \Big) \cdot \Big( \frac{\partial u^{(k)}}{\partial x_i} - \frac{\partial u^{(k-1)}}{\partial x_i} \Big) \\
& + (D^{(k)} - D^{(k-1)}) E^{(k)} \cdot E^{(k)} \\
& + D^{(k-1)} (E^{(k)} - E^{(k-1)}) \cdot (E^{(k)} - E^{(k-1)}) \\
& + (\beta^{(k)} - \beta^{(k-1)}) \Big\{ |H^{(k)} \times \eta|^2 + \frac{\beta^{(k)}}{\beta^{(k-1)}} |H^{(k)} \cdot \eta|^2 \Big\} \Bigg\}
\end{aligned}
\tag{3.11}
$$

*holds for $k = 1, 2, ..., n$.*

Let us choose a convenient function $\varphi(x)$: Let $\Phi(x)$ be the solution of the Neumann problem

$$
\begin{cases}
\Delta \Phi = 1 & \text{in } \Omega \\
\dfrac{\partial \Phi}{\partial \eta} = \dfrac{\text{measure}(\Omega)}{\text{area}(S)} & \text{on } \partial\Omega
\end{cases}
$$

which admits a solution $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Let $\delta > 0$ and $x_0 \in \Omega$ (to be chosen later) and define

$$
\varphi(x) \;=\; \delta\, \Phi(x) + \frac{1}{2} |x - x_0|^2 \;. \tag{3.12}
$$

Thus, the normal derivative of $\varphi$ is given by

$$
\frac{\partial \varphi}{\partial \eta} \;=\; \delta\, \frac{\partial \Phi(x)}{\partial \eta} + (x - x_0) \cdot \eta \;.
$$

Now, we concentrate our discussion in estimating the term $\displaystyle\sum_{k=0}^{n} \int_0^T \!\! \int_{\Omega_k} J_k(x,t)\, dx\, dt$ in (3.9), where $J_k$ is given by (3.8).

**Lemma 3.2.** *Under the assumptions of Lemma 3.1 and Hypothesis I (with $\rho \equiv 1$) and choosing $\varphi(x)$ as in (3.12), then, the following estimate*

$$\sum_{k=0}^{n} \int_0^T\!\!\int_{\Omega_k} J_k(x,t)\; dx\, dt \;\le$$

$$\le\; \delta\, c_5 \sum_{k=0}^{n} \int_0^T\!\!\int_{\Omega_k} \left\{ \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} + D^{(k)} E^{(k)} \cdot E^{(k)} + \beta^{(k)} |H^{(k)}|^2 \right\} dx\, dt$$

*holds for any $\delta > 0$ and some positive constant $c_5$ which depends only on $\Phi$ and the norms of the matrices $A_{ij}$, $A_i$ and $D$.*

We will impose some geometric assumptions on $\Omega$ and $S_k$:

**HYPOTHESIS II.** There exists a positive constant $\delta_1 \ge 0$ such that

**a**) $\delta_1\, c_5 < 1$,

**b**) $\delta_1 \dfrac{\partial \Phi}{\partial \eta} + (x - x_0) \cdot \eta \ge 0$ for some point $x_0 \in \Omega$ and all $x \in S_k$,

**c**) $\delta_1 \dfrac{\text{measure}(\Omega)}{\text{area}(S)} + (x - x_0) \cdot \eta > 0$ for all $x \in S$,

where $c_5$ is given as in the conclusion of Lemma 3.2 and $\eta = \eta(x)$ denotes the unit outward normal to $S_k$ (or to $S$ in c)). □

**Remark 3.3.** We note that the above assumptions on Hypothesis II hold with $\delta_1 = 0$ for star-shaped surfaces $S_1, S_2, S_3, ..., S_n$ and strictly star-shaped surface $S$ with respect to $x_0$, i.e.

$$(x - x_0) \cdot \eta > 0 \quad \text{for all } x \in S\,.$$

If all surfaces $S_1, S_2, ..., S_n$ are strictly star-shaped with respect to a point $x_0 \in \Omega$, then conditions a) and b) hold with $\delta_1 > 0$ for a class of domains $\Omega$ which includes star-shaped regions. □

**Lemma 3.4.** *Under the assumptions of Lemma 3.1, Hypothesis I and II (with $\rho = 1$) then, the following estimate*

$$\int_0^T\!\!\int_S V_n\, dS\, dt \;\le$$

$$\begin{aligned}
\le\; & -(t+t_0)\int_S \left\{ b|u|^2 + \gamma \left| \int_0^t [H\,\mathbf{x}\,\eta]\, \exp(-\sigma(t-\tau))\, d\tau \right|^2 \right\} dS \,\Bigg|_{t=0}^{t=T} \\
& - \int_S a\,|u|^2\, dS \,\Bigg|_{t=0}^{t=T} - \int_0^T\!\!\int_S (1-c_6\, b)\, b\, |u|^2\, dS\, dt \\
& - \int_0^T\!\!\int_S \left\{ 2\,(t+t_0)\, a - \frac{\partial \varphi}{\partial \eta} - c_7 \right\} |u_t|^2\, dS\, dt \\
& - \int_0^T\!\!\int_S \left\{ \frac{\partial \varphi}{\partial \eta} - \delta_0\, |\nabla \varphi| \right\} \sum_{i,j=1}^{3} A_{ij}\, \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i}\, dS\, dt \\
& - \int_0^T\!\!\int_S \left\{ 2\,(t+t_0)\,\alpha - (\beta + c_8\,\alpha^2)\,(3+\delta_0^{-1})\,|\nabla\varphi| - c_9 \right\} |H\,\mathbf{x}\,\eta|^2\, dS\, dt \\
& - \int_0^T\!\!\int_S \left\{ 2\,(t+t_0)\,\sigma - 1 - \gamma\, c_{10}\,(3+\delta_0^{-1})\,|\nabla\varphi| - c_{11} \right\} \\
& \qquad \cdot\, \gamma \left| \int_0^t [H\,\mathbf{x}\,\eta]\, \exp\Big(-\sigma(t-\tau)\Big)\, d\tau \right|^2 dS\, dt
\end{aligned}$$

*holds, for some positive constants $c_j$, $6 \le j \le 11$ (which will be defined in the proof of Lemma 3.4).*

Finally, we will requere the following monotonicity assumptions:

**HYPOTHESIS III.** We assume the monotonicity conditions on $\{A_{ij}^{(k)}\}$, $\{D^{(k)}\}$ and $\{\beta^{(k)}\}$:

**1)** $\sum_{i,j=1}^{3} \left(A_{ij}^{(k-1)} - A_{ij}^{(k)}\right) v_j \cdot v_i \ge 0$ for any $v_i \in \mathbb{R}^3$ and all $1 \le k \le n$.

**2)** $\left(D^{(k)} - D^{(k-1)}\right) v \cdot v \ge 0$ for any $v \in \mathbb{R}^3$ and all $1 \le k \le n$ and $k = 1, 2, ..., n$.

**3)** $\beta^{(k)} \ge \beta^{(k-1)}$ for all $1 \le k \le n$. □

Let us consider the following quantities: Let $\delta_0 > 0$ be such that

$$\frac{\partial \varphi}{\partial \eta} \ge \delta_0 |\nabla \varphi| \quad \text{for any } \; x \in S \tag{3.13}$$

which is possible because $\dfrac{\partial \varphi}{\partial \eta} > 0$ on $S$ and $S$ is compact. Let

$$\lambda_0 \;=\; \max_{x \in \overline{\Omega}} \Big\{ |x - x_0| + \delta_1 |\nabla \Phi| \Big\}\,, \tag{3.14}$$

where $x_0 \in \Omega$ and $\delta_1$ are as in Hypothesis II.

With the help of the above Lemmas now we can prove the main result of this paper.

**Theorem 3.5.** *Let us assume Hypothesis I, II and III and*

$$b(x) \le \frac{c_0\,\delta_0}{2\,\lambda_0} \tag{3.15}$$

*where the constants $\delta_1$ and $c_5$ appeared in Hypothesis II, $\delta_0$ in (3.13), $c_0$ in Hypothesis I and $\lambda_0$ in (3.14). Let $f = (f_1, f_2, f_3, f_4, 0)$ belong to $M_1 \cap \mathcal{D}(\mathcal{A})$ and $\{u, E, H\}$ be the unique solution of problem (1.1)–(1.4) obtained in Theorem 2.5. Then, there exist positive constants $c$ and $w$ such that*

$$\mathcal{E}(t) \le c \exp(-wt)\,\mathcal{E}(0)$$

*for any $t \ge 0$ where $\mathcal{E}(t)$ is given by (1.5).*

**Proof:** We will use identity (3.9). First, we observe that we need to get a bound for the term

$$\begin{aligned} I = 2\sum_{k=0}^{n} \int_{\Omega_k} \Bigg\{ & u_t^{(k)} \cdot (\nabla\varphi \cdot \nabla) u^{(k)} + u_t^{(k)} \cdot u^{(k)} + \beta^{(k)} (\nabla\varphi \times H^{(k)}) \cdot D^{(k)} E^{(k)} \\ & + \beta^{(k)} (\nabla\varphi \times H^{(k)}) \cdot \left( \sum_{i=1}^{3} A_i \frac{\partial u^{(k)}}{\partial x_i} \right) \Bigg\}\, dx \Bigg|_{t=0}^{t=T} . \end{aligned} \tag{3.16}$$

Each term on the integrand of (3.16) can be bound in the same way as in the proof (which we will give later of Lemma 3.4). Except that will appear the term $\sum_{k=0}^{n} \int_{\Omega_k} |u^{(k)}|^2\, dx$. However, since $u^{(k)} \in [H^1(\Omega_k)]^3$ for $k = 0, 1, ..., n$ then, we know that the following inequality

$$c_{12} \sum_{k=0}^{n} \|u^{(k)}\|^2_{[L^2(\Omega_k)]^3} \le \sum_{k=0}^{n} \int_{\Omega_k} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i}\, dx + \int_S b\,|u|^2\, dS$$

holds for some positive constant $c_{12}$. Here $u^{(n)} = u$. Thus, the term $I$ in (3.16) can be estimated by

$$|I| \le c_{13}\,\mathcal{E}(T) \le c_{13}\,\mathcal{E}(0) \tag{3.17}$$

for some positive constant $c_{13}$. Observe that all terms on the right hand side of the conclusion of Lemma 3.3 can assume to be with a fixed sign provided we take

$t_0 > 0$ large enough. In fact, $1 - c_6\, b \geq 0$ by assumption (3.15), $\dfrac{\partial\varphi}{\partial\eta} - \delta_0|\nabla\varphi| \geq 0$ on $S$ by (3.13). The coefficient $\left\{2\,(t+t_0)\,a - \dfrac{\partial\varphi}{\partial\eta} - c_7\right\}$ as well as the last two coefficients on the inequality in Lemma 3.3 will be positive for all $t \geq 0$ as long as we choose $t_0 = T_0$ large enough. Now, we use Lemmas 3.1, 3.2 and 3.3 together with (3.17) to conclude from identity (3.9) that

$$(T+T_0)\,\mathcal{E}(T) \;\leq\; c_{13}\,\mathcal{E}(0) + \delta_1 c_5 \int_0^T \mathcal{E}(t)\,dt \tag{3.18}$$

for any $T > 0$. Recall that $\delta_1 c_5 < 1$. Let us denote by $g(T)$ the right hand side of (3.18). Clearly $\dfrac{g'(T)}{g(T)} \leq \dfrac{\delta_1 c_5}{T+T_0}$ which implies that $g(T) \leq \dfrac{(T+T_0)^p}{T_0^p}\, g(0)$ where $p = \delta_1 c_5 < 1$. Returning to (3.18) we obtain that

$$\mathcal{E}(T) \;\leq\; \frac{c_{14}}{(T+T_0)^{1-p}}\;\mathcal{E}(0) \tag{3.19}$$

where $c_{14} = c_{13}\, T_0^{-p}$. Now, we can choose $T > 0$ large enough in (3.19) so that $c_{14}/(T+T_0)^{1-p}$ is strictly less than one. The semigroup property then implies the conclusion of Theorem 3.5.

**Corollary 3.6.** *Under the assumptions of Theorem 3.5, let* $f = (f_1, f_2, f_3, f_4, 0) \in M_1$, *then*

**a)** *The same conclusion as in Theorem 3.5 holds.*

**b)** *If* $\gamma \equiv 0$, *then, the semigroup* $\{U(t)\}_{t\geq 0}$ *associated with problem (1.1)–(1.4) takes the closed subspace* $M_1$ *into itself and* $\|U(t)\|_{\mathcal{L}(Z,Z)} < 1$ *for any* $t > T_0\left[\left(\dfrac{c_{13}}{T_0}\right)^{1/1-p} - 1\right]$.

**Proof:** a) follows from a density argument and Theorem 3.5. Item b) is a consequence of (3.19), again by a density argument. ∎

Now, we will prove the technical Lemmas 3.1, 3.2 and 3.4.

**Proof of Lemma 3.1:** The idea is to use the interface conditions (1.4). In order to simplify notations let us denote by $E^{(k-1)} = E$, $E^{(k)} = \widetilde{E}$, $H^{(k-1)} = H$, $H^{(k)} = \widetilde{H}$, $D^{(k-1)} = D$, $D^{(k)} = \widetilde{D}$, $A_{ij}^{(k-1)} = P_{ij}$, $A_{ij}^{(k)} = \widetilde{P}_{ij}$, $\beta^{(k-1)} = \beta$, $\beta^{(k)} = \tilde{\beta}$, $u^{(k-1)} = u$ and $u^{(k)} = \tilde{u}$. Using the interface conditions (1.4) and (3.10) we find that

$$
\begin{aligned}
V_{k-1} - V_k \;=\; & L + \frac{\partial\varphi}{\partial\eta}\,\beta|H|^2 - \frac{\partial\varphi}{\partial\eta}\,\tilde\beta|\widetilde H|^2 - 2\,\beta(H\cdot\eta)\,(H\cdot\nabla\varphi)\\
& + 2\,\tilde\beta(\widetilde H\cdot\eta)\,(\widetilde H\cdot\nabla\varphi) + \frac{\partial\varphi}{\partial\eta}\,DE\cdot E - \frac{\partial\varphi}{\partial\eta}(\widetilde D\widetilde E\cdot\widetilde E)\\
& - 2\left\{\left(DE + \sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\right)\cdot\eta\right\}\{E\cdot\nabla\varphi\}\\
& + 2\left\{\left(\widetilde D\widetilde E + \sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\right)\cdot\eta\right\}\{\widetilde E\cdot\nabla\varphi\}\\
& + 2\,\frac{\partial\varphi}{\partial\eta}\,E\cdot\left\{\sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\right\} - 2\,\frac{\partial\varphi}{\partial\eta}\,\widetilde E\cdot\left\{\sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\right\}
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
L \;=\; & 2\Big\{(\nabla\varphi\cdot\nabla)u\Big\}\cdot\left\{\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^* E\,\eta_i\right\}\\
& - \frac{\partial\varphi}{\partial\eta}\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i} - 2\Big\{|\nabla\varphi\cdot\nabla)\tilde u\Big\}\cdot\left\{\sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\,\eta_i - \sum_{i=1}^{3} A_i^*\widetilde E\,\eta_i\right\}\\
& + \frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\cdot\frac{\partial \tilde u}{\partial x_i}\right\}.
\end{aligned}
$$

Using (2.12) we obtain the identities

$$
\beta|H\,\mathbf{x}\,\eta|^2 + \beta|H\cdot\eta|^2 \;=\; \beta|\widetilde H\,\mathbf{x}\,\eta|^2 + \frac{\tilde\beta^2}{\beta}|\widetilde H\cdot\eta|^2 \tag{3.21}
$$

$$
\beta(H\cdot\eta)(H\cdot\nabla\varphi) \;=\; \tilde\beta(\widetilde H\cdot\eta)\Big\{\eta(H\cdot\eta) + \eta\,\mathbf{x}\,(H\,\mathbf{x}\,\eta)\Big\}\cdot\nabla\varphi \tag{3.22}
$$

because $H = \eta(H\cdot\eta) + \eta\,\mathbf{x}\,(H\,\mathbf{x}\,\eta)$ since $|\eta| = 1$. Observe also that (3.21) it is equal to $\beta|H|^2$ because $|H|^2 = |H\,\mathbf{x}\,\eta|^2 + |H\cdot\eta|^2$. Furthermore (3.22) can be written as

$$
\begin{aligned}
& \tilde\beta(\widetilde H\cdot\eta)\left\{\eta\,\frac{\tilde\beta}{\beta}\,\widetilde H\cdot\eta + \eta\,\mathbf{x}\,(\widetilde H\,\mathbf{x}\,\eta)\right\}\cdot\nabla\varphi \;=\\
& \qquad = \tilde\beta(\widetilde H\cdot\eta)\left\{\eta\,\frac{\tilde\beta}{\beta}\,\widetilde H\cdot\eta + \widetilde H - \eta(\widetilde H\cdot\eta)\right\}\cdot\nabla\varphi\\
& \qquad = \tilde\beta(\widetilde H\cdot\eta)\,(\widetilde H\cdot\nabla\varphi) + \frac{\tilde\beta}{\beta}(\tilde\beta - \beta)\,(\nabla\varphi\cdot\eta)\,|\widetilde H\cdot\eta|^2\;.
\end{aligned}
$$

From the above discussion, we can write the identity

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial\eta}\,\beta|H|^2-\frac{\partial\varphi}{\partial\eta}\,\tilde\beta|\widetilde H|^2-2\,\beta(H\,.\,\eta)\,(H\,.\nabla\varphi)+2\,\tilde\beta(\widetilde H\,.\,\eta)\,(\widetilde H\,.\nabla\varphi)\;=\\
&\quad=\;\frac{\partial\varphi}{\partial\eta}\left\{\beta|\widetilde H\,\mathbf{x}\,\eta|^2+\frac{\tilde\beta^2}{\beta}|\widetilde H\,.\,\eta|^2\right\}-\frac{\partial\varphi}{\partial\eta}\left\{\tilde\beta|\widetilde H\,\mathbf{x}\,\eta|^2+\tilde\beta|\widetilde H\,.\,\eta|^2\right\}\\
&\qquad-\,2\,\tilde\beta(\widetilde H\,.\,\eta)\,(\widetilde H\,.\nabla\varphi)-2\,\frac{\tilde\beta}{\beta}(\tilde\beta-\beta)\,\frac{\partial\varphi}{\partial\eta}|\widetilde H\,.\,\eta|^2+2\,\tilde\beta(\widetilde H\,.\,\eta)\,(\widetilde H\,.\nabla\varphi)\\
&\quad=\;-\frac{\partial\varphi}{\partial\eta}\left\{(\tilde\beta-\beta)|\widetilde H\,\mathbf{x}\,\eta|^2+\frac{\tilde\beta}{\beta}(\tilde\beta-\beta)|\widetilde H\,.\,\eta|^2\right\}.
\end{aligned}
\tag{3.23}
$$

Using the interface conditions

$$
\begin{cases}
u=\tilde u\\
\displaystyle\sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*E\,\eta_i\;=\;\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial\tilde u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*\widetilde E\,\eta_i
\end{cases}
\tag{3.24}
$$

on $S_k$ and the fact that $\dfrac{\partial}{\partial x_i}(u-\tilde u)=\eta_i\,\dfrac{\partial}{\partial\eta}(u-\tilde u)$ on $S_k$ because $u-\tilde u=0$ for $x\in S_k$, we deduce the following identities

$$
\begin{aligned}
&2\,(\nabla\varphi\,.\nabla)\,u\,.\left(\sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*E\,\eta_i\right)\;-\\
&\qquad\qquad-\;2\,(\nabla\varphi\,.\nabla)\,\tilde u\,.\left(\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial\tilde u}{\partial x_j}\,\eta_i-\sum_{i=1}^{3}A_i^*\widetilde E\,\eta_i\right)\;=\\
&\quad=\;(\nabla\varphi\,.\nabla)\,(u-\tilde u)\,.\left(\sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*E\,\eta_i\right)\\
&\qquad+\;(\nabla\varphi\,.\nabla)\,(u-\tilde u)\,.\left(\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial\tilde u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*\widetilde E\,\eta_i\right)\\
&\quad=\;\frac{\partial\varphi}{\partial\eta}\,\frac{\partial}{\partial\eta}(u-\tilde u)\,.\left(\sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*E\,\eta_i\right)\\
&\qquad+\;\frac{\partial\varphi}{\partial\eta}\,\frac{\partial}{\partial\eta}(u-\tilde u)\,.\left(\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial\tilde u}{\partial x_j}\,\eta_i\;-\sum_{i=1}^{3}A_i^*\widetilde E\,\eta_i\right)\;=
\end{aligned}
\tag{3.25}
$$

$$= \frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial}{\partial x_i}(u-\tilde u) - \sum_{i=1}^{3} A_i^* E\boldsymbol{\cdot}\frac{\partial}{\partial x_i}(u-\tilde u)\right\}$$
$$+ \frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial}{\partial x_i}(u-\tilde u) - \sum_{i=1}^{3} A_i^* \widetilde E\boldsymbol{\cdot}\frac{\partial}{\partial x_i}(u-\tilde u)\right\}.$$

Substitution of identity (3.25) into the expression of $L$ (given after (3.20)) give us that

$$\begin{aligned} L = &-\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial \tilde u}{\partial x_i} - \sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i}\right\} \\ &- \frac{\partial\varphi}{\partial\eta}\sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\boldsymbol{\cdot} E + \frac{\partial\varphi}{\partial\eta}\sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\boldsymbol{\cdot} E \\ &- \frac{\partial\varphi}{\partial\eta}\sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\boldsymbol{\cdot} \widetilde E + \frac{\partial\varphi}{\partial\eta}\sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\boldsymbol{\cdot} \widetilde E\,. \end{aligned} \tag{3.26}$$

The following identities will be useful:

$$\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial \tilde u}{\partial x_i} - \sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} =$$
$$\begin{aligned} &= \sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\left(\frac{\partial u}{\partial x_j}-\frac{\partial \tilde u}{\partial x_j}\right)\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} \\ &= \sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right) \\ &= \sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right) \\ &\quad + \sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right). \end{aligned} \tag{3.27}$$

Substitution of (3.27) into (3.26) give us that

$$\begin{aligned} L = &-\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{3}P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right)\right. \\ &+ \sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right) + \sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\boldsymbol{\cdot} E - \sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\boldsymbol{\cdot} E \\ &\left. + \sum_{i=1}^{3} A_i\,\frac{\partial u}{\partial x_i}\boldsymbol{\cdot} \widetilde E - \sum_{i=1}^{3} A_i\,\frac{\partial \tilde u}{\partial x_i}\boldsymbol{\cdot} \widetilde E\right\}. \end{aligned} \tag{3.28}$$

Again, we use the interface conditions (3.24) on $S_k$ to obtain

$$\begin{aligned}
&\sum_{i,j=1}^{3} P_{ij}\,\frac{\partial u}{\partial x_j}\cdot\left(\frac{\partial \tilde u}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)-\sum_{i=1}^{3} A_i^* E\cdot\left(\frac{\partial \tilde u}{\partial x_i}-\frac{\partial u}{\partial x_i}\right) = \\
&\qquad = \sum_{i,j=1}^{3}\left(\frac{\partial \tilde u}{\partial \eta}-\frac{\partial u}{\partial \eta}\right)\cdot P_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i-\sum_{i=1}^{3}\left(\frac{\partial \tilde u}{\partial \eta}-\frac{\partial u}{\partial \eta}\right)\cdot A_i^* E\,\eta_i \\
&\qquad = \left\{\sum_{i,j=1}^{3} \widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\,\eta_i-\sum_{i=1}^{3} A_i^* \widetilde E\,\eta_i\right\}\cdot\left\{\frac{\partial \tilde u}{\partial \eta}-\frac{\partial u}{\partial \eta}\right\}.
\end{aligned}\tag{3.29}$$

Substitution of (3.29) into (3.30) give us that

$$\begin{aligned}
L \;=\; &-\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}\right. \\
&+\left[\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\,\eta_i-\sum_{i=1}^{3}A_i^*\widetilde E\,\eta_i\right]\cdot\left(\frac{\partial \tilde u}{\partial \eta}-\frac{\partial u}{\partial \eta}\right) \\
&\left.+\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial u}{\partial x_j}\cdot\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right)+\sum_{i=1}^{3}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right)\cdot A_i^*\widetilde E\right\} \\
=\; &-\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}+\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial \tilde u}{\partial x_j}\cdot\left(\frac{\partial \tilde u}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)\right. \\
&+\sum_{i=1}^{3}A_i^*\widetilde E\cdot\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right)+\sum_{i,j=1}^{3}\widetilde P_{ij}\,\frac{\partial u}{\partial x_j}\cdot\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right) \\
&\left.+\sum_{i=1}^{3}\left(\frac{\partial u}{\partial x_i}-\frac{\partial \tilde u}{\partial x_i}\right)\cdot A_i^*\widetilde E\right\} \\
=\; &-\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3}(P_{ij}-\widetilde P_{ij})\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}\right. \\
&+\sum_{i,j=1}^{3}\widetilde P_{ij}\left(\frac{\partial \tilde u}{\partial x_j}-\frac{\partial u}{\partial x_j}\right)\cdot\left(\frac{\partial \tilde u}{\partial x_i}-\frac{\partial u}{\partial x_i}\right) \\
&\left.-\;2\sum_{i=1}^{3}A_i\,\frac{\partial \tilde u}{\partial x_i}\cdot\widetilde E+2\sum_{i=1}^{3}A_i\,\frac{\partial u}{\partial x_i}\cdot\widetilde E\right\}.
\end{aligned}\tag{3.30}$$

Now, we return to (3.20) and use (3.23) with (3.30) to obtain that

$$
\begin{aligned}
V_{k-1} - V_k \;=\; & -\frac{\partial \varphi}{\partial \eta}\Bigg\{ \sum_{i,j=1}^{3} (P_{ij} - \widetilde{P}_{ij})\, \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \\
& + \sum_{i,j=1}^{3} \widetilde{P}_{ij} \Big( \frac{\partial \tilde{u}}{\partial x_j} - \frac{\partial u}{\partial x_j} \Big) \cdot \Big( \frac{\partial \tilde{u}}{\partial x_i} - \frac{\partial u}{\partial x_i} \Big) \\
& - 2 \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \cdot \widetilde{E} + 2 \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \cdot \widetilde{E} \\
& + (\tilde{\beta} - \beta)\, |\widetilde{H} \,\mathbf{x}\, \eta|^2 + \frac{\tilde{\beta}}{\beta} (\tilde{\beta} - \beta)\, |\widetilde{H} \cdot \eta|^2 \\
& - 2\, E \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Big] + 2\, \widetilde{E} \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \Big] - DE \cdot E + \widetilde{D}\widetilde{E} \cdot \widetilde{E} \Bigg\} \\
& + 2 \Bigg\{ \Bigg( \widetilde{D}\widetilde{E} + \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \Bigg) \cdot \eta \Bigg\} \{ \widetilde{E} \cdot \nabla \varphi \} \\
& - 2 \Bigg\{ \Bigg( DE + \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Bigg) \cdot \eta \Bigg\} \{ E \cdot \nabla \varphi \} \,.
\end{aligned} \tag{3.31}
$$

Let us write in a more convenient form some of the terms in (3.31):

$$
\begin{aligned}
K \;\equiv\; & \frac{\partial \varphi}{\partial \eta} DE \cdot E - \frac{\partial \varphi}{\partial \eta} \widetilde{D}\widetilde{E} \cdot \widetilde{E} \\
& + 2 \frac{\partial \varphi}{\partial \eta} E \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Big] - 2 \frac{\partial \varphi}{\partial \eta} \widetilde{E} \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Big] \\
& + 2 \Bigg\{ \Bigg( \widetilde{D}\widetilde{E} + \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \Bigg) \cdot \eta \Bigg\} \{ \widetilde{E} \cdot \nabla \varphi \} \\
& - 2 \Bigg\{ \Bigg( DE + \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Bigg) \cdot \eta \Bigg\} \{ E \cdot \nabla \varphi \} \\
=\; & 2 \Bigg\{ \frac{\partial \varphi}{\partial \eta} DE \cdot E - \frac{\partial \varphi}{\partial \eta} \widetilde{D}\widetilde{E} \cdot \widetilde{E} + (\widetilde{D}\widetilde{E} \cdot \eta)\, (\widetilde{E} \cdot \nabla \varphi) - (DE \cdot \eta)\, (E \cdot \nabla \varphi) \\
& + \frac{\partial \varphi}{\partial \eta} E \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Big] - \frac{\partial \varphi}{\partial \eta} \widetilde{E} \cdot \Big[ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \Big] \\
& + \Big[ \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \cdot \eta \Big] \{ \widetilde{E} \cdot \nabla \varphi \} - \Big[ \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \cdot \eta \Big] \{ E \cdot \nabla \varphi \} \Bigg\} \\
& - \frac{\partial \varphi}{\partial \eta} DE \cdot E + \frac{\partial \varphi}{\partial \eta} \widetilde{D}\widetilde{E} \cdot \widetilde{E} \,.
\end{aligned} \tag{3.32}
$$

Next we use the identity

$$(a \mathbf{x} b) \centerdot (c \mathbf{x} d) \;=\; (a \centerdot c)\,(b \centerdot d) - (b \centerdot c)\,(a \centerdot d)$$

valid for any vectors $a, b, c, d \in \mathbb{R}^3$ to obtain the following identities

$$\begin{aligned}
2\,(\nabla\varphi \,\mathbf{x}\, DE) \centerdot (\eta \,\mathbf{x}\, E) \;&=\; 2\Big\{(\nabla\varphi \centerdot \eta)\,(DE \centerdot E) - (DE \centerdot \eta)\,(E \centerdot \nabla\varphi)\Big\} \\
&=\; 2\,\frac{\partial\varphi}{\partial\eta}\, DE \centerdot E \;-\; 2\,(DE \centerdot \eta)\,(E \centerdot \nabla\varphi)\;,
\end{aligned}$$

$$2\,(\nabla\varphi \,\mathbf{x}\, \widetilde{D}\widetilde{E}) \centerdot (\eta \,\mathbf{x}\, \widetilde{E}) \;=\; 2\,\frac{\partial\varphi}{\partial\eta}\, \widetilde{D}\widetilde{E} \centerdot \widetilde{E} \;-\; 2\,(\widetilde{D}\widetilde{E} \centerdot \eta)\,(E \centerdot \nabla\varphi)\;,$$

$$2\left(\nabla\varphi \,\mathbf{x} \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) \centerdot (\eta \,\mathbf{x}\, E) \;=\; 2\,\frac{\partial\varphi}{\partial\eta}\left(E \centerdot \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) - 2\left(\sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) \centerdot \eta(E \centerdot \nabla\varphi)$$

and

$$2\left(\nabla\varphi \,\mathbf{x} \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i}\right) \centerdot (\eta \,\mathbf{x}\, \widetilde{E}) \;=\; 2\,\frac{\partial\varphi}{\partial\eta}\left(\widetilde{E} \centerdot \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) - 2\left(\sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i}\right) \centerdot \eta(\widetilde{E} \centerdot \nabla\varphi)\;.$$

Substitution of the above identities in (3.32) give us that

$$\begin{aligned}
K \;&=\; 2\,(\nabla\varphi \,\mathbf{x}\, DE) \centerdot (\eta \,\mathbf{x}\, E) - 2\,(\nabla\varphi \,\mathbf{x}\, \widetilde{D}\widetilde{E}) \centerdot (\eta \,\mathbf{x}\, \widetilde{E}) \\
&\quad + 2\left(\nabla\varphi \,\mathbf{x} \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\right) \centerdot (\eta \,\mathbf{x}\, E) \;-\; 2\left(\nabla\varphi \,\mathbf{x} \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i}\right) \centerdot (\eta \,\mathbf{x}\, \widetilde{E}) \\
&\quad - \frac{\partial\varphi}{\partial\eta}\, DE \centerdot E \;+\; \frac{\partial\varphi}{\partial\eta}\, \widetilde{D}\widetilde{E} \centerdot \widetilde{E} \\
&\quad + 2\,\frac{\partial\varphi}{\partial\eta}\, \widetilde{E} \centerdot \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \;-\; 2\,\frac{\partial\varphi}{\partial\eta}\, \widetilde{E} \centerdot \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \\
&=\; 2\,(\eta \,\mathbf{x}\, E) \centerdot \left(\nabla\varphi \;\mathbf{x} \left\{ DE + \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} \right\}\right) \qquad (3.33) \\
&\quad - 2\,(\eta \,\mathbf{x}\, \widetilde{E}) \centerdot \left(\nabla\varphi \;\mathbf{x} \left\{ \widetilde{D}\widetilde{E} + \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial x_i} \right\}\right) \\
&\quad - \frac{\partial\varphi}{\partial\eta}\, DE \centerdot E \;+\; \frac{\partial\varphi}{\partial\eta}\, \widetilde{D}\widetilde{E} \centerdot \widetilde{E} \\
&\quad + 2\,\frac{\partial\varphi}{\partial\eta}\, \widetilde{E} \centerdot \sum_{i=1}^{3} A_i \frac{\partial \tilde{u}}{\partial \eta}\widetilde{E} \;-\; 2\,\frac{\partial\varphi}{\partial\eta}\, \widetilde{E} \centerdot \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i}\;.
\end{aligned}$$

If we use the interface conditons $\eta\,\mathbf{x}\,E = \eta\,\mathbf{x}\,\widetilde{E}$ together with (2.12) we can simplify some terms of $K$:

$$
\begin{aligned}
&2\,(\eta\,\mathbf{x}\,E)\,\boldsymbol{.}\Big(\nabla\varphi\,\mathbf{x}\Big\{DE+\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}\Big\}\Big) - 2\,(\eta\,\mathbf{x}\,\widetilde{E})\,\boldsymbol{.}\Big(\nabla\varphi\,\mathbf{x}\Big\{\widetilde{D}\widetilde{E}+\sum_{i=1}^{3}A_i\frac{\partial \tilde u}{\partial x_i}\Big\}\Big) = \\
&\quad = 2\,(\eta\,\mathbf{x}\,\widetilde{E})\,\boldsymbol{.}\Big(\nabla\varphi\,\mathbf{x}\Big\{DE+\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}\Big\}\Big) \\
&\qquad - 2\,(\eta\,\mathbf{x}\,\widetilde{E})\,\boldsymbol{.}\Big(\nabla\varphi\,\mathbf{x}\Big\{\widetilde{D}\widetilde{E}+\sum_{i=1}^{3}A_i\frac{\partial \tilde u}{\partial x_i}\Big\}\Big) \\
&\quad = 2\,\frac{\partial\varphi}{\partial\eta}\,\widetilde{E}\,\boldsymbol{.}\Big\{DE+\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}\Big\} - 2\,(\widetilde{E}\,\boldsymbol{.}\nabla\varphi)\Big\{DE+\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}\Big\}\boldsymbol{.}\,\eta \\
&\qquad - 2\,\frac{\partial\varphi}{\partial\eta}\,\widetilde{E}\,\boldsymbol{.}\Big\{\widetilde{D}\widetilde{E}+\sum_{i=1}^{3}A_i\frac{\partial \tilde u}{\partial x_i}\Big\} + 2\,(\widetilde{E}\,\boldsymbol{.}\nabla\varphi)\Big\{\widetilde{D}\widetilde{E}+\sum_{i=1}^{3}A_i\frac{\partial \tilde u}{\partial x_i}\Big\}\boldsymbol{.}\,\eta \\
&\quad = 2\,\frac{\partial\varphi}{\partial\eta}\,\widetilde{E}\,\boldsymbol{.}\Big\{DE+\sum_{i=1}^{3}A_i\frac{\partial u}{\partial x_i}\Big\} - 2\,\frac{\partial\varphi}{\partial\eta}\,\widetilde{E}\,\boldsymbol{.}\Big\{\widetilde{D}\widetilde{E}+\sum_{i=1}^{3}A_i\frac{\partial \tilde u}{\partial x_i}\Big\}\,.
\end{aligned}
\tag{3.34}
$$

Substitution of identity (3.34) into (3.33) give us that

$$
K = -\frac{\partial\varphi}{\partial\eta}\Big\{D(E-\widetilde{E})\,\boldsymbol{.}\,(E-\widetilde{E}) + (\widetilde{D}-D)\widetilde{E}\,\boldsymbol{.}\,\widetilde{E}\Big\}\,. \tag{3.35}
$$

Using (3.35) in (3.31) we finally deduce that

$$
\begin{aligned}
V_{k-1}-V_k = -\frac{\partial\varphi}{\partial\eta}\Bigg\{&\sum_{i,j=1}^{3}(P_{ij}-\widetilde{P}_{ij})\,\frac{\partial u}{\partial x_j}\,\boldsymbol{.}\,\frac{\partial u}{\partial x_i} \\
&+\sum_{i,j=1}^{3}\widetilde{P}_{ij}\Big(\frac{\partial \tilde u}{\partial x_j}-\frac{\partial u}{\partial x_j}\Big)\boldsymbol{.}\Big(\frac{\partial \tilde u}{\partial x_i}-\frac{\partial u}{\partial x_i}\Big) \\
&+(\tilde\beta-\beta)\Big[|\widetilde{H}\,\mathbf{x}\,\eta|^2+\frac{\tilde\beta}{\beta}\,|\widetilde{H}\,\boldsymbol{.}\,\eta|^2\Big] \\
&+D(E-\widetilde{E})\,\boldsymbol{.}\,(E-\widetilde{E}) + (\widetilde{D}-D)\widetilde{E}\,\boldsymbol{.}\,\widetilde{E}\Bigg\}
\end{aligned}
$$

which completes the proof of Lemma 3.1. ∎

**Proof of Lemma 3.2:** Straightforward calculations using (3.8) and $\varphi(x)$ chosen as in (3.12) lead us to the identity

$$
\begin{aligned}
J_k \;=\;& \delta \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \;-\; 2\,\delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p\, \partial x_i}\, A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \\
&-\; \delta |u_t^{(k)}|^2 + 2\,\delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_i\, \partial x_p}\, d_{ij}^{(k)} E_j^{(k)} E_p^{(k)} \;+\; 2\,\delta \sum_{i,j=1}^{3} \frac{\partial^2 \Phi}{\partial x_i\, \partial x_j}\, \beta^{(k)} H_i^{(k)} H_j^{(k)} \\
&-\; \delta\, D^{(k)} E^{(k)} \cdot E^{(k)} \;-\; \delta\, \beta^{(k)} |H^{(k)}|^2 \\
&+\; 2\,\delta\, E^{(k)} \cdot \left\{ \sum_{i,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_i\, \partial x_p} A_p \frac{\partial u^{(k)}}{\partial x_i} + \left( \sum_{j=1}^{3} A_j \frac{\partial u^{(k)}}{\partial x_j} \cdot \nabla \right) \nabla \Phi - \sum_{j=1}^{3} A_j \frac{\partial u^{(k)}}{\partial x_j} \right\} .
\end{aligned}
\tag{3.36}
$$

Let us estimate the terms on the right hand side of (3.36): We claim that for any $\varepsilon > 0$ we have that

$$
\begin{aligned}
&- 2\,\delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p\, \partial x_i}\, A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \;\le \\
&\qquad \le\; \delta\, \varepsilon \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \\
&\qquad\quad +\; \delta\, \varepsilon^{-1} \sum_{i,j=1}^{3} A_{ij}^{(k)} \left( \sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p\, \partial x_j} \frac{\partial u^{(k)}}{\partial x_p} \right) \cdot \left( \sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p\, \partial x_i} \frac{\partial u^{(k)}}{\partial x_p} \right) .
\end{aligned}
\tag{3.37}
$$

In fact, let $v_i = \displaystyle\sum_{p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p\, \partial x_i} \frac{\partial u^{(k)}}{\partial x_p}$ and $\varepsilon > 0$ then, we can write

$$
\begin{aligned}
-2 \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot v_i \;=\;& - \sum_{i,j=1}^{3} A_{ij}^{(k)} \left( \sqrt{\varepsilon}\, \frac{\partial u^{(k)}}{\partial x_j} + \frac{1}{\sqrt{\varepsilon}}\, v_j \right) \cdot \left( \sqrt{\varepsilon}\, \frac{\partial u^{(k)}}{\partial x_i} + \frac{1}{\sqrt{\varepsilon}}\, v_i \right) \\
&+\; \varepsilon \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \;+\; \frac{1}{\varepsilon} \sum_{i,j=1}^{3} A_{ij}^{(k)} v_j \cdot v_i \\
\le\;& \varepsilon \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \;+\; \varepsilon^{-1} \sum_{i,j=1}^{3} A_{ij}^{(k)} v_j \cdot v_i
\end{aligned}
$$

because $A_{ij}^{(k)}$ satisfies assumption 3) of Hypothesis I. This proves (3.37). Let

$$
c_2 \;=\; \max_{\substack{x \in \overline{\Omega} \\ i,j=1,2,3}} \|A_{ij}(x)\| \,, \qquad c_3 \;=\; \max_{\substack{x \in \overline{\Omega} \\ i,j=1,2,3}} \left| \frac{\partial^2 \Phi}{\partial x_i\, \partial x_j} \right|
$$

where $\| \ \|$ denotes the norm of the matrix. With this notations, we have that

$$|v_i| \le c_3 \left\{ \sum_{j=1}^{3} \left| \frac{\partial u^{(k)}}{\partial x_j} \right| \right\}$$

and

$$\begin{aligned} \left| \sum_{i,j=1}^{3} A_{ij}^{(k)} v_j \cdot v_i \right| &\le \sum_{i,j=1}^{3} \|A_{ij}^{(k)}(x)\| \, |v_j| \, |v_i| \le c_2 \left( \sum_{j=1}^{3} |v_j| \right)^2 \\ &\le 9 \, c_2 \, c_3^2 \left\{ \sum_{i=1}^{3} \left| \frac{\partial u^{(k)}}{\partial x_i} \right| \right\}^2 \le 27 \, c_2 \, c_3^2 \sum_{i=1}^{3} \left| \frac{\partial u^{(k)}}{\partial x_i} \right|^2 . \end{aligned} \tag{3.38}$$

From (3.38) we deduce that

$$\left| \sum_{i,j=1}^{3} A_{ij}^{(k)} v_j \cdot v_i \right| \le 27 \, c_2 \, c_3^2 \, c_0^{-1} \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} \tag{3.39}$$

where $c_0$ is the positive constant in Hypothesis I (item 3)). Using (3.39) into (3.37) we get that

$$\begin{aligned} &- 2 \, \delta \sum_{i,j,p=1}^{3} \frac{\partial^2 \Phi}{\partial x_p \, \partial x_i} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_p} \le \\ &\qquad \le \delta \left\{ \varepsilon + 27 \, c_2 \, c_3^2 \, c_0^{-1} \varepsilon^{-1} \right\} \sum_{i,j=1}^{3} A_{ij}^{(k)} \frac{\partial u^{(k)}}{\partial x_j} \cdot \frac{\partial u^{(k)}}{\partial x_i} . \end{aligned} \tag{3.40}$$

Let us bound the last term on the right hand side of (3.36). Let

$$c_4 = \max_{j=1,2,3} \|A_j\|$$

then

$$\sum_{i,p=1}^{3} \left| \frac{\partial^2 \Phi}{\partial x_i \, \partial x_p} A_p \frac{\partial u^{(k)}}{\partial x_i} \right| \le 3 \, c_4 \, c_3 \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right| \tag{3.41}$$

and

$$\sum_{j=1}^{3} \left| A_j \frac{\partial u^{(k)}}{\partial x_j} \right| \le c_4 \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right| . \tag{3.42}$$

Also

$$\sum_{j=1}^{3}\left|\left(A_j\,\frac{\partial u^{(k)}}{\partial x_j}\,\boldsymbol{.}\nabla\right)\nabla\Phi\right| = \sum_{j=1}^{3}\left\{\left(a_{1j}\,\frac{\partial^2\Phi}{\partial x_1^2}+a_{2j}\,\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}+a_{3j}\,\frac{\partial^2\Phi}{\partial x_1\,\partial x_3}\right)^2 + \left(a_{1j}\,\frac{\partial^2\Phi}{\partial x_1\,\partial x_2}+a_{2j}\,\frac{\partial^2\Phi}{\partial x_2^2}+a_{3j}\,\frac{\partial^2\Phi}{\partial x_2\,\partial x_3}\right)^2 + \left(a_{1j}\,\frac{\partial^2\Phi}{\partial x_1\,\partial x_3}+a_{2j}\,\frac{\partial^2\Phi}{\partial x_2\,\partial x_3}+a_{3j}\,\frac{\partial^2\Phi}{\partial x_3^2}\right)^2\right\}^{1/2}$$

where $(a_{1j},a_{2j},a_{3j}) = A_j\dfrac{\partial u^{(k)}}{\partial x_j}\cdot$

Thus

$$\sum_{j=1}^{3}\left|\left(A_j\,\frac{\partial u^{(k)}}{\partial x_j}\,\boldsymbol{.}\nabla\right)\nabla\Phi\right| \le 3\,c_3\,c_4\sum_{j=1}^{3}\left|\frac{\partial u^{(k)}}{\partial x_j}\right| . \tag{3.43}$$

From (3.41)–(3.43) we obtain the estimate

$$\begin{aligned} 2\,\delta\,E^{(k)}\,\boldsymbol{.}\left\{\sum_{i,p=1}^{3}\frac{\partial^2\Phi}{\partial x_i\,\partial x_p}\,A_p\,\frac{\partial u^{(k)}}{\partial x_i} + \left(\sum_{j=1}^{3}A_j\,\frac{\partial u^{(k)}}{\partial x_j}\,\boldsymbol{.}\nabla\right)\nabla\Phi - \sum_{j=1}^{3}A_j\,\frac{\partial u^{(k)}}{\partial x_j}\right\} &\le \\ \le\; 2\,\delta\,|E^{(k)}|\,\{6\,c_3\,c_4+c_4\}\sum_{j=1}^{3}\left|\frac{\partial u^{(k)}}{\partial x_j}\right| & \\ \le\; \delta\,\{6\,c_3\,c_4+c_4\}\,\varepsilon_1|E^{(k)}|^2 + \delta\,\varepsilon_1^{-1}\{6\,c_3\,c_4+c_4\}\left(\sum_{j=1}^{3}\left|\frac{\partial u^{(k)}}{\partial x_j}\right|\right)^2 & \end{aligned} \tag{3.44}$$

for any $\varepsilon_1 > 0$. Since $A_{ij}$ satisfies assumption 3) in Hypothesis I, we get the bound

$$\left(\sum_{j=1}^{3}\left|\frac{\partial u^{(k)}}{\partial x_j}\right|\right)^2 \le 3\sum_{j=1}^{3}\left|\frac{\partial u^{(k)}}{\partial x_j}\right|^2 \le 3\,c_0^{-1}\sum_{j=1}^{3}A_{ij}^{(k)}\,\frac{\partial u^{(k)}}{\partial x_j}\,\boldsymbol{.}\,\frac{\partial u^{(k)}}{\partial x_i} .$$

Thus, the left hand side of (3.44) can be bound by

$$\delta\,(6\,c_3\,c_4+c_4)\,\varepsilon_1|E^{(k)}|^2 + 3\,\delta\,\varepsilon_1^{-1}c_0^{-1}(6\,c_3+1)\,c_4\sum_{i,j=1}^{3}A_{ij}^{(k)}\,\frac{\partial u^{(k)}}{\partial x_j}\,\boldsymbol{.}\,\frac{\partial u^{(k)}}{\partial x_i} \tag{3.45}$$

for any $\varepsilon_1 > 0$. Finally,

$$\begin{aligned} 2\,\delta\sum_{i,j,p=1}^{3}\frac{\partial^2\Phi}{\partial x_i\,\partial x_p}\,d_{ij}^{(k)}E_j^{(k)}E_p^{(k)} + 2\,\delta\sum_{i,j=1}^{3}\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}\,\beta^{(k)}H_i^{(k)}H_j^{(k)} &\le \\ \le\; 6\,\delta\,c_4\|D^{(k)}\|\,d_0^{-1}(DE^{(k)}\,\boldsymbol{.}\,E^{(k)}) + 6\,\delta\,c_4\,\beta^{(k)}|H^{(k)}|^2 & \end{aligned} \tag{3.46}$$

where $\|D^{(k)}\|$ denotes the norm of the matrix $D^{(k)}$ and $d_0 > 0$ is as in Hypothesis I (item 2)).

Using (3.40), (3.44), (3.45) and (3.46) we deduce the following estimate for $J_k$ given by (3.29):

$$\begin{aligned} J_k \,\le\, & \delta\Big\{1+\varepsilon+27\,c_2\,c_3^2\,c_0^{-1}\varepsilon^{-1}+3\,(6\,c_3+1)\,c_4\,\varepsilon_1^{-1}c_0^{-1}\Big\}\sum_{i,j=1}^{3}A_{ij}^{(k)}\,\frac{\partial u^{(k)}}{\partial x_j}\cdot\frac{\partial u^{(k)}}{\partial x_i}\\ & +\delta\Big\{6\,c_4\|D^{(k)}\|\,d_0^{-1}+(6\,c_3+1)\,c_4\,\varepsilon_1\,d_0^{-1}-1\Big\}\,(D^{(k)}E^{(k)}\cdot E^{(k)})\\ & +\delta\,\beta^{(k)}\,(6\,c_2-1)\,|H^{(k)}|^2\,-\,\delta\,|u_t^{(k)}|^2 \end{aligned} \tag{3.47}$$

for any $\varepsilon > 0$, $\varepsilon_1 > 0$, where we use Hypothesis I, (item 2)). Let us choose $\varepsilon = 3\,c_3(c_2\,c_0^{-1})^{1/2}$ and $\varepsilon_1 = (3\,d_0\,c_0^{-1})^{1/2}$ in (3.47) to obtain the desired estimate of Lemma 3.2 with

$$\begin{aligned} c_5 \,=\, \max\Big\{ & 1+12\,c_3(c_2\,c_0^{-1})^{1/2}+\sqrt{3}\,c_4(6\,c_3+1)\,(c_0\,d_0)^{-1/2},\\ & 6\,c_4\max_k\|D^{(k)}\|+\sqrt{3}\,c_4(6\,c_3+1)\,(c_0\,d_0)^{-1/2},\quad 6\,c_4\Big\}\,. \ \blacksquare \end{aligned}$$

**Proof of Lemma 3.4:** From now on we will choose $\delta = \delta_1$ in the definition of $\varphi(x)$ in (3.12). Now, let us get a bound for the term $\displaystyle\int_0^T\!\!\int_S V_n\,dS\,dt$ in (3.9). Using the boundary conditions (1.3) we can rewrite $V_n$ as

$$\begin{aligned} V_n \,=\, & -\frac{\partial}{\partial t}\left\{(t+t_0)\left[b|u|^2+\gamma\left|\int_0^T\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(x)\,(t-\tau)\Big)\,d\tau\right|^2\right]\right\}\\ & -\frac{\partial}{\partial t}\Big\{a|u|^2\Big\}-b|u|^2-\Big\{2\,(t+t_0)\,a-\frac{\partial\varphi}{\partial\eta}\Big\}\,|u_t|^2\\ & -\frac{\partial\varphi}{\partial\eta}\left\{\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}\right\}-2\,(\nabla\varphi\cdot\nabla)\,u\cdot(au_t+bu)\\ & -2\,(t{+}t_0)\,\alpha\,|H\,\mathbf{x}\,\eta|^2-\Big\{2\,(t{+}t_0)\,\sigma-1\Big\}\,\gamma\left|\int_0^t[H\,\mathbf{x}\,\eta]\,\exp\Big(-\sigma(1{-}\tau)\Big)\,d\tau\right|^2\\ & +\frac{\partial\varphi}{\partial\eta}\,DE\cdot E+\frac{\partial\varphi}{\partial\eta}\,\beta|H|^2-2\,(DE\cdot\eta)\,(E\cdot\nabla\varphi)\\ & -2\,\beta(H\cdot\eta)\,(H\cdot\nabla\varphi)+2\,(\eta\,\mathbf{x}\,E)\cdot\left(\nabla\varphi\,\mathbf{x}\sum_{i=1}^{3}A_i\,\frac{\partial u}{\partial x_i}\right). \end{aligned} \tag{3.48}$$

Let $c_j$, $j = 7, 8, ..., 11$, be the following constants

$$c_6 = 2\,\lambda_0\, c_0^{-1}\delta_0^{-1}\,, \qquad c_7 = 8\,\lambda_0\, a_1^2\, c_0^{-1}\delta_0^{-1}\,, \qquad c_8 = c_{10} = 2\,d_1\,,$$
$$c_9 = 16\,\alpha_1^2\, c_4^2\, c_0^{-1}\delta_0^{-1}\,, \qquad c_{11} = 16\,\lambda_0\,\gamma_1\, c_4^2\, c_0^{-1}\delta_0^{-1}$$

where $a_1 = \max_{x\in S} a(x)$, $d_1 = \max_{x\in\overline{\Omega}} \|D(x)\|$, $\alpha_1 = \max_{x\in S} \alpha(x)$ and $\gamma_1 = \max_{x\in S} \gamma(x)$. The constant $c_4$ was defined in the proof of Lemma 3.2 (see below (3.40)) and $c_0$ appeared in Hypothesis I (item 3)).

We will get a bound for some of the terms on the right hand side of (3.41). We use the identity

$$H \;=\; \eta\,\mathbf{x}\,(H\,\mathbf{x}\,\eta) \;+\; \eta\, H\,\boldsymbol{.}\,\eta$$

to rewrite the expression

$$-2\,\beta(H\,\boldsymbol{.}\,\eta)\,(H\,\boldsymbol{.}\nabla\varphi) \;=\; -2\,\beta(\nabla\varphi\,\boldsymbol{.}\,\eta)\,|H\,\boldsymbol{.}\,\eta|^2 -\, 2\,\beta(H\,\boldsymbol{.}\,\eta)\,(H\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta)\;.$$

Now we can obtain a bound for the term

$$\begin{aligned}
&\frac{\partial\varphi}{\partial\eta}\,\beta|H|^2 -\, 2\,\beta(H\,\boldsymbol{.}\,\eta)\,(H\,\boldsymbol{.}\nabla\varphi) \;= \\
&\quad= \frac{\partial\varphi}{\partial\eta}\,\beta|H\,\mathbf{x}\,\eta|^2 - \frac{\partial\varphi}{\partial\eta}\,\beta|H\,\boldsymbol{.}\,\eta|^2 -\, 2\,\beta(H\,\boldsymbol{.}\,\eta)\,(H\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta) \\
&\quad\le \frac{\partial\varphi}{\partial\eta}\,\beta|H\,\mathbf{x}\,\eta|^2 - \frac{\partial\varphi}{\partial\eta}\,\beta|H\,\boldsymbol{.}\,\eta|^2 + \beta\,\delta_0|\nabla\varphi|\,|H\,\boldsymbol{.}\,\eta|^2 + \beta\,\delta_0^{-1}|\nabla\varphi|\,|H\,\mathbf{x}\,\eta|^2 \\
&\quad\le \Big(\nabla\varphi\,\boldsymbol{.}\,\eta + \delta_0^{-1}|\nabla\varphi|\Big)\,\beta|H\,\mathbf{x}\,\eta|^2\;.
\end{aligned}\tag{3.49}$$

Next, we use the identity

$$E \;=\; \eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta) + \eta\, E\,\boldsymbol{.}\,\eta$$

in order to obtain that

$$\begin{aligned}
\frac{\partial\varphi}{\partial\eta}\,(DE\,\boldsymbol{.}\,E) \;=\; &\frac{\partial\varphi}{\partial\eta}\,(D\eta\,\boldsymbol{.}\,\eta)\,|E\,\boldsymbol{.}\,\eta|^2 + 2\,\frac{\partial\varphi}{\partial\eta}\,(E\,\boldsymbol{.}\,\eta)\,\Big(D\eta\,\boldsymbol{.}\,\{\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\}\Big) \\
&+ \frac{\partial\varphi}{\partial\eta}\,D\{\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\}\,\boldsymbol{.}\,\{\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\}
\end{aligned}\tag{3.50}$$

and

$$\begin{aligned}
-2\,(DE\,\boldsymbol{.}\,\eta)(E\,\boldsymbol{.}\nabla\varphi) \;=\; &-2\,(D\eta\,\boldsymbol{.}\,\eta)\,(E\,\boldsymbol{.}\,\eta)\,(E\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta) \\
&- 2\,\frac{\partial\varphi}{\partial\eta}\,(D\eta\,\boldsymbol{.}\,\eta)\,|E\,\boldsymbol{.}\,\eta|^2 - 2\,\frac{\partial\varphi}{\partial\eta}\,(E\,\boldsymbol{.}\,\eta)\,\Big(D\eta\,\boldsymbol{.}\,\{\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\}\Big) \\
&- 2\,D\,\eta\,\boldsymbol{.}\,\{\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\}\,(E\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta)\;.
\end{aligned}\tag{3.51}$$

Using (3.43), (3.50) and (3.51) we get a representation and therefore an inequality as follows

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial\eta}(DE\,\boldsymbol{.}\,E) - 2\,(DE\,\boldsymbol{.}\,\eta)\,(E\,\boldsymbol{.}\nabla\varphi) \;= \\
&\qquad = -\frac{\partial\varphi}{\partial\eta}(D\eta\,\boldsymbol{.}\,\eta)\,|E\,\boldsymbol{.}\,\eta|^2 - 2\,(D\eta\,\boldsymbol{.}\,\eta)\,(E\,\boldsymbol{.}\,\eta)\,(E\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta) \\
&\qquad\quad - 2\,(D\eta\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(E\,\mathbf{x}\,\eta)\,(E\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta) \\
&\qquad\quad + \frac{\partial\varphi}{\partial\eta}\,D\Big(\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\Big)\,\boldsymbol{.}\,\Big(\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\Big) \\
&\qquad \le \delta_0^{-1}|\nabla\varphi|\,(D\eta\,\boldsymbol{.}\,\eta)\,|E\,\mathbf{x}\,\eta|^2 - 2\,(D\eta\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(E\,\mathbf{x}\,\eta)\,(E\,\mathbf{x}\,\eta)\,\boldsymbol{.}\,(\nabla\varphi\,\mathbf{x}\,\eta) \\
&\qquad\quad + \frac{\partial\varphi}{\partial\eta}\,D\Big(\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\Big)\,\boldsymbol{.}\,\Big(\eta\,\mathbf{x}\,(E\,\mathbf{x}\,\eta)\Big) \\
&\qquad \le \Big\{(\nabla\varphi\,\boldsymbol{.}\,\eta) + \delta_0^{-1}|\nabla\varphi| + 2\,|\nabla\varphi|\Big\}\,d_1|E\,\mathbf{x}\,\eta|^2\;.
\end{aligned}
\tag{3.52}
$$

From the boundary conditions (1.3) it follows that

$$
|E\,\mathbf{x}\,\eta|^2 \;\le\; 2\,\alpha^2|H\,\mathbf{x}\,\eta|^2 + 2\,\gamma^2\left|\int_0^T\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(x)\,(t-\tau)\Big)\,d\tau\right|^2
$$

which together with (3.52) and (3.49) give us the estimate

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial\eta}(DE\,\boldsymbol{.}\,E) + \frac{\partial\varphi}{\partial\eta}\beta|H|^2 - 2\,(DE\,\boldsymbol{.}\,\eta)\,(E\,\boldsymbol{.}\nabla\varphi) - 2\beta(H\boldsymbol{.}\,\eta)\,(H\boldsymbol{.}\nabla\varphi) \;\le \\
&\qquad\quad \le \Big[\nabla\varphi\,\boldsymbol{.}\,\eta + \delta_0^{-1}|\nabla\varphi| + 2\,|\nabla\varphi|\Big]\,[2\,d_1\,\alpha^2 + \beta]\,|H\,\mathbf{x}\,\eta|^2 \\
&\qquad\qquad + \Big[\nabla\varphi\,\boldsymbol{.}\,\eta + \delta_0^{-1}|\nabla\varphi| + 2\,|\nabla\varphi|\Big] \\
&\qquad\qquad\quad \cdot\, 2\,d_1\gamma^2\left|\int_0^t\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(x)\,(t-\tau)\Big)\,d\tau\right|^2 \\
&\qquad\quad \le (3+\delta_0^{-1})\,|\nabla\varphi|\,(2\,d_1\alpha^2+\beta)\,|H\,\mathbf{x}\,\eta|^2 \\
&\qquad\qquad + 2\,d_1\gamma^2(3+\delta_0^{-1})\,|\nabla\varphi|\left|\int_0^t\Big[H(x,\tau)\,\mathbf{x}\;\eta\Big]\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right|^2 .
\end{aligned}
\tag{3.53}
$$

Let $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ positive real numbers. With the notations given above we have the following estimates

$$
\begin{aligned}
-2\,(\nabla\varphi\,\boldsymbol{.}\nabla)u\,\boldsymbol{.}\,(au_t+bu) \;\le\; &\lambda_0\,a_1\varepsilon_2^{-1}|u_t|^2 + \lambda_0\,b^2\varepsilon_3^{-1}|u|^2 \\
&+ (\varepsilon_2\,a_1c_0^{-1} + \varepsilon_3\,c_0^{-1})\,|\nabla\varphi|\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\,\boldsymbol{.}\,\frac{\partial u}{\partial x_i}
\end{aligned}
\tag{3.54}
$$

where we used Cauchy–Schwarz inequality and Hypothesis I (item 3)). By the same reasons and the boundary conditions (1.3) we deduce that

$$
\begin{aligned}
2\,(\eta\,\mathbf{x}\,E)\,\boldsymbol{\cdot}\left(\nabla\varphi\,\mathbf{x}\sum_{i=1}^{3}A_i\,\frac{\partial u}{\partial x_i}\right) &\le \\
&\le |\nabla\varphi|\,c_4\,\varepsilon_1^{-1}|\eta\,\mathbf{x}\,E|^2 + 3\,|\nabla\varphi|\,c_4\,\varepsilon_1\,c_0^{-1}\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} \\
&\le 2\,\lambda_0\,c_4\,\varepsilon_1^{-1}\left\{\alpha_1^2\,|H\,\mathbf{x}\,\eta|^2+\gamma_1\gamma\left|\int_0^t\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right|^2\right\} \qquad (3.55)\\
&\quad + 3\,|\nabla\varphi|\,c_4\,\varepsilon_1\,c_0^{-1}\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i}\,.
\end{aligned}
$$

Now, we choose $\varepsilon_1 = \dfrac{1}{8}\,c_0\,\delta_0\,c_4^{-1}$, $\varepsilon_2 = \dfrac{1}{8}\,c_0\,\delta_0\,a_1^{-1}$ and $\varepsilon_3 = \dfrac{1}{2}\,c_0\,\delta_0$ in (3.54) and (3.55). Thus, the summation of the left hand sides is less than or equal to

$$
\begin{aligned}
&(8\,\lambda_0\,a_1^2\,c_0^{-1}\delta_0^{-1})\,|u_t|^2 + 2\,\lambda_0\,b^2c_0^{-1}\delta_0^{-1}|u|^2+\delta_0|\nabla\varphi|\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i}\;+\\
&\quad+\;(16\,\lambda_0\,c_4^2\,c_0^{-1}\delta_0^{-1})\left\{\alpha_1^2\,|H\,\mathbf{x}\,\eta|^2+\gamma_1\gamma\left|\int_0^t\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right|^2\right\}.
\end{aligned}
$$

Hence, from (3.48), (3.53) and the above discussion, we obtain the estimate

$$
\begin{aligned}
V_n \le\; &-\frac{\partial}{\partial t}\left\{(t+t_0)\,b|u|^2+\gamma\left|\int_0^t\Big[H(x,\tau)\,\mathbf{x}\,\eta\Big]\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right|^2\right\}\\
&-\frac{\partial}{\partial t}\{a|u|\}^2-\{1-2\,\lambda_0\,b\,c_0^{-1}\delta_0^{-1}\}\,b|u|^2\\
&-\left\{2\,(t+t_0)a-\frac{\partial\varphi}{\partial\eta}-8\,\lambda_0\,a_1^2\,c_0^{-1}\delta_0^{-1}\right\}|u_t|^2\\
&-\left\{\frac{\partial\varphi}{\partial\eta}-\delta_0|\nabla\varphi|\right\}\sum_{i,j=1}^{3}A_{ij}\,\frac{\partial u}{\partial x_j}\boldsymbol{\cdot}\frac{\partial u}{\partial x_i} \qquad (3.56)\\
&-\left\{2\,(t+t_0)\alpha-(3+\delta_0^{-1})\,(2\,d_1\,\alpha^2+\beta)\,|\nabla\varphi|-16\,\lambda_0\,c_4^2\,\delta_0^{-1}\alpha_1^2\right\}|H\,\mathbf{x}\,\eta|^2\\
&-\left\{2\,(t+t_0)-1-2\,d_1\gamma_1(3+\delta_0^{-1})\,|\nabla\varphi|-16\,\lambda_0\,c_4^2\,c_0^{-1}\delta_0^{-1}\,\gamma_1\right\}\\
&\quad\cdot\,\gamma\left|\int_0^t[H\,\mathbf{x}\,\eta]\,\exp\Big(-\sigma(t-\tau)\Big)\,d\tau\right|^2.
\end{aligned}
$$

Integration of (3.56) in $S\times[0,T]$ proves Lemma 3.4. ∎

## 4 – Exact controllability

In this section, we use the result of Theorem 3.5 to prove exact boundary controllability to an arbitrary state of solutions of (1.1), (1.2), (1.4) and (1.7) when $\gamma \equiv 0$.

**Theorem 4.1.** *Under the assumptions of Theorem 3.5 and $\gamma \equiv 0$, there exists $\widetilde{T} > 0$ such that for any $T > \widetilde{T}$, given any initial data $f \in M_1$ and any terminal state $g \in M_1$, there exists a boundary control $\{\vec{p}(x,t), \vec{q}(x,t)\}$ belonging to $[L^2(S\times(0,T))]^6$ driving the system (1.1), (1.2), (1.4), (1.7) to the terminal state $g(x)$ at time $T$:*

$$u(x,T) = g_1(x)\,, \quad u_t(x,T) = g_2(x)\,, \quad E(x,T) = g_3(x) \quad \text{and} \quad H(x,T) = g_4(x)\,.$$

*Moreover*

$$\|\vec{p}\|_W^2 + \|\vec{q}\|_W^2 \;\leq\; c\Big\{\|f\|_Z^2 + \|g\|_Z^2\Big\} \tag{4.1}$$

*for positive constant $c$ where $W = [L^2(S\times(0,T))]^3$.*

**Proof:** Let $\widetilde{T} = T_0 \left[\left(\dfrac{c_{13}}{T_0}\right)^{1/1-p} - 1\right] > 0$. We consider the following equation in $M_1$:

$$v - U^*(T)\,U(T)\,v \;=\; f - U^*(T)\,g \tag{4.2}$$

where $\{U(t)\}_{t\geq 0}$ is the semigroup associated with problem (1.1)–(1.4). The operator $F(T) = U^*(T)\,U(T)$ takes $M_1$ into itself and $\|F(T)\| < 1$ for any $T > \widetilde{T}$ by Corollary 3.6. Thus we can solve (4.2) for any $f, g \in M_1$ and

$$\|v\|_Z \;\leq\; c\Big\{\|f\|_Z + \|g\|_Z\Big\}\,.$$

Consequently, if we choose $v = (I - F(T))^{-1}(f - U^*(T)g)$ then we will have that

$$\begin{aligned}(u_1, u_2, u_3, u_4) \;&=\; U(t)v - U^*(T-t)\,(U(T)v - g)\\ &\equiv\; (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4) - (w_1, w_2, w_3, w_4)\end{aligned}$$

is a weak solution of (1.1), (1.2), (1.4) and (1.7) with

$$\vec{p}(x,t) = -a\,\tilde{v}_2 - a\,w_2\,, \qquad \vec{q}(x,t) = \alpha\,\eta\,\mathbf{x}\,\{(\tilde{v}_4 + w_4)\,\mathbf{x}\,\eta\}\,.$$

We observe that

$$(u_1, u_2, u_3, u_4)\big|_{t=T} \;=\; g(x)$$

therefore by the energy identity we obtain (4.1).

*ACKNOWLEDGEMENT* – We would like to express our sincere thanks to the Referee of this journal for several suggestions which improved the final version of our manuscript. The first author (B.K.) was supported by a Grant of FAPERJ (Brasil) in the project E-26/151.523 and he is a Visiting Researcher at the National Laboratory of Scientific Computation (LNCC/MCT, Brasil). The second author (G.P.M.) was partially supported by a Grant of CNPq and PRONEX (MCT, Brasil). Part of this work was done while he was visiting the Centro de Modelamiento Matematico (CMM) of the Universidad de Chile, as part of the Chair "Marcel Dassault". He express his gratitude for the kind hospitality of the members of the CMM and especially to Prof. Carlos Conca.

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