PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 4 – 1999

ON SUMS OF POWERS OF TERMS IN A LINEAR RECURRENCE

R.S. Melham

1 – Introduction

Define the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ by

$$\begin{cases} U_n = p\,U_{n-1} - U_{n-2}\,, & U_0=0, \quad U_1=1\,, \\ V_n = p\,V_{n-1} - V_{n-2}\,, & V_0=2, \quad V_1=p\,, \end{cases} \tag{1.1}$$

where $p \geq 2$ is an integer. For $p=2$ $\{U_n\}$ becomes the sequence of non-negative integers, and for this reason we may look upon $\{U_n\}$ as a generalization of the non-negative integers. The sequence $\{V_n\}$ bears the same relation to $\{U_n\}$ as does the Lucas sequence to the Fibonacci sequence. For $p>2$ the Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha-\beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n\,,$$

where

$$\alpha = \frac{p+\sqrt{p^2-4}}{2} \quad \text{and} \quad \beta = \frac{p-\sqrt{p^2-4}}{2}$$

are the roots of $x^2-px+1=0$. We put $\Delta = (\alpha-\beta)^2 = p^2-4$.

Clary and Hemenway [2] proved

Theorem 1.

$$(p+1)\sum_{k=1}^{n} U_k^3 = (U_{n+1}-U_n+2)\Big(\sum_{k=1}^{n} U_k\Big)^2\,. \tag{1.2}$$

For $p=2$ this reduces to the well known identity

$$1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2\,. \tag{1.3}$$

Received: February 12, 1998; *Revised*: May 8, 1998.
AMS Subject Classification: 11B37, 11B39.

Similar results on sums of powers of integers have a long history. If k is a positive integer write

$$T_k(n) = (1+2+\cdots+n)^k \quad \text{and} \quad S_k(n) = 1^k + 2^k + \cdots + n^k \ .$$

Then a result which extends (1.3) is

Theorem 2.

$$T_k(n) = \frac{1}{2^{k-1}} \sum \binom{k}{2i-1} S_{2k+1-2i}(n) \ , \tag{1.4}$$

the sum being taken over those i for which $2 \le 2\,i \le k+1$.

The first few instances of (1.4) are

$$T_2(n) = S_3(n) \ , \tag{1.5}$$

$$T_3(n) = \frac{1}{4}\,S_3(n) + \frac{3}{4}\,S_5(n) \ , \tag{1.6}$$

$$T_4(n) = \frac{1}{2}\,S_5(n) + \frac{1}{2}\,S_7(n) \ . \tag{1.7}$$

Theorem 1 has been rediscovered many times. It occurs in a 1952 paper of Piza [5], and according to MacDougall [4] it was known as far back as 1877 (Lampe) and 1878 (Stern). In 1997 G.L. Cohen, a colleague of the present writer, also rediscovered Theorem 1, and thus provided the motivation for this paper. Chapter xiv of Lucas [3] contains an excellent historical survey on sums of powers of integers. In a recent paper, which also contains a wealth of historical material, Beardon [1] generalized (1.3) by describing all polynomial relations that exist between any two of the S_i.

Our object in this paper is to produce further identities like (1.2) which involve higher powers. Our main results are stated as Theorems 3, 4 and 5 in Section 3.

2 – Some preliminary results

We require the following:

$$\Delta U_n^3 = U_{3n} - 3\,U_n \ , \tag{2.1}$$

$$\Delta^2 U_n^5 = U_{5n} - 5\,U_{3n} + 10\,U_n \ , \tag{2.2}$$

$$\Delta^3 U_n^7 = U_{7n} - 7\,U_{5n} + 21\,U_{3n} - 35\,U_n \ , \tag{2.3}$$

$$U_{5n} = \Delta^2 U_n^5 + 5\,\Delta U_n^3 + 5\,U_n \ , \tag{2.4}$$

$$U_{7n} = \Delta^3 U_n^7 + 7\,\Delta^2 U_n^5 + 14\,\Delta U_n^3 + 7\,U_n \ , \tag{2.5}$$

$$V_m\,U_n = U_{m+n} - U_{m-n} \ , \tag{2.6}$$

$$\Delta U_m U_n = V_{m+n} - V_{m-n} \ , \tag{2.7}$$

$$U_{2n} = U_n\,V_n \ , \tag{2.8}$$

$$V_{2m} - 2 = \Delta U_m^2 \ , \tag{2.9}$$

$$U_n^2 + U_{n+1}^2 = p\,U_n\,U_{n+1} + 1 \ , \tag{2.10}$$

$$U_n^4 + U_{n+1}^4 = (p^2-2)\,U_n^2\,U_{n+1}^2 + 2p\,U_n\,U_{n+1} + 1 \ , \tag{2.11}$$

$$U_n^6 + U_{n+1}^6 = (p^3-3p)\,U_n^3\,U_{n+1}^3 + (3p^2-3)\,U_n^2\,U_{n+1}^2 + 3p\,U_n\,U_{n+1} + 1 \ . \tag{2.12}$$

Identities (2.1)–(2.3) are obtained from the Binet form for U_n by taking the appropriate power. Identities (2.4) and (2.5) are obtained from (2.1)–(2.3). Identities (2.6) and (2.7) are special cases of (8) and (10) respectively in [2], while (2.8)–(2.10) follow immediately from the Binet forms. Identities (2.11) and (2.12) follow from (2.10) after taking appropriate powers.

If, for the sequences U_n and V_n, we highlight the dependence on the parameter p by writing $U_n(p)$ and $V_n(p)$, then we have the following composition formulas which appear as (17) and (18) in [2]

$$V_{rn}(p) = V_n(V_r(p)) \ , \tag{2.13}$$

$$U_{rn}(p) = U_r(p)\,U_n(V_r(p)) \ . \tag{2.14}$$

In the work which follows we need the following lemmas.

Lemma 1. *If m is a positive integer, then*

$$\sum_{k=1}^{n} U_{2mk} = \frac{U_{mn}\,U_{m(n+1)}}{U_m} \ .$$

Proof: By using the Binet form for U_{2mk} and the formula for the sum of a geometric progression, we obtain

$$\begin{aligned} \sum_{k=1}^{n} U_{2mk} &= \frac{U_{2mn+2m} - U_{2mn} - U_{2m}}{V_{2m} - 2} \\ &= \frac{V_{2mn+m}\,U_m - U_m\,V_m}{\Delta U_m^2} \qquad \text{(by (2.6), (2.8) and (2.9))} \ , \end{aligned}$$

and the result follows from (2.7). ∎

Lemma 2.

$$U_{3n}\,U_{3(n+1)} = \Delta^2 (U_n\,U_{n+1})^3 + 3p\,\Delta (U_n\,U_{n+1})^2 + (3\Delta+9)\,U_n\,U_{n+1}\ .$$

Proof: From (2.1) we have

$$\begin{aligned} U_{3n}\,U_{3(n+1)} &= U_n\,U_{n+1}\,(\Delta U_n^2+3)\,(\Delta U_{n+1}^2+3) \\ &= U_n\,U_{n+1}\Big(\Delta^2 U_n^2 U_{n+1}^2 + 3\,\Delta (U_n^2+U_{n+1}^2) + 9\Big)\ , \end{aligned}$$

and the result follows from (2.10). ∎

Lemma 3.

$$\begin{aligned} U_{5n}\,U_{5(n+1)} = {}& \Delta^4 (U_n\,U_{n+1})^5 + 5p\,\Delta^3 (U_n\,U_{n+1})^4 + 5\,(2p^2-1)\,\Delta^2 (U_n\,U_{n+1})^3 \\ &+ 5p\,(2p^2-3)\,\Delta (U_n\,U_{n+1})^2 + 5\,(\Delta^2+5\Delta+5)\,U_n\,U_{n+1}\ . \end{aligned}$$

Proof: From (2.4) we have

$$U_{5n}\,U_{5(n+1)} = U_n\,U_{n+1}\,(\Delta^2 U_n^4 + 5\,\Delta U_n^2 + 5)\,(\Delta^2 U_{n+1}^4 + 5\,\Delta U_{n+1}^2 + 5)\ .$$

We complete the proof by multiplying the terms in the brackets and using (2.10) and (2.11). ∎

In precisely the same manner, using (2.5) and (2.10)–(2.12), we can prove

Lemma 4.

$$\begin{aligned} U_{7n}\,U_{7(n+1)} = {}& \Delta^6 (U_n\,U_{n+1})^7 + 7p\,\Delta^5 (U_n\,U_{n+1})^6 + 7\,(3p^2-1)\,\Delta^4 (U_n\,U_{n+1})^5 \\ &+ 35p\,(p^2-1)\,\Delta^3 (U_n\,U_{n+1})^4 + 7\,(5p^4-10p^2+2)\,\Delta^2 (U_n\,U_{n+1})^3 \\ &+ 7p\,\Delta (3\Delta^2+14\Delta+14)\,(U_n\,U_{n+1})^2 \\ &+ 7\,(\Delta^3+7\Delta^2+14\Delta+7)\,U_n\,U_{n+1}\ . \end{aligned}$$

3 – The main results

From (2.1) we have $\Delta U_{2k}^3 = U_{6k} - 3\,U_{2k}$, and using Lemma 1 we obtain

$$\Delta \sum_{k=1}^{n} U_{2k}^3 = \frac{U_{3n}\,U_{3(n+1)}}{U_3} - 3\,U_n\,U_{n+1}\ .$$

By Lemma 2 this becomes

$$U_3 \sum_{k=1}^{n} U_{2k}^3 = \Delta(U_n\, U_{n+1})^3 + 3p\,(U_n\, U_{n+1})^2 \ ,$$

and Lemma 1 with $m = 1$ yields

$$U_3 \sum_{k=1}^{n} U_{2k}^3 = \Delta\Bigl(\sum_{k=1}^{n} U_{2k}\Bigr)^3 + 3p\,\Bigl(\sum_{k=1}^{n} U_{2k}\Bigr)^2 \ . \tag{3.1}$$

To convert (3.1) to a form involving consecutive subscripts we use (2.14) with $r = 2$. That is, in (3.1) we make the substitution $U_{2k}(p) = p\,U_k(p^2 - 2)$. Finally, if we put $U_3 = p^2 - 1$ and $\Delta = p^2 - 4$, and replace p by $\sqrt{p+2}$ in order to restore the original parameter p, we obtain

Theorem 3.

$$(p+1) \sum_{k=1}^{n} U_k^3 \ = \ (p-2)\Bigl(\sum_{k=1}^{n} U_k\Bigr)^3 + 3\Bigl(\sum_{k=1}^{n} U_k\Bigr)^2 \ . \tag{3.2}$$

Now (3.2) reduces to (1.3) when $p = 2$. We also note that (3.2) is equivalent to (1.2). Indeed we can obtain (1.2) if we first factorise the right side of (3.1) and *then* convert to a form involving consecutive subscripts.

Next we obtain an analogue of (3.2) involving fifth powers. From (2.2) we have $\Delta^2 U_{2k}^5 = U_{10k} - 5\,U_{6k} + 10\,U_{2k}$, and using Lemma 1 we obtain

$$U_3\, U_5\, \Delta^2 \sum_{k=1}^{n} U_{2k}^5 \ = \ U_3\, U_{5n}\, U_{5(n+1)} - 5\,U_5\, U_{3n}\, U_{3(n+1)} + 10\,U_3\, U_5\, U_n\, U_{n+1} \ .$$

After we make the necessary substitutions using Lemma 2 and Lemma 3 this identity becomes

$$\begin{aligned} U_3\, U_5 \sum_{k=1}^{n} U_{2k}^5 \ &= \ U_3\, \Delta^2 (U_n\, U_{n+1})^5 + 5p\, U_3\, \Delta (U_n\, U_{n+1})^4 \\ &\quad + 5p^4\,(U_n\, U_{n+1})^3 - 5p^3\,(U_n\, U_{n+1})^2 \ . \end{aligned}$$

Next we use Lemma 1 to replace each occurrence of $U_n\, U_{n+1}$ by $\sum_{k=1}^{n} U_{2k}$. Finally if we note that $U_5 = p^4 - 3p^2 + 1$, and convert to consecutive subscripts as before, we obtain

Theorem 4.

$$
\begin{aligned}
(p+1)\,(p^2+p-1)\sum_{k=1}^{n} U_k^5 \;&=\; (p+1)\,(p-2)^2\left(\sum_{k=1}^{n} U_k\right)^5 \\
&\quad + 5\,(p+1)\,(p-2)\left(\sum_{k=1}^{n} U_k\right)^4 \qquad (3.3)\\
&\quad + 5\,(p+2)\left(\sum_{k=1}^{n} U_k\right)^3 - 5\left(\sum_{k=1}^{n} U_k\right)^2 .
\end{aligned}
$$

When $p = 2$ (3.3) becomes

$$S_5(n) = \frac{4}{3}\,T_3(n) - \frac{1}{3}\,T_2(n)\ ,$$

which can be obtained from (1.5) and (1.6).

Next we obtain an identity involving seventh powers. Since the algebra is lengthy (but straightforward) we omit the details. Using (2.3) together with Lemma 1 we have

$$
\begin{aligned}
U_3\,U_5\,U_7\,\Delta^3 \sum_{k=1}^{n} U_{2k}^7 \;&=\; U_3\,U_5\,U_{7n}\,U_{7(n+1)} - 7\,U_3\,U_7\,U_{5n}\,U_{5(n+1)} \\
&\quad + 21\,U_5\,U_7\,U_{3n}\,U_{3(n+1)} - 35\,U_3\,U_5\,U_7\,U_n\,U_{n+1}\ .
\end{aligned}
$$

Then we use Lemmas 2–4, together with Lemma 1, to express the right side as a polynomial in $\sum_{k=1}^{n} U_{2k}$. Finally, noting that $U_7 = p^6 - 5p^4 + 6p^2 - 1$, we change to consecutive subscripts to obtain

Theorem 5.

$$
\begin{aligned}
&(p+1)\,(p^2+p-1)\,(p^3+p^2-2p-1)\sum_{k=1}^{n} U_k^7 \;= \\
&\quad = (p{+}1)\,(p{-}2)^3\,(p^2{+}p{-}1)\left(\sum_{k=1}^{n} U_k\right)^7 + 7\,(p{+}1)\,(p{-}2)^2\,(p^2{+}p{-}1)\left(\sum_{k=1}^{n} U_k\right)^6 \\
&\quad + 7\,(p+1)\,(2p-1)\,(p^2-4)\left(\sum_{k=1}^{n} U_k\right)^5 + 35\,p\,(p+1)\left(\sum_{k=1}^{n} U_k\right)^4 \qquad (3.4)\\
&\quad - 7\,(p+2)\,(2p+1)\left(\sum_{k=1}^{n} U_k\right)^3 + 7\,(2p+1)\left(\sum_{k=1}^{n} U_k\right)^2 .
\end{aligned}
$$

When $p=2$ (3.4) becomes

$$S_7(n) = 2\,T_4(n) - \frac{4}{3}\,T_3(n) + \frac{1}{3}\,T_2(n) \ ,$$

which can be obtained from (1.5)–(1.7).

4 – Concluding remarks

Interestingly, in each of (3.2)–(3.4) the sum of the polynomial coefficients on the right side is equal to the polynomial coefficient on the left side. We have not been able to detect any other pattern in these coefficients. Our method of deriving these identities suggests that there are higher power analogues. Is there a more direct way to derive them? Is there a general formula which encompasses all such identities?

We conclude by making an unusual observation. If we denote the k-th derivative (with respect to p) of V_n by $V_n^{(k)}$, where $V_n^{(0)} = V_n$, then (2.10)–(2.12) can be written respectively as

$$U_n^2 + U_{n+1}^2 = V_1^{(0)}\,U_n\,U_{n+1} + V_1^{(1)} \ ,$$

$$U_n^4 + U_{n+1}^4 = V_2^{(0)}\,U_n^2\,U_{n+1}^2 + V_2^{(1)}\,U_n\,U_{n+1} + \frac{V_2^{(2)}}{2!} \ ,$$

$$U_n^6 + U_{n+1}^6 = V_3^{(0)}\,U_n^3\,U_{n+1}^3 + V_3^{(1)}\,U_n^2\,U_{n+1}^2 + \frac{V_3^{(2)}}{2!}\,U_n\,U_{n+1} + \frac{V_3^{(3)}}{3!} \ .$$

After checking that this pattern continues for several more cases, we make the following conjecture.

Conjecture: If k is a positive integer then

$$U_n^{2k} + U_{n+1}^{2k} = \sum_{r=0}^{k} \frac{V_k^{(r)}}{r!}\,U_n^{k-r}\,U_{n+1}^{k-r} \ .$$

REFERENCES

[1] Beardon, A.F. – Sums of powers of integers, *The American Mathematical Monthly,* 103(3) (1996), 201–213.

[2] Clary, S. and Hemenway, P.D. – *On sums of cubes of Fibonacci numbers.* In "Applications of Fibonacci Numbers", 5 (G.E. Bergum et al., Eds.), Kluwer Academic Publishers, 1993, 123–136.

[3] Lucas, Édouard – *Théorie des Nombres*, Paris, Gauthier-Villars, 1891.

[4] MacDougall, J.A. – Identities relating the sums of powers of integers – An exercise in generalization, *Australian Senior Mathematical Journal,* 2(1) (1988), 53–62.

[5] Piza, P.A. – Powers of sums and sums of powers, *Mathematics Magazine,* 25(3) (1952), 137–142.

R.S. Melham,
School of Mathematical Sciences, University of Technology, Sydney,
PO Box 123, Broadway, NSW 2007 – AUSTRALIA