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GLOBAL EXISTENCE OF 2D SLIGHTLY COMPRESSIBLE VISCOUS MAGNETO-FLUID MOTION

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Abstract: In this paper we study the equations of magneto-hydrodynamics for a 2D compressible viscous fluid, under periodic boundary conditions. It is well known that, as the Mach number goes to zero, the solutions of the compressible model approximate that of the incompressible one. In dimension 2 such incompressible limit solution is global in time. We prove that also the compressible solution exists for all time, provided that the Mach number is sufficiently small and the initial data are almost incompressible.

1 – Introduction

In this paper we study the following problem (see the Appendix for a derivation from the usual MHD equations and the meaning of all quantities)

$$\varepsilon^2(\rho_t + u\cdot\nabla\rho) + (1+\varepsilon^2\rho)\,\nabla\cdot u \;=\; 0\;, \tag{1}$$

$$\begin{aligned} u_t + (u\cdot\nabla)\,u + (1+\varepsilon^2\rho)^{\gamma-2}\,\nabla\rho + \frac{1}{2}\,(1+\varepsilon^2\rho)^{-1}\,\nabla|H|^2 \;-&\\ -\;(1+\varepsilon^2\rho)^{-1}\,(H\cdot\nabla)\,H - (1+\varepsilon^2\rho)^{-1}\,(\nu\,\Delta u + \eta\,\nabla\nabla\cdot u) \;=\; 0\;,& \end{aligned} \tag{2}$$

$$H_t + (u\cdot\nabla)H - (H\cdot\nabla)\,u + H\,\nabla\cdot u - \mu\,\Delta H \;=\; 0\;, \tag{3}$$

$$\nabla\cdot H \;=\; 0\;, \tag{4}$$

where ν, η, μ are given constants such that

$$\nu > 0\,, \qquad \eta+\nu > 0\,, \qquad \mu > 0\;.$$

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If the parameter ε, which represents the Mach number, is formally set to zero, one obtains the equations of magneto-hydrodynamics for the incompressible flow

$$U_t + (U \cdot \nabla)U + \frac{1}{2}\nabla|B|^2 - (B \cdot \nabla)B + \nabla P - \nu\,\Delta U = 0\,, \tag{5}$$

$$B_t + (U \cdot \nabla)B - (B \cdot \nabla)U - \mu\,\Delta B = 0\,, \tag{6}$$

$$\nabla \cdot B = 0\,, \tag{7}$$

$$\nabla \cdot U = 0\,. \tag{8}$$

In this work we restrict the discussion to two-space variables

$$(x, y) \in (\mathbb{R} \bmod 2\pi)^2 := T^2\ ;$$

hence, we assume that the functions

$$\rho(x,y,t), \quad u(x,y,t), \quad H(x,y,t), \quad U(x,y,t), \quad B(x,y,t), \quad P(x,y,t)$$

are 2π-periodic in x and y.

For $\varepsilon > 0$, the equations (1)–(4) are supplemented by the following initial conditions

$$\rho(x,y,0) = \rho_0(x,y)\,, \qquad u(x,y,0) = u_0(x,y)\,, \qquad H(x,y,0) = H_0(x,y)\ , \tag{9}$$

with $u_0, \rho_0, H_0 \in C^\infty(T^2)$. We assume that

$$\int_{T^2} \rho_0(x,y)\,dx\,dy = 0\ , \tag{10}$$

$$\int_{T^2} u_0(x,y)\,dx\,dy = 0\ , \tag{11}$$

$$\int_{T^2} H_0(x,y)\,dx\,dy = 0 \quad \text{and} \quad \nabla \cdot H_0 = 0\ . \tag{12}$$

For the incompressible equations (5)–(8) we consider the following initial conditions

$$U(x,y,0) = U_0(x,y)\,, \qquad B(x,y,0) = B_0(x,y)\ ,$$

with $U_0, B_0 \in C^\infty(T^2)$. We assume that

$$\int_{T^2} U_0(x,y)\,dx\,dy = 0 \quad \text{and} \quad \nabla \cdot U_0 = 0\ , \tag{13}$$

$$\int_{T^2} B_0(x,y)\,dx\,dy = 0 \quad \text{and} \quad \nabla \cdot B_0 = 0\ . \tag{14}$$

Moreover, to eliminate the free (time-depending) constant in the incompressible pressure, we impose the side condition

$$\int_{T^2} P(x,y,t)\,dx\,dy \;=\; 0\,, \qquad \forall\, t \ge 0\;. \tag{15}$$

It is well known that solutions of the compressible MHD equations (see Appendix) converge to the incompressible solution (U,P,B) as the Mach number tends to zero (see [1], [5]).

Moreover, it is known that the incompressible problem (5)–(8), (13)–(15) has a unique global classical solution $(U,P,B) \in C^\infty(T^2 \times [0,+\infty))$, and this solution, together with all its derivatives, tends to zero at an exponential rate as $t \to +\infty$. More precisely, we have that

$$\|U(\cdot,t)\|_j + \|P(\cdot,t)\|_j + \|B(\cdot,t)\|_j \;\le\; C_j\, e^{-c_j t}\,, \qquad t \ge 0, \quad j = 0,1,2,\ldots\,, \tag{16}$$

with $C_j > 0$, $c_j > 0$ independent of t, and where $\|\cdot\|_j$ denotes the usual norm in the Sobolev space $H^j(T^2)$, i.e.

$$\|u(\cdot,t)\|_j \;=\; \sum_{|\alpha|\le j} \int_{T^2} |D^\alpha u(x,y)|\,dx\,dy\;,$$

where $\alpha = (\alpha_1,\alpha_2)$ is a multi-index of order $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$.

The aim of this paper is to show all-time existence also for the compressible problem (1)–(4), provided that $\varepsilon > 0$ is sufficiently small, and the initial data u_0, ρ_0, H_0 are *almost* incompressible.

The main result of this paper is given by the following theorem.

Theorem 1.1. *Consider the compressible problem (1)–(4), supplemented by the initial conditions (9), with* $u_0, \rho_0, H_0 \in C^\infty(T^2)$, $\nu > 0$, $\varepsilon > 0$, $\eta + \nu > 0$, $\mu > 0$. *Assume also (10)–(14).*

There are $\varepsilon_0 = \varepsilon_0(U_0, B_0, \nu, \mu, \eta) > 0$ *and* $\delta_0 = \delta_0(U_0, B_0, \nu, \mu, \eta) > 0$ *so that the following holds. If* $0 < \varepsilon \le \varepsilon_0$ *and*

$$\|u_0 - U_0\|_3^2 + \varepsilon^2\,\|\rho_0 - P_0\|_3^2 + \|H_0 - B_0\|_3^2 \;\le\; \delta_0^2\;, \tag{17}$$

then the solution (u,ρ,H) *is in* $C^\infty(T^2 \times [0,+\infty))$. ∎

For the proof we adapt to our problem the approach of [3].

The paper is organized as follows. In Section 2 we introduce some notations and preliminaries; in Section 3 we introduce the problem obtained by the difference between the incompressible problem and the compressible one, which will be used as an auxiliary problem in order to obtain the existence result; in Section 4 we study the associated linear case; in Section 5 we obtain some properties of the spatial mean of the solutions of the auxiliary problem; in Section 6 we derive some a-priori estimates for the nonlinear terms of the equations and we prove the main result of the paper; in Section 7 we derive the equations we study from the usual MHD equations.

2 – Notation and preliminaries

With $\langle u,v\rangle = \sum \overline{u}_j\, v_j$ we denote the Euclidean inner product in any finite dimensional space, and with $|u| = \langle u,u\rangle^{\frac{1}{2}}$ the associated norm.

Also, if $A \in \mathbf{C}^{n\times n}$ is an $n\times n$ matrix, then $|A|$ denotes the corresponding matrix norm.

If $H=H^*$ and $G=G^*$ are Hermitian matrices in $\mathbf{C}^{n\times n}$, then we write $H\le G$ if and only if $G-H$ is positive semidefinite, i.e.

$$u^* H u \le u^* G u\,, \qquad \forall\, u\in \mathbf{C}^n \;.$$

If $u,v\colon T^2\to\mathbb{R}^n$ are functions with components $u_j,v_j\in L^2(T^2)$, then their L^2-inner product is

$$(u,v) \;=\; \int_{T^2} \Big\langle u(x,y), v(x,y)\Big\rangle\; dx\, dy \;,$$

and the corresponding norm is denoted by $\|\cdot\|$.

We recall Parseval's identity

$$(u,v) \;=\; \sum_{k\in\mathbf{Z}^2} \Big\langle \widehat{u}(k), \widehat{v}(k)\Big\rangle \;,$$

where

$$\widehat{w}(k) \;=\; \frac{1}{2\pi}\int_{T^2} e^{-i(k_1x+k_2y)}\, w(x,y)\; dx\, dy \;,$$

for every $w\in L^2(T^2)$.

We shall use some general estimates holding in the classical Sobolev spaces and based on the chain and the Leibniz' rule. For the proof of these results we refer to [2] and [3].

1. *Sobolev's Inequality*: Let $s>\dfrac{N}{2}$ and let $u\in H^s(T^N)$. Then, $u\in C(T^N)$ and

$$|u|_\infty \le\; c\,\|u\|_s \;. \tag{18}$$

Moreover,

(19) $$\forall\, u \in H^3(T^2) \quad |Du|_\infty \le c\,\|u\|_3 \;;$$

(20) $$\forall\, u \in H^4(T^2) \quad |D^2 u|_\infty \le C\,\|u\|_4 \;.$$

2. *Estimate based on the chain rule*: Let $s > \dfrac{N}{2}$ and let $u \in H^s(T^N, \mathbb{R}^n)$, $\phi \in C^s(\mathbb{R}^n)$, then

(21) $$\|D^\alpha(\phi\circ u)\| \;\le\; C\Big(1 + |u|_\infty^{|\alpha|-1}\Big)\,\|u\|_{|\alpha|} \;,$$

for each multi-index α, $1 \le |\alpha| \le s$.

3. *Estimates based on the Leibniz'rule*: Let $s > \dfrac{N}{2}$ and let $f, g \in H^s(T^N)$. Then,

(22) $$\|D^\alpha(fg)\| \;\le\; C\Big(|f|_\infty\,\|g\|_{|\alpha|} + |g|_\infty\,\|f\|_{|\alpha|}\Big) \;;$$

(23) $$\|D^\alpha(fg) - f D^\alpha g\| \;\le\; C\Big(|Df|_\infty\,\|g\|_{|\alpha|-1} + |g|_\infty\,\|f\|_{|\alpha|}\Big) \;,$$

for every multi-index α, $0 \le |\alpha| \le s$.

3 – Equations for the perturbed variables

Let us denote by (U, P, B) the solution of the incompressible problem (5)–(8). It is convenient to subtract (U, P, B) from the solution of (1)–(3) and to write the resulting system of equations in a symmetric form. We define the new variables r, u', H' by

$$\rho' = \rho - P\,, \qquad r = \varepsilon\,\rho'\,, \qquad u' = u - U\,, \qquad H' = H - B\;.$$

Straightforward calculations yield

(24) $$r_t + (U + u')\cdot\nabla r + \frac{1}{\varepsilon}\,\nabla\cdot u' \;=\; F_1 \;,$$

(25) $$u'_t + \Big((U+u')\cdot\nabla\Big)u' + \frac{1}{\varepsilon}\,\nabla r \,+ \\ +\,\nabla H'\cdot B - (B\cdot\nabla)H' - \nu\,\Delta u' - \eta\,\nabla\nabla\cdot u' \;=\; F_2 \;,$$

(26) $$H'_t + \Big((U+u')\cdot\nabla\Big)H' - (B\cdot\nabla)u' + B\,\nabla\cdot u' - \mu\,\Delta H' \;=\; F_3 \;,$$

where, by denoting $A = 1+\varepsilon^2 P + \varepsilon\, r$, we have

$$
\begin{aligned}
g_1 &= -(P_t + U\cdot\nabla P)\ ;\\
F_1 &= -\varepsilon\, u'\cdot\nabla P - \varepsilon P\,\nabla\cdot u' - r\,\nabla\cdot u' + \varepsilon\, g_1\ ;\\
F_2 &= -U_t - \Big((U+u')\cdot\nabla\Big)U - A^{\gamma-2}\,\nabla P + \frac{1}{\varepsilon}\,(1-A^{\gamma-2})\,\nabla r\\
&\quad - A^{-1}\,\nabla B\cdot(B+H') - A^{-1}\,\nabla H'\cdot H' + (1-A^{-1})\,\nabla H'\cdot B\\
&\quad + A^{-1}\Big((B+H')\cdot\nabla\Big)B + A^{-1}(H'\cdot\nabla)H' - (1-A^{-1})(B\cdot\nabla)H'\\
&\quad - (1-A^{-1})\,(\nu\,\Delta u' + \eta\,\nabla\nabla\cdot u') + A^{-1}\,\nu\,\Delta U\ ;\\
F_3 &= -B_t - \Big((U+u')\cdot\nabla\Big)B + \Big((B+H')\cdot\nabla\Big)U + (H'\cdot\nabla)u' - H'\,\nabla\cdot u' + \mu\,\Delta B\ .
\end{aligned}
$$

If we introduce the 5-vector $w^T = (r, u_1', u_2', H_1', H_2')$ and take account of (5)–(17), equations (24)–(26) can be written as

$$
w_t + \Big((U+u')\cdot\nabla\Big)w + \mathcal{B}w \;=\; A_\varepsilon w + \varepsilon\, G + Q_1 + Q_2\ , \tag{27}
$$

where

$$
\mathcal{B} \;=\; \begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -B_2\partial_2 & B_2\partial_1\\
0 & 0 & 0 & B_1\partial_2 & -B_1\partial_1\\
0 & -B_2\partial_2 & B_1\partial_2 & 0 & 0\\
0 & B_2\partial_1 & -B_1\partial_1 & 0 & 0
\end{pmatrix}; \tag{28}
$$

$$
A_\varepsilon \;=\; -\frac{1}{\varepsilon}\begin{pmatrix}
0 & \partial_1 & \partial_2 & 0 & 0\\
\partial_1 & 0 & 0 & 0 & 0\\
\partial_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 0\\
0 & \nu\Delta+\eta\partial_1^2 & \eta\partial_1\partial_2 & 0 & 0\\
0 & \eta\partial_1\partial_2 & \nu\Delta+\eta\partial_2^2 & 0 & 0\\
0 & 0 & 0 & \mu\Delta & 0\\
0 & 0 & 0 & 0 & \mu\Delta
\end{pmatrix}; \tag{29}
$$

$$
G \;=\; \begin{pmatrix} g_1\\ g_2\\ 0\end{pmatrix} \;=\; \begin{pmatrix} -(P_t + U\cdot\nabla P)\\ \dfrac{1}{\varepsilon}\,\nabla P(1-E^{\gamma-2}) - \dfrac{\varepsilon\,\nu\, P\Delta U}{E}\\ 0\end{pmatrix}, \quad \text{with } E = 1+\varepsilon^2 P\ ; \tag{30}
$$

$$Q_1 = \begin{pmatrix} 0 \\ \tilde{Q}_1 \\ 0 \\ 0 \end{pmatrix}, \tag{31}$$

where $\tilde{Q}_1$ is the 2 vector defined as follows

$$\begin{aligned} \tilde{Q}_1 = {} & \left(\frac{1}{A} - 1\right)\Big[\nu\,\Delta u' + \eta\,\nabla\nabla\cdot u'\Big] + \frac{1}{\varepsilon}\,(1 - A^{\gamma-2})\,\nabla r \\ & + \left(\frac{1}{A} - 1\right)\Big[(B\cdot\nabla)B - \nabla H'\cdot B + (B\cdot\nabla)H' - \nabla B\cdot B\Big] \\ & + \frac{1}{A}\Big[-\nabla(B+H')\cdot H' + (H'\cdot\nabla)\,(B+H')\Big]\ ; \end{aligned}$$

and finally

$$Q_2 = \begin{pmatrix} -\varepsilon\,u'\cdot\nabla P - (\varepsilon P + r)\,\nabla\cdot u' \\ \\ -(u'\cdot\nabla)U - [A^{\gamma-2} - E^{\gamma-2}]\,\nabla P - \dfrac{\varepsilon\,\nu\,r\,\Delta U}{A\,E} \\ \\ -(u'\cdot\nabla)B + (H'\cdot\nabla)U - H'(\nabla\cdot u') + (H'\cdot\nabla)u' \end{pmatrix}. \tag{32}$$

4 – The linear case $w_t = A_\varepsilon w$

To analyze the all-time existence question for equation (27) it is natural to consider first the linear system $w_t = A_\varepsilon w$, and, secondly, to apply Fourier expansion in space. One obtains a linear O.D.E. system

$$\widehat{w}_t(k,t) = \widehat{A}_\varepsilon(k)\,\widehat{w}(k,t)\ , \tag{33}$$

for each vector $k \in \mathbf{Z}^2$.

The previous matrix $\widehat{A}_\varepsilon(k)$ is the so-called "symbol of A_ε" and it is defined as follows

$$\widehat{A}_\varepsilon(k) = -\begin{pmatrix} 0 & \dfrac{i}{\varepsilon}\,k_1 & \dfrac{i}{\varepsilon}\,k_2 & 0 & 0 \\ \dfrac{i}{\varepsilon}\,k_1 & \nu|k|^2 + \eta\,k_1^2 & \eta\,k_1k_2 & 0 & 0 \\ \dfrac{i}{\varepsilon}\,k_2 & \eta\,k_1k_2 & \nu|k|^2 + \eta\,k_2^2 & 0 & 0 \\ 0 & 0 & 0 & \mu|k|^2 & 0 \\ 0 & 0 & 0 & 0 & \mu|k|^2 \end{pmatrix}. \tag{34}$$

The matrix $\widehat{A}_\varepsilon(k)$ allows us to construct a new matrix $\tilde{H}$, and consequently a new norm, which is equivalent to the L^2-norm, in which we have the exponential decay of w, except for its spatial averages. Recall that the spatial averages of the solutions of $w_t = A_\varepsilon w$ are constant in time.

We now prove that the eigenvalues of $\widehat{A}_\varepsilon(k)$ have negative real parts for all $k \neq 0$. This implies that all solutions $\widehat{w}(k,t)$, $k \neq 0$, tend to zero as $t \to +\infty$.

Lemma 4.1. *Let $\lambda_j = \lambda_j(\nu,\eta,\varepsilon,\mu,k)$ be the eigenvalues of the matrix $\widehat{A}_\varepsilon(k)$. Then, we have*

$$\operatorname{Re}\lambda_j \;\leq\; \max\left\{-\nu,\, -\frac{\nu+\eta}{2},\, -\frac{1}{(\nu+\eta)\varepsilon^2},\, -\mu\right\} \;<\; 0\,, \qquad j=1,2,...,5\ ,$$

for all $\nu > 0,\ \nu+\eta > 0,\ \mu > 0,\ \varepsilon > 0,\ k \in \mathbf{Z}^2,\ k \neq 0.$

Proof: For

$$\phi_1 \;=\; \frac{1}{|k|}\begin{pmatrix} 0 \\ -k_2 \\ k_1 \\ 0 \\ 0 \end{pmatrix},$$

we have

$$\widehat{A}_\varepsilon(k)\,\phi_1 \;=\; -\nu\,|k|^2\,\phi_1\ ,$$

thus $\lambda_1 = -\nu|k|^2$. Let us consider $\phi_2, \phi_3 \in \mathbb{R}^5$ as follows

$$\phi_2 \;=\; \frac{1}{|k|}\begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \\ 0 \end{pmatrix}, \qquad \phi_3 \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We note that ϕ_2 and ϕ_3 are orthogonal to ϕ_1, and that

$$\widehat{A}_\varepsilon(k)\,\phi_2 \;=\; -\frac{i}{\varepsilon}\,|k|\,\phi_3 - (\nu+\eta)\,|k|^2\,\phi_2\ ,$$

$$\widehat{A}_\varepsilon(k)\,\phi_3 \;=\; -\frac{i}{\varepsilon}\,|k|\,\phi_2\ .$$

It easily follows that the eigenvalues λ_2 and λ_3 of $\widehat{A}_\varepsilon(k)$ are the eigenvalues of

$$\tilde{B} \;=\; \begin{pmatrix} -(\nu+\eta)\,|k|^2 & -\frac{i}{\varepsilon}\,|k| \\ -\frac{i}{\varepsilon}\,|k| & 0 \end{pmatrix}. \tag{35}$$

Exactly as in [3], Lemma 4.1, we have

$$\operatorname{Re}\lambda_2, \lambda_3 \;\le\; \max\left\{-\frac{1}{(\nu+\eta)\varepsilon^2}, \, -\frac{\nu+\eta}{2}\right\} .$$

Finally, if we consider

$$\phi_4 \;=\; \frac{1}{|k|}\begin{pmatrix} 0\\ 0\\ 0\\ k_1\\ k_2 \end{pmatrix} \qquad \text{and} \qquad \phi_5 \;=\; \frac{1}{|k|}\begin{pmatrix} 0\\ 0\\ 0\\ -k_2\\ k_1 \end{pmatrix} ,$$

we have

$$\widehat{A}_\varepsilon(k)\,\phi_4 \;=\; -\mu\,|k|^2\,\phi_4 \;,$$

$$\widehat{A}_\varepsilon(k)\,\phi_5 \;=\; -\mu\,|k|^2\,\phi_5 \;.$$

Then, $\lambda_4 = \lambda_5 = -\mu|k|^2$ and the lemma is proved. ∎

Lemma 4.2. *For fixed $\nu+\eta > 0$, consider the family of matrices $\tilde{B}$ given in (35) for $k \in \mathbf{Z}^2$, $k \neq 0$. There are positive constants $\tilde{c}_0$, C_1, C_2, ε_0, $0 < \varepsilon_0 < 1$, depending only on $\nu+\eta$, and there are Hermitian matrices $W = W(\nu+\eta, \varepsilon, k) \in \mathbf{C}^{2\times 2}$ for $0 < \varepsilon \le \varepsilon_0$ with the following properties:*

$$0 \;<\; (1 - C_1\varepsilon)I \;\le\; W \;\le\; (1 + C_1\varepsilon)I \;; \tag{36}$$

$$q^*(W\tilde{B} + \tilde{B}^* W)q \;\le\; -\tilde{c}_0\,q^*\,Wq - (\nu+\eta)\,|k|^2\,|q_2|^2\,, \qquad \forall\, q \in \mathbf{C}^2 \;; \tag{37}$$

$$|W - I| \;\le\; C_2\,\frac{\varepsilon}{|k|} \;. \tag{38}$$

Proof: For the proof we refer to [3], Lemma 4.2. ∎

Lemma 4.3. *For fixed $\nu > 0$, $\nu+\eta > 0$, $\mu > 0$, and ε_0 sufficiently small, $0 < \varepsilon_0 < 1$, consider the family of matrices $\widehat{A}_\varepsilon(k)$, given by (34), for $0 < \varepsilon \le \varepsilon_0$, $k \in \mathbf{Z}^2$, $k \neq 0$. There are positive constants c_0, c_1, C_1, C_2 depending only on ν, η, μ, and there are Hermitian matrices $Z = Z(\nu, \mu, \eta, \varepsilon, k) \in \mathbf{C}^{5\times 5}$ with the following properties*

$$0 \;<\; (1 - C_1\varepsilon)I \;\le\; Z \;\le\; (1 + C_1\varepsilon)I \;; \tag{39}$$

$$q^*(Z\widehat{A}_\varepsilon + \widehat{A}^*_\varepsilon Z)q \;\le\; -2\,c_0\,q^*\,Z\,q - c_1\,|k|^2\Big(|q_2|^2 + |q_3|^2 + |q_4|^2 + |q_5|^2\Big)\,, \qquad \forall\, q \in \mathbf{C}^5 \;; \tag{40}$$

$$|Z - I| \;\le\; C_2\,\frac{\varepsilon}{|k|} \;. \tag{41}$$

Proof: Let $W \in \mathbf{C}^{2\times 2}$ denote the symmetrizer for the matrix $\tilde{B}$ defined as

$$W = \begin{pmatrix} 1 & i\,\dfrac{\varepsilon\, l}{|k|} \\ -i\,\dfrac{\varepsilon\, l}{|k|} & 1 \end{pmatrix},$$

where l is a suitable constant. We set

$$Z = S \begin{pmatrix} 1 & & & 0 \\ & W & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} S^* := S\widehat{H}S^* ,$$

where S is the orthogonal matrix containing as columns the vectors ϕ_j defined in the proof of Lemma 4.1, and $\widehat{H}$ is defined by this equation.

Let q be an arbitrary vector in $\mathbf{C}^5$, and let $p = S^*q$. We now show (39). Let us consider

$$Z - I = S\widehat{H}S^* - SIS^* = S \begin{pmatrix} 0 & & & 0 \\ & W - I & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} S^* := SRS^* .$$

We have

$$q^*(Z - I)\, q = q^*SRS^*q = p^*R\,p ,$$

and for (36) we obtain

$$-C_1\,\varepsilon\,|p|^2 \le p^*\,R\,p \le C_1\,\varepsilon\,|p|^2 .$$

We now prove (40). Let us consider

$$\begin{aligned} q^*(Z\widehat{A}_\varepsilon + \widehat{A}_\varepsilon^* Z)\, q &= q^*(S\widehat{H}S^*\widehat{A}_\varepsilon + \widehat{A}_\varepsilon^* S\widehat{H}S^*)\, q \\ &= p^*\widehat{H}(S^*\widehat{A}_\varepsilon S)\, p + p^*(S^*\widehat{A}_\varepsilon^* S)\,\widehat{H}p \\ &= p^* \begin{pmatrix} -\nu\,|k|^2 & & & 0 \\ & W\tilde{B} + \tilde{B}^*W & & \\ & & -\mu\,|k|^2 & \\ 0 & & & -\mu\,|k|^2 \end{pmatrix} p . \end{aligned}$$

By using (37), we can estimate the right hand side of the above identity as follows

$$\begin{aligned} q^*(Z\widehat{A}_\varepsilon + \widehat{A}_\varepsilon^* Z)\, q \le\; & -\nu\,|k|^2\,|p_1|^2 - \tilde{c}_0(p_2, p_3)\, W \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} \\ & - (\nu + \eta)\,|k|^2\,|p_2|^2 - \mu\,|k|^2\,(|p_4|^2 + |p_5|^2) \le \end{aligned}$$

$$\leq -2\,c_0\,p^*\widehat{H}p - c_1\,|k|^2\left(|p_1|^2+|p_2|^2+|p_4|^2+|p_5|^2\right)$$
$$= -2\,c_0\,q^*\,Z\,q - c_1\,|k|^2\left(|q_2|^2+|q_3|^2+|q_4|^2+|q_5|^2\right)\,,$$

for suitable $0 < c_0 \leq \dfrac{\tilde{c}_0}{2}$ and $0 < c_1 < \min\{\nu, \nu+\eta, \mu\}$.

To conclude the proof we remark that inequality (41) follows directly from (38). ∎

Remark. By the previous Lemma the symmetrizer $Z = Z(\mu,\nu,\eta,\varepsilon,k)$ of (33) is constructed for $\nu > 0$, $\nu+\eta > 0$, $\mu > 0$, $0 < \varepsilon \leq \varepsilon_0$, $k \in \mathbf{Z}^2$, $k \neq 0$. For $k = 0$ we set $Z = I$. □

By using the matrices $Z(k)$, $k \in \mathbf{Z}^2$, and the Fourier coefficients we define a new inner product on $L^2(T^2)$ by

$$(u,v)_Z = \sum_{k\in\mathbf{Z}^2} \widehat{u}^*(k)\,Z(k)\,\widehat{v}(k)\;.$$

We denote by $\|\cdot\|_Z$ the corresponding norm.

As an easy consequence of Lemma 4.3 and Parseval's relation, we obtain the following results.

Lemma 4.4. *The inner product $(\cdot,\cdot)_Z$ has the following properties.*

(i) *The norm $\|\cdot\|_Z$ and the L^2-norm are equivalent, i.e. there exists a constant C_1 such that*

$$(1-C_1\varepsilon)\,\|w\|^2 \leq \|w\|_Z^2 \leq (1+C_1\varepsilon)\,\|w\|^2\;.$$

(ii) *If $w = (r, u', H') \in L^2(T^2)$ satisfies $\widehat{w}(0) = 0$, i.e. if w has zero spatial mean, there exist two positive constants c_0, c_1 such that*

$$(w, A_\varepsilon w)_Z + (A_\varepsilon w, w)_Z \leq -2\,c_0\,\|w\|_Z^2 - c_1\Big(\|\nabla u'\|^2 + \|\nabla H'\|^2\Big)\;.$$

(iii) *If $w_1 \in L^2(T^2)$, $w_2 \in H^1(T^2)$, then there exists a real positive constant C_2 such that*

$$\Big|(w_1, Dw_2)_Z - (w_1, Dw_2)\Big| \leq \varepsilon\,C_2\,\|w_1\|\,\|w_2\|\;,$$

where D denotes ∂_1 or ∂_2.

(**iv**) *If $w_1, w_2 \in H^1(T^2)$, then the rule of integration by parts holds in Z-inner product:*

$$(w_1, Dw_2)_Z \;=\; -(Dw_1, w_2)_Z \;. \;\blacksquare$$

In order to prove Theorem (1.1) we have essentially to estimate the non linear terms. To obtain these estimates we need some bounds concerning the L^2-inner product and consequently the Z-inner product.

Lemma 4.5. *Let $w \in H^1(T^2, \mathbb{R}^5)$ and let $v \in C^1(T^2)$ be a real valued function. Then, we have*

$$|(w, vDw)| \;\leq\; C\,|Dv|_\infty\,\|w\|^2 \;, \tag{42}$$

for $D = \partial_1$ or $D = \partial_2$.

Proof: The proof easily follows by integration by parts. $\blacksquare$

Lemma 4.6. *Under the assumptions of Lemma 4.5, we have that*

$$|(w, vDw)_Z| \;\leq\; C\Big(\varepsilon\,|v|_\infty + |Dv|_\infty\Big)\,\|w\|^2 \;. \tag{43}$$

Proof: For the proof we refer to [3], Lemma 4.6. $\blacksquare$

5 – Spatial mean

In this Section we derive some properties of the spatial mean of the solution w of problem (27): they come from the conservation laws.

Lemma 5.1. *Under assumptions (10)–(15) and if $w = (r', u'_1, u'_2, H'_1, H'_2)$ is a solution of (27), we have that*

$$\widehat{r}(0,t) \;=\; 0\,, \qquad \forall\, t \geq 0 \;, \tag{44}$$

$$\widehat{H'_1}(0,t) \;=\; \widehat{H'_2}(0,t) \;=\; 0\,, \qquad \forall\, t \geq 0 \;. \tag{45}$$

Proof: The proof is an easy consequence of the conservation laws. From (1) and (10) we have that

$$\int_{T^2} (1 + \varepsilon^2\,\rho)\,dx\,dy \;=\; \int_{T^2} (1 + \varepsilon^2\,\rho_0)\,dx\,dy \;=\; \int_{T^2} dx\,dy \;. \tag{46}$$

Recalling that $\rho = \rho' + P$, we have from (46) and (15) that

$$0 \;=\; \int_{T^2} (\rho' + P)\, dx\, dy \;=\; \int_{T^2} \rho'\, dx\, dy \ ,$$

which implies (44). By integrating equation (3) on T^2, we find that

$$\frac{d}{dt} \int_{T^2} H(x,y,t)\, dx\, dy \;=\; 0\,, \qquad \forall\, t \geq 0 \ . \tag{47}$$

Then, the conclusion of the proof follows from (12). ∎

We now set

$$\overline{u'} \;=\; \frac{1}{(2\pi)^2} \int_{T^2} u'\, dx\, dy$$

and we consider the spatial mean:

$$a(t) = (0, \overline{u'}, 0)^T \ ,$$

and

$$w^c = \; w - a(t) \ .$$

Exactly as in [3], the following lemma holds. We repeat the proof for the sake of completness. Without loss of generality we may assume that δ_0 in (17) is less than 1. From now on we also assume that $\varepsilon \leq \varepsilon_0 < 1$.

Lemma 5.2. *Under the assumptions of Lemma 5.1 and* (17)*, there exist some constants C, c depending on U_0 and ν, but independent of u_0, ρ_0, ε, such that*

$$|\overline{u'}| \;\leq\; \varepsilon\, C\Big(\delta_0 + \varepsilon + \|w^c\|_Z \Big(e^{-ct} + \|w^c\|_Z\Big)\Big)\,, \qquad \forall\, t \geq 0 \ . \tag{48}$$

Proof: We have

$$\int_{T^2} (1 + \varepsilon^2 \rho' + \varepsilon^2 P)\,(u' + U)\, dx\, dy \;=\; \varepsilon^2 \int_{T^2} \rho_0\, u_0\, dx\, dy \ .$$

Recalling that ρ', U and P all have mean zero and solving for $\overline{u}'$, we obtain

$$\overline{u}' \;=\; \varepsilon^2 (2\pi)^{-2} \int_{T^2} (\rho_0 u_0 - PU)\, dx\, dy - \varepsilon (2\pi)^{-2} \int_{T^2} \Big(\varepsilon(\rho' + P)(u' - \overline{u}') + \varepsilon\, \rho' U\Big)\, dx\, dy \ .$$

From (17) we conclude

$$\begin{aligned} \varepsilon \int_{T^2} |\rho_0 u_0 - PU|\, dx\, dy \;&\leq\; \varepsilon \int_{T^2} |\rho_0 u_0 - P_0 U_0|\, dx\, dy \;+\; \varepsilon \int_{T^2} |P_0 U_0 - PU|\, dx\, dy \\ &\leq\; C\,\delta_0 + C\,\varepsilon \ . \end{aligned}$$

Using the Cauchy–Schwarz inequality on the remaining terms, the estimate follows. ∎

Remark. By the definition of the Z-inner product and of w^c, we have

$$(w^c, a(t))_Z = 0\,, \qquad \forall\, t \geq 0\ .$$

Consequently, as an immmediate corollary of these Lemmas, one has

$$\|w^c\|_Z^2 \;\geq\; \frac{1}{2}\,\|w\|_Z^2 - \varepsilon^2\, C\Big(\varepsilon^2 + \delta_0^2 + \|w^c\|_Z^2\Big(e^{-ct} + \|w^c\|_Z^2\Big)\Big)\,, \qquad \forall\, t \geq 0\ . \square \tag{49}$$

6 – Nonlinear estimates

Standard arguments about coupled parabolic-hyperbolic system (see [4]) imply that a unique solution w of (27) exists and it belongs to C^∞ in some time interval $[0, T)$. Moreover, if T is finite, then

$$\sup_{0 \leq t < T} \|w(\cdot, t)\|_3 \;=\; +\infty\ . \tag{50}$$

Our *a-priori* estimates will show, however, that the left hand-side of (50) remains bounded if the initial data of the problem have a norm in H^3 sufficiently small. Consequently, the solution exists for all $t \geq 0$.

To show the first *a-priori* estimates for the solution w, we introduce the following two functions:

$$\phi^2(t) \;=\; \frac{1}{2}\sum_{|\alpha| \leq 3} \|D^\alpha w(\cdot, t)\|_Z^2\ ,$$

$$h^2(t) \;=\; \sum_{|\alpha| = 4} \Big(\|D^\alpha u'(\cdot, t)\|^2 + \|D^\alpha H'(\cdot, t)\|^2\Big)\ .$$

We remark that

$$\frac{d}{dt}\phi^2 \;=\; \mathrm{Re}\sum_{|\alpha| \leq 3} (D^\alpha w, D^\alpha w_t)_Z\ , \tag{51}$$

where

$$\begin{aligned} D^\alpha w_t \;=\; & -D^\alpha\Big(\Big((U + u')\cdot\nabla\Big)\, w\Big) - D^\alpha(\mathcal{B}w) + A_\varepsilon\, D^\alpha w \\ & + \varepsilon\, D^\alpha G + D^\alpha Q_1 + D^\alpha Q_2\ . \end{aligned} \tag{52}$$

We bound the right-hand side of (51) term by term in a sequence of lemmas.

Lemma 6.1. *Under the assumptions of Theorem 1.1, let $T > 0$ such that in the interval $0 \le t < T$ there exists a smooth solution w of (27), such that $\phi(t) \le 1$. Then,*

$$\left|\Big(D^\alpha w,\, D^\alpha\Big(\big((U+u')\cdot\nabla\big)w\Big)\Big)_Z\right| \le C\, e^{-ct}\,\phi^2(t) + C\,\phi^3(t)\,, \qquad \forall\,|\alpha| \le 3\,. \blacksquare \tag{53}$$

The proof of the previous assert follows the same lines as in [3], Lemma 6.2: the inequality comes from Lemma 4.6 and by the Sobolev's inequalities (18)–(20).

In order to estimate the term $D^\alpha(\mathcal{B}w)$, we need the following result.

Lemma 6.2. *For each $w \in H^1(T^2)$, we have that*

$$|(\mathcal{B}w, w)_Z| \le C\Big(\varepsilon\,|B|_\infty + |DB|_\infty\Big)\,\|w\|^2\,. \tag{54}$$

Proof: By using the definition of the matrix $\mathcal{B}$, we have

$$(\mathcal{B}w, w)_Z = \left(\begin{pmatrix} 0 \\ -B_2\,\partial_2 H_1' + B_2\,\partial_1 H_2' \\ B_1\,\partial_2 H_1' - B_1\,\partial_1 H_2' \\ -B_2\,\partial_2 u_1' + B_1\,\partial_2 u_2' \\ B_2\,\partial_1 u_1' - B_1\,\partial_1 u_2' \end{pmatrix}, \begin{pmatrix} r \\ u_1' \\ u_2' \\ H_1' \\ H_2' \end{pmatrix}\right)_Z . \tag{55}$$

We now set

$$\mathcal{B}_1 = \begin{pmatrix} 0 \\ -\partial_2(B_2 H_1') + \partial_1(B_2 H_2') \\ \partial_2(B_1 H_1') - \partial_1(B_1 H_2') \\ -\partial_2(B_2\, u_1') + \partial_2(B_1 u_2') \\ \partial_1(B_2\, u_1') - \partial_1(B_1 u_2') \end{pmatrix},$$

and

$$\Lambda_1 = \left(\begin{pmatrix} 0 \\ -H_1'\partial_2 B_2 + H_2'\,\partial_1 B_2 \\ H_1'\partial_2 B_1 - H_2'\,\partial_1 B_1 \\ -u_1'\partial_2 B_2 + u_2'\,\partial_2 B_1 \\ u_1'\partial_1 B_2 - u_2'\,\partial_1 B_1 \end{pmatrix}, \begin{pmatrix} r \\ u_1' \\ u_2' \\ H_1' \\ H_2' \end{pmatrix}\right)_Z .$$

Then, identity (55) can be written as

$$(\mathcal{B}w, w)_Z = (\mathcal{B}_1, w)_Z - \Lambda_1\,. \tag{56}$$

From (iii) of Lemma 4.4, we can write

$$(\mathcal{B}_1, w)_Z = (\mathcal{B}_1, w) + \Lambda_2\,,$$

where Λ_2 satisfies

$$|\Lambda_2| \,\le\, C\,\varepsilon\,|B|_\infty\,\|w\|^2 \;.$$

Moreover, by the definition of Λ_1 and by (i) of Lemma 4.4, we have

$$|\Lambda_1| \,\le\, C\,|DB|_\infty\,\|w\|^2 \;.$$

To conclude the proof, it is sufficiently to compute explicitly the L^2-inner product $(\mathcal{B}_1, w)$. By integrating by parts, it easily follows that

$$|(\mathcal{B}_1, w)| \,\le\, C\,|DB|_\infty\,\|w\|^2 \;,$$

which concludes the proof. ∎

Lemma 6.3. *Under the assumptions of Lemma 6.1 we have*

$$|(D^\alpha w, D^\alpha(\mathcal{B}w))_Z| \,\le\, C\,e^{-ct}\,\phi^2(t)\,, \qquad \forall\,|\alpha|\le 3\;.$$

Proof: We have

$$(D^\alpha w, D^\alpha(\mathcal{B}w))_Z \,=\, (D^\alpha w, \mathcal{B}D^\alpha w)_Z \,+\, \sum_{0\le\beta<\alpha} c_{\alpha,\beta}(D^\alpha w, D^{\alpha-\beta}(\mathcal{B}D^\beta w))_Z \;.$$

By using Lemma 6.2, we find

$$|(D^\alpha w, \mathcal{B}D^\alpha w)_Z| \,\le\, C\Big(\varepsilon\,|B|_\infty + |DB|_\infty\Big)\,\|D^\alpha w\|^2 \,\le\, C\,e^{-ct}\,\phi^2(t)\;. \tag{57}$$

In order to estimate

$$\Lambda_3 \;=\; \sum_{0\le\beta<\alpha} c_{\alpha,\beta}(D^\alpha w, D^{\alpha-\beta}(\mathcal{B}D^\beta w))_Z \;,$$

we observe that, by the definition of the matrix $D^{\alpha-\beta}\mathcal{B}$, we have to estimate

$$(D^{\alpha-\beta-\gamma}B_i\,\partial_j D^{\beta+\gamma}w,\; D^\alpha w)_Z \;,$$

where $0\le\gamma\le\alpha-\beta$. Hence, by using the Cauchy–Schwarz inequality, we easily obtain

$$|\Lambda_3| \,\le\, C\,e^{-ct}\phi^2(t)\;. \tag{58}$$

Collecting (57) and (58), we conclude the proof. ∎

In order to estimate the term concerning the matrix A_ε, we use the estimates obtained on the function w^c.

Lemma 6.4. *Under the assumptions of Lemma 6.1, we have*

$$\mathrm{Re}\sum_{|\alpha|\le 3}(D^\alpha w, A_\varepsilon D^\alpha w)_Z \;\le\; -c_0\,\phi^2(t) - c_1\,h^2(t) + \varepsilon^2\,C\Big(\varepsilon^2+\delta_0^2+e^{-ct}\,\phi^2(t)+\phi^4(t)\Big)\;. \tag{59}$$

Proof: The inequality comes by using (ii) of Lemma 4.4, and (49). For the details we refer to [3], Lemma 6.3. ∎

Lemma 6.5. *Under the assumptions of Lemma 6.1, we have*

$$|(D^\alpha w, \varepsilon D^\alpha G)_Z| \;\le\; C\,e^{-ct}\,\phi^2(t)+\varepsilon^2\,C\,e^{-ct}\,, \qquad \forall\,|\alpha|\le 3\;. \tag{60}$$

Proof: By using the Cauchy–Schwarz inequality and the decay at infinite of U and P we have

$$|(D^\alpha w, \varepsilon D^\alpha G)_Z| \;\le\; \phi(t)\,\varepsilon\,C\,e^{-ct}\;.$$

Hence, by the Cauchy inequality, the proof easily follows. ∎

Lemma 6.6. *Under the assumptions of Lemma 6.1, we have*

$$\sum_{|\alpha|\le 3}|(D^\alpha w, D^\alpha Q_1)_Z| \;\le\; C\,e^{-ct}\Big(\phi^2(t)+\varepsilon^2\,h^2(t)\Big)+C\phi^3(t) + C\,\phi^2(t)\,h(t)+\varepsilon\,C\,\phi(t)\,h^2(t)+C\,\varepsilon^2\,e^{-ct}\;. \tag{61}$$

Proof: By recalling that $A = 1+\varepsilon^2 P+\varepsilon\,r$, we decompose Q_1 as

$$Q_1 = M_1+M_2+M_3+M_4\;,$$

where

$$M_1 = \begin{pmatrix} 0 \\ \Big(\dfrac{1}{A}-1\Big)\Big[\nu\,\Delta u' + \eta\,\nabla\nabla\cdot u'\Big] + \dfrac{1}{\varepsilon}\,(1-A^{\gamma-2})\,\nabla r \\ 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 \\ \Big(\dfrac{1}{A}-1\Big)\Big[-\nabla H'\cdot B + (B\cdot\nabla)H'\Big] \\ 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 \\ \left(\frac{1}{A} - 1\right)\Big[(B\cdot\nabla)B - \nabla B\cdot B\Big] \\ 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 \\ \frac{1}{A}\Big[-\nabla H\cdot H' + (H'\cdot\nabla)H\Big] \\ 0 \end{pmatrix}.$$

We now estimate separately the vectors $M_1, ..., M_4$. For M_1 we refer to [3], Lemma 6.5. In particular, we find that M_1 is majored exactly by the left hand side of (61).

By following the same lines as in the proof of Lemma 6.3, and by observing that

$$D^\alpha\left(\frac{1}{A}\right) \le C\Big(\varepsilon\,\phi(t) + \varepsilon^2\, e^{-ct}\Big) \tag{62}$$

and, finally, by applying the Cauchy–Schwarz inequality, we get

$$\sum_{|\alpha|\le 3} |(D^\alpha M_2, D^\alpha w)_Z| \le e^{-ct}\Big(\varepsilon\,\phi(t) + \varepsilon^2\, e^{-ct}\Big)\,\phi(t)\Big(\phi(t) + h(t)\Big)\,.$$

As an easy consequence of the exponential decay of B and by using (62), one has the following estimate concerning the vector M_3

$$\begin{aligned} \sum_{|\alpha|\le 3} |(D^\alpha M_3, D^\alpha w)_Z| &\le C\, e^{-ct}\phi(t)\Big(\varepsilon\,\phi(t) + \varepsilon^2\, e^{-ct}\Big) \\ &\le C\, e^{-ct}\,\phi^2(t) + C\,\varepsilon^2\, e^{-ct}\,. \end{aligned} \tag{63}$$

In order to estimate the vector M_4, we observe that

$$\sum_{|\alpha|\le 3}\Big(D^\alpha\Big[\nabla H\cdot H' + (H'\cdot\nabla)H\Big],\, D^\alpha w\Big)_Z \le C\,\phi(t)\,\Big(\phi^2(t) + C\, e^{-ct}\,\phi(t) + h(t)\,\phi(t)\Big)\,.$$

Now, by (62), we get

$$\sum_{|\alpha|\le 3} |(D^\alpha M_4, D^\alpha w)_Z| \le C\,\phi^3(t) + C\,\phi^2(t)\,h(t) + C\, e^{-ct}\,\phi^2(t)\,.$$

The proof is covered by collecting the previous estimates on the four terms $M_1, ..., M_4$. ∎

Lemma 6.7. *Under the assumptions of Lemma 6.1, we have*

$$\sum_{|\alpha|\le 3} |(D^\alpha Q_2, D^\alpha w)_Z| \le C\, e^{-ct} \Big(\phi^2(t) + \varepsilon^2\, h^2(t)\Big) + C\,\phi^2(t)\,h(t) + C\,\phi^3(t)\ . \tag{64}$$

Proof: We recall that $E = (1+\varepsilon^2 P)$ and we split the matrix Q_2 as

$$Q_2 = N_1 + N_2\ ,$$

with

$$N_1 \;=\; \begin{pmatrix} -\varepsilon\, u'\cdot\nabla P - \varepsilon(P+\rho')\,\nabla\cdot u' \\[2ex] -(u'\cdot\nabla)U - (A^{\gamma-2} - E^{\gamma-2})\nabla P - \dfrac{\varepsilon^2\,\nu\,\rho'\,\Delta U}{A\,E} \\[2ex] 0 \end{pmatrix},$$

$$N_2 \;=\; \begin{pmatrix} 0 \\[2ex] 0 \\[2ex] -(u'\cdot\nabla)B + (H'\cdot\nabla)U - H'(\nabla\cdot u') + (H'\cdot\nabla)u' \end{pmatrix}.$$

By using Lemma 6.6 in [3], we find

$$\sum_{|\alpha|\le 3} |(D^\alpha N_1, D^\alpha w)_Z| \le C\, e^{-ct}\,\phi^2(t) + \varepsilon^2\, C\, e^{-ct}\, h^2(t) + C\,\phi^2(t)\,h(t)\ . \tag{65}$$

The estimate for the matrix N_2 simply follows by using the exponential decay of B and its derivatives, and by the definitions of the functions $h(t)$ and $\phi(t)$. More precisely, we obtain

$$\sum_{|\alpha|\le 3} |(D^\alpha N_2, D^\alpha w)_Z| \le C\,\phi(t)\,\Big(C\,\phi(t)\,e^{-ct} + \phi^2(t) + h(t)\,\phi(t)\Big)\ . \tag{66}$$

By summing (65) and (66), we obtain inequality (64). ∎

We now prove the following fundamental result.

Lemma 6.8. *Under the assumptions of Theorem 1.1, let $T>0$ such that, in the interval $0\le t<T$, there exists a smooth solution w of (27), with $\phi(t)\le 1$. Then, the function $\phi^2(t)$ satisfies*

$$\begin{aligned} \frac{d}{dt}\phi^2(t) \le\; & (C\,e^{-ct} - c_0)\,\phi^2(t) + \left(\varepsilon^2\, C\, e^{-ct} - \frac{1}{2}\,c_1\right) h^2(t) \\ & + \varepsilon^2\, C\, e^{-ct} + \varepsilon^4\, C + \varepsilon^2\,\delta_0^2\, C + C\,\phi^3(t) + \varepsilon\, C\,\phi(t)\,h^2(t)\ , \end{aligned} \tag{67}$$

$$\phi^2(0) \;\le\; \frac{1}{2}\,(1+C_1\,\varepsilon)\,\delta_0^2\;, \tag{68}$$

where all the constants are independent of ε*, for* $0<\varepsilon\le\varepsilon_0$.

Proof: The proof is an easy consequence of (51), (52) and the results of the previous Lemmata 6.1, 6.3–6.7. ∎

We are now in position to prove Theorem 1.1. We give only the idea of the proof. For all details we refer to [3], Theorem 2.1.

Assume that δ_0 and ε_0 are chosen so that $\phi(0)<1$. From the existence theory of hyperbolic-parabolic systems, it is well-known that there is a maximal interval of existence $[0,T)$, with $\phi(t)<1$, $\forall\, t\in[0,T)$. Moreover, if $T<\infty$, then

$$\limsup_{t\to T^-}\phi^2(t) \;=\; 1\;.$$

The basic idea to prove all-time existence is to show that $\phi^2(t)<1$ in arbitrary intervals of existence. We consider the scalar ordinary differential inequality

$$\frac{d}{dt}y(t) \;\le\; \Big(C\,e^{-ct}-\frac{1}{2}\,c_0\Big)\,y(t) \;+\; \varepsilon^2\,C(e^{-ct}+\varepsilon^2+\delta_0^2)\;, \tag{69}$$

$$y(0) \;\le\; \frac{1}{2}\,(1+C_1\,\varepsilon)\,\delta_0^2\;. \tag{70}$$

The following result holds (see [3], Lemma 6.7).

Lemma 6.9. *There exists* K *depending only on* C, $\tilde c_0$, c, C_1 *such that any solution* $y(t)$ *of (69), (70) satisfies*

$$y(t) \;<\; K^2(\varepsilon^2+\delta_0^2)\,, \qquad 0\le t<+\infty\;, \tag{71}$$

$$\limsup_{t\to+\infty}\, y(t) \;\le\; \varepsilon^2K^2(1+\varepsilon^2+\delta_0^2)\;. \;∎ \tag{72}$$

We now choose $\varepsilon_0(\nu,\mu,\eta,U_0,B_0)$ and $\delta_0(\nu,\mu,\eta,U_0,B_0)$ sufficiently small such that all previous lemmas hold and such that

$$K\,\sqrt{\varepsilon_0^2+\delta_0^2} \;\le\; 1\;, \tag{73}$$

$$2\Big(\varepsilon_0^2\,C+\varepsilon_0\,CK\,\sqrt{\varepsilon_0^2+\delta_0^2}\Big) \;\le\; c_1\;, \tag{74}$$

$$2\,CK\,\sqrt{\varepsilon_0^2+\delta_0^2}, \;\le\; c_0\;. \tag{75}$$

Hence from (67)–(68) the function $\phi^2(t)$ satisfies the inequalities (69)–(70).

Our claim is to show that $\phi^2(t) < K^2(\varepsilon^2+\delta_0^2)$ in any interval of existence and for $0<\varepsilon\le\varepsilon_0$. By contradiction, we suppose that there exists $T^*>0$ such that

$$\phi^2(t) \;<\; K^2(\varepsilon^2+\delta_0^2)\,, \qquad \forall\, 0\le t<T^* \,, \tag{76}$$

$$\phi^2(T^*) \;\ge\; K^2(\varepsilon^2+\delta_0^2) \;. \tag{77}$$

By using inequality (69)–(70), we obtain that

$$\phi^2(T^*) \;<\; K^2(\varepsilon^2+\delta_0^2) \;.$$

The last inequality contradicts (77), hence all-time existence is proved.

7 – Appendix

We derive in this Section equations (1)–(3). We consider the equations of magneto-hydrodynamics of isentropic compressible flow:

$$\rho_t + u\cdot\nabla\rho + \rho\,\nabla\cdot u \;=\; 0 \,, \tag{78}$$

$$\rho\Big(u_t+(u\cdot\nabla)u\Big) + \nabla p + \mu' H\times(\nabla{\times}H) - \nu_0\,\Delta u - \eta_0\nabla\nabla\cdot u \;=\; 0 \,, \tag{79}$$

$$H_t - \nabla\times(u{\times}H) - \mu_0\,\Delta H \;=\; 0 \,, \tag{80}$$

$$\frac{p}{p_*} = \left(\frac{\rho}{\rho_*}\right)^{\gamma}, \qquad \gamma\ge 1 \,, \tag{81}$$

where ρ is the density, u the velocity field, H the magnetic field, p the pressure, ν_0, η_0 the viscosity coefficients, μ_0 the resistivity, μ' the magnetic permeability.

We introduce the new variables $t=t_*\,\tilde t$, $x=x_*\,\tilde x$, $\rho=\rho_*\,\tilde\rho$, $H=H_*\,\tilde H$, where t_*, x_*, ρ_*, H_* are the units of time, length, density and magnetic field. We set

$$u_* = \frac{x_*}{t_*} \,,$$

hence $u=u_*\tilde u$. If we rewrite equation (78)–(80) in terms of $\tilde\rho$, $\tilde u$, $\tilde H$, we obtain the adimensional form of the compressible fluid equations. More precisely, dropping $\tilde{}$ in order to simplify the notation, we obtain that the equation (78) remains invariant. Equation (79) becomes:

$$\rho_*\,\rho\left(\frac{u_*}{t_*}\,u_t + \frac{u_*^2}{x_*}\,(u\cdot\nabla)\,u\right) + \frac{p_*}{x_*}\,\nabla\rho^\gamma \;+\; \mu'\,\frac{H_*^2}{x_*}\,H\times(\nabla{\times}H) \;- \\ -\;\frac{\nu_0}{x_*^2}\,u_*\,\Delta u \;-\; \frac{\eta_0}{x_*^2}\,u_*\,\nabla(\nabla\cdot u) \;=\; 0 \;.$$

We divide all terms by $\dfrac{\rho_* \, x_*}{t_*^2}$, hence:

$$\rho\Big(u_t + (u\cdot\nabla)\, u\Big) + \frac{p_*}{{u_*}^2\,\rho_*}\,\nabla\rho^\gamma + \mu' \frac{H_*^2}{{u_*}^2\,\rho_*}\, H\times(\nabla{\times}H) - \\ - \frac{\nu_0}{u_*\,\rho_*\,x_*}\,\Delta u - \frac{\eta_0}{u_*\,\rho_*\,x_*}\,\nabla(\nabla\cdot u) \;=\; 0\;.$$

We denote by a_* the sound speed corresponding to the state ρ_*, p_*, such that

$$a_*^2 = \frac{d}{d\rho} p(\rho_*) \;=\; \gamma\,\rho_*^{\gamma-1} = \gamma\,\frac{p_*}{\rho_*}$$

and by $\varepsilon := \dfrac{u_*}{a_*}$ the Mach number.

Hence, dividing the momentum equation by ρ, we obtain:

$$u_t + (u\cdot\nabla)u + \frac{1}{\varepsilon^2}\,\rho^{\gamma-2}\,\nabla\rho + \frac{A}{\rho}\,H\times(\nabla{\times}H) - \frac{\nu}{\rho}\,\nabla u - \frac{\eta}{\rho}\,\nabla\nabla\cdot u \;=\; 0\;,$$

where

$$A := \mu'\,\frac{H_*^2}{\rho_*\,u_*^2}\;,\qquad \nu := \frac{\nu_0}{x_*\,u_*\,\rho_*}\;,\qquad \eta := \frac{\eta_0}{x_*\,u_*\,\rho_*}\;.$$

Equation (80) becomes:

$$H_t - \nabla\times(u{\times}H) - \frac{\mu_0}{x_*\,u_*}\,\Delta H \;=\; 0\;.$$

By setting $\mu := \dfrac{\mu_0}{x_* u_*}$, the system (78)–(81) becomes:

$$\begin{aligned} \rho_t + u\cdot\nabla\rho + \rho\,\nabla\cdot u \;&=\; 0\;,\\ u_t + (u\cdot\nabla)u + \frac{1}{\varepsilon^2}\,\rho^{\gamma-2}\,\nabla\rho + \frac{A}{\rho}\,H\times(\nabla{\times}H) - \frac{\nu}{\rho}\,\Delta u - \frac{\eta}{\rho}\,\nabla\nabla\cdot u \;&=\; 0\;,\\ H_t - \nabla\times(u{\times}H) - \mu\,\Delta H \;&=\; 0\;. \end{aligned} \tag{82}$$

If we introduce a new variable $r = r(x,y,t)$ by setting $\rho = 1+\varepsilon^2 r$, we obtain

$$\varepsilon^2\Big(r_t + (u\cdot\nabla)\,r\Big) + (1+\varepsilon^2 r)\,\nabla\cdot u \;=\; 0\;,$$

$$u_t + (u\cdot\nabla)\,u + (1+\varepsilon^2 r)^{\gamma-2}\,\nabla r + \frac{A}{1+\varepsilon^2\,r}\,H\times(\nabla{\times}H) - \\ - \frac{\nu}{1+\varepsilon^2 r}\,\Delta u - \frac{\eta}{1+\varepsilon^2 r}\,\nabla\nabla\cdot u \;=\; 0\;,$$

$$H_t - \nabla\times(u{\times}H) - \mu\,\Delta H \;=\; 0\;.$$

Then, we obtain equations (1)–(3) if we write ρ instead of r, and take $A = 1$. We introduce the variable r for the following reason. We suppose that there exists the formal asymptotic expansion:

$$\rho(x, y, t) \;=\; \rho_0(x, y, t) + \varepsilon\, \rho_1(x, y, t) + 0(\varepsilon^2)$$

and a similar expansion for u and H. We substitute the expansions in (82), and equalize the coefficients of powers of ε. It follows that ρ_0 is constant in space and time and that ρ_1 may be taken equal to zero. Hence by choosing $\rho_0 = 1$, we have

$$\rho \;=\; 1 + 0(\varepsilon^2) \;,$$

(see [4], [5] for more details).

REFERENCES

[1] Browning, G. and Kreiss, H.O. – Problems with different time scales for non-linear partial differential equations, *SIAM J. Appl. Math.,* 42(4) (1982), 704–718.

[2] Hagstrom, T. and Lorenz, J. – All-time existence of smooth solutions to PDEs of mixed type and the invariant subspace of uniform states, *Adv. Appl. Math.,* 16 (1995), 219–257.

[3] Hagstrom, T. and Lorenz, J. – All-time existence of classical solutions for slightly compressible flows, *SIAM J. Math. Anal.,* 29(3) (1998), 652–672.

[4] Kreiss, H.O. and Lorenz, J. – *Initial-Boundary Value Problems and The Navier–Stokes Equations*, Academic Press, New York, 1989.

[5] Klainermam, S. and Majda, A. – Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.,* 34 (1981), 481–525.

[6] Klainerman, S. and Majda, A. – Compressible and incompressible fluids, *Comm. Pure Appl. Math.,* 35 (1982), 629–651.

[7] Temam, R. – *Navier–Stokes Equations: Theory and Numerical Analysis*, 3rd Ed., North Holland, Amsterdam, 1984.

[8] Temam, R. – *Infinite Dimensional Dynamic System in Mechanics and Physics*, 2nd Ed., Springer, New York, 1997.

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