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MICROLOCAL TEMPERED INVERSE IMAGE AND CAUCHY PROBLEM

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Abstract: We prove an inverse image formula for the functor $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$ of Andronikof [A], that is, the microlocalization of the functor $\mathcal{T}hom(\cdot, \mathcal{O})$ of tempered cohomology introduced by Kashiwara. As an application, following an approach initiated by D'Agnolo and Schapira, we study the tempered ramified linear Cauchy problem. We deal with ramifications of logarithmic type, or along a swallow's tail subvariety, or at the boundary of the data existence domain.

1 – Introduction

The construction of $\mathcal{T}hom(\cdot, \mathcal{D}b_X)$ has been introduced by Kashiwara in [K 1] and has been microlocalized by Andronikof in [A]. A slightly different and more systematic exposition has been provided by Kashiwara and Schapira in [K-S 2]. In the present paper we give a microlocal version of the inverse image formula [K-S 2, Theorem 4.5] (with p submersive)

$$(1) \qquad R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{D}_{X\to Y},\, \mathcal{T}hom(p^{-1}F, \mathcal{D}b_Y)\Big) \xrightarrow{\sim} p^{-1}\,\mathcal{T}hom_{\mathcal{D}_X}(F, \mathcal{D}b_X)\;,$$

namely, we prove the formula (with p submersive)

$$(2) \qquad \mathrm{R}^t\, p'_*\, p_\pi^{-1}\, \mathcal{T}\mu\mathrm{hom}(L, \mathcal{D}b_Y) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\, \mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{D}b_X)\Big)\;.$$

By the aid of (2) it is possible to obtain a sheaf theoretical Cauchy–Kowalewsky type theorem. This last makes in particular possible, adapting a method due

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to [D'A-S 1], to recover some known results on the tempered ramified Cauchy problem (see [La, Th. 3.2.6], [Le 2], [Sc]).

In §2 we recall all the notions and the machinery we will need in the paper.

In §3 we present an inverse image theorem for the functor $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$. The main tools for proving it are formula (1) and Andronikof's stalk formula for $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$ ([A, Prop. 2.3.3]).

In §4 we prove a Cauchy–Kowalewski type formula for the functor $\mathcal{T}hom(\cdot, \mathcal{O})$. The idea of the proof is to use Sato's distinguished triangle and the microlocalization of the functor $\mathcal{T}hom(\cdot, \mathcal{O})$ as developed in [A]. This allows us to shift the problem to the cotangent space, where by use of contact transforms the system is reduced to a partial de Rham system. Thorough use of the theory of $\mathcal{E}_X$-modules is made. We then need the result of §3 to complete the proof.

Applications follow, which are the tempered analogues of results already obtained by [D'A-S 1] and [D'A-S 2] (see [La], [Le 2], [Sc]).

In the first one we deal with tempered holomorphic functions with ramifications of logarithmic type along a hypersurface. This application involves the notion of perverse sheaf. A proof of this result has been obtained by use of second microlocalization (see [La, Th. 3.2.6]). Our proof just uses microlocalization.

A second application is concerned with the Cauchy problem for tempered holomorphic functions ramified along singularities of "swallow's tail" type as in [Le 2]. Again, the notion of perverse sheaf is used.

Remark that the functor $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$ is only defined on $\mathbb{R}$-constructible sheaves. So its employment only allows us to treat ramified functions of finite determination (see [H-L-W], [Le 1] for the case of data with general ramification).

We prove also a theorem on the Cauchy problem for tempered holomorphic functions defined in domains determined by the characteristic real hypersurfaces issuing from the boundary of the domain where the data are defined. This last application extends results of [Sc].

We are deeply indebted to Professor Andrea D'Agnolo and Professor Pierre Schapira for a many useful discussion.

2 – Notations and definitions

We shall follow the notations of [K-S 1]. We also refer to [S 1] for an exposition of $\mathcal{E}$-modules theory, to [K 1] for the functor $\mathcal{T}hom$ (see also [K-S 2]) and to [A] for the functor $\mathcal{T}\mu\mathrm{hom}$.

Geometry.

Given two sets X, Y we will denote by $q_1: X \times Y \to X$ and $q_2: X \times Y \to Y$ the projections on the first and second factor; similar notations will be used for the product of more than two sets. Given a function $f: X \to Y$, denote by Δ_f its graph, a subset of $X \times Y$. Let X be a complex manifold of complex dimension n. Define $X_{\mathbb{R}}$ the real underlying manifold to X and $\overline{X}$ the complex conjugate to X. If no confusion arises we will sometimes write X instead of $X_{\mathbb{R}}$. Denote by $\tau_X: TX \to X$ the tangent bundle, $\pi_X: T^*X \to X$ the cotangent bundle. If $(x) = (x_1, ..., x_n)$ is a local coordinate system on X, we denote by $(x; \xi)$ the associated coordinate system on T^*X. If no confusion arises we will write π instead of π_X. We will identify X with $T^*_X X$, the zero section of T^*X. We also write $\dot{T}^*X = T^*X \setminus T^*_X X$ and $\dot{\pi}_X = \pi_X|_{\dot{T}^*X}$. If M is a closed submanifold of X, we denote by $T^*_M X$ the conormal bundle to M in X.

Let $f: Y \to X$ be a morphism of manifolds. Define ${}^t f'$ and f_π the natural mappings associated to f:

$$T^*Y \xleftarrow{\ {}^t f'\ } Y \times_X T^*X \xrightarrow{\ f_\pi\ } T^*X\ . \tag{3}$$

Set $T^*_Y X := {}^t f'^{-1}(T^*_Y Y)$. Let A be a closed conic subset of T^*X. One says f is *non-characteristic* for A if $T^*_Y X \cap f_\pi^{-1}(A) \subset Y \times_X T^*_X X$.

We consider T^*X endowed with its canonical symplectic structure. We will denote by $C_S(K)$ the Whitney normal cone of K along S; it is a subset of $T_{T^*_S} T^*X$ (see [K-S 1, Def. 4.1.1]). Given a function $f: Y \to X$, we define the projection $q: T^*_Y(X \times Y) \to Y$. Let W be a subset of T^*Y. We say that f is *non-characteristic* for A on W (see [K-S 1, Def. 6.2.7]) if

$$\dot{q}^t_\pi\, \dot{q}'^{-1}\Big(C_{T^*_Y(X\times Y)}(A \times T^*_Y Y)\Big) \cap V = \emptyset\ . \tag{4}$$

Let V and T be two conic smooth involutive manifolds in a neighborhood of $\xi \in \dot{T}^*X$, V being regular involutive. We say that V is *non glancing* with respect to T if for any function ϕ defined in a neighborhood of ξ, such that $\phi|_T = 0$ and $\mathrm{d}\phi \neq 0$, the vector H_ϕ is not tangent to V. If Y is a submanifold of X we say that V is *non glancing with respect to Y* if V is non glancing with respect to $Y \times_X T^*X$.

If E is a vector space and γ is a cone in E, we denote by $Z(\gamma)$ the set $\{(e, e') \in E \times E;\ e' - e \in \gamma\} \subset E \times E$. If γ is a conic subset of TX, one denotes $\gamma^a = -\gamma$. One says that γ is proper if its fibers contain no lines. One denotes by γ° the polar cone to γ, a convex conic subset of T^*X (see [K-S 1, (3.7.6)]).

Sheaves.

Let X be a real analytic manifold. Recall that the dualizing complex ω_X is isomorphic to $or_X[\dim X]$, where or_X is the orientation sheaf. We denote by $D^b(\mathbb{C}_X)$ (respectively $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$) the derived category of the category of complexes of sheaves with bounded (resp. bounded and $\mathbb{R}$-constructible) cohomology. To an object F in $D^b(\mathbb{C}_X)$ we associate its *micro-support* $\mathrm{SS}(F)$, a closed conic involutive subset of T^*X (see [K-S 1, Chapter 5]). Let V be a subset of T^*X. One says that f is *non-characteristic* for F on V if f is non-characteristic for $\mathrm{SS}(F)$ on V.

For a sheaf F on X, when considering its inverse image $\pi_X^{-1}F$ on T^*X we will often simply write F instead of $\pi_X^{-1}F$.

We shall use the language of kernels, as developed in [K-S 1, §3.6]. Given two objects $F \in D^b(\mathbb{C}_{X\times Y})$ and $G \in D^b(\mathbb{C}_{Y\times Z})$, $F \circ G$ will denote the composition of kernels $\mathrm{R}q_{13!}(q_{12}^{-1}F \otimes q_{23}^{-1}G)$ (see [K-S 1, (3.6.2)]); it is an object in $D^b(\mathbb{C}_{X\times Z})$.

We will make use of the notion of *perverse sheaf* (see [K-S 1, Chapter 10] for an exposition).

Let $\Omega \subset T^*X$; one denotes by $D^b(\mathbb{C}_X;\Omega)$ the localization of $D^b(\mathbb{C}_X)$ by the full subcategory of objects whose microsupport is disjoint from Ω (see [K-S 1, Chapter 6]). Recall that a morphism $u: F \to G$ in $\mathrm{D}^b(X)$ becomes an isomorphism in $\mathrm{D}^b(X;\Omega)$ if $\Omega \cap \mathrm{SS}(H) = \emptyset$, H being the third term of a distinguished triangle

$$F \xrightarrow{u} G \to H \xrightarrow{+1} \quad . \tag{5}$$

Let $f: Y \to X$ be a morphism of complex manifolds; let $\xi \in T^*X$, $\pi(\xi) \in f(Y)$. Let $F \in D^b(\mathbb{C}_X;\xi)$. According to [K-S 1, Prop. 6.1.9], [D'A-S 1, Lemma 1.3.4], we consider the condition

$$ {}^tf'^{-1}({}^tf'(\xi)) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{\xi\} \quad \text{in a neighborhood of } \xi \ , \tag{6}$$

Lemma 2.1 ([K-S 1, Prop. 6.1.9]). *Let $F \in \mathrm{D}^b(\mathbb{C}_X;\xi)$ satisfy (6); then there exists $F' \in \mathrm{D}^b(\mathbb{C}_X)$ with ${}^tf'^{-1}({}^tf'(\xi)) \cap f_\pi^{-1}(\mathrm{SS}(F_{Y,\xi})) \subset \{\xi\}$ and that f is noncharacteristic for $F_{Y,\xi}$, and there exists a morphism $F' \to F$ (resp. $F \to F'$) in $\mathrm{D}^b(\mathbb{C}_X)$ which is an isomorphism at ξ. Moreover, for $F \in \mathrm{D}^b(\mathbb{C}_X;\xi)$ satisfying (6), the object $F_{Y,\xi} := f^{-1}F'$ (resp. $f^!F'$) of $\mathrm{D}^b(\mathbb{C}_Y; {}^tf'^{-1}(\xi)$ does not depend (up to isomorphism) on the choice of F'.*

Definition 2.2. Let $F \in \mathrm{D}^b(\mathbb{C}_X;\xi)$ satisfy (6). We define the microlocal inverse image (resp. extraordinary inverse image) of F by $f_{\mu,\xi}^{-1}F := f^{-1}F'$ (resp. $f_{\mu,\xi}^!F := f^!F'$), where F' is the complex constructed in Lemma 2.1. The functor

$f^{-1}_{\mu,\xi}(\cdot)$ (resp. $f^{!}_{\mu,\xi}(\cdot)$) is a functor from the full subcategory of $\mathrm{D}^b(\mathbb{C}_X;\xi)$ whose objects verify (6) to $\mathrm{D}^b(\mathbb{C}_Y;{}^t f'(\xi))$.

$\mathcal{D}$ and $\mathcal{E}$-modules. (See [S-K-K], and also [S 1] for an exposition.)

Let X be a complex manifold and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. By $\mathcal{D}_X$ (respectively $\mathcal{E}_X$) we denote the sheaf of holomorphic finite order differential operators on X (resp. microdifferential operators on X). We denote by $D^b(\mathcal{D}_X)$ (resp. $D^b_{coh}(\mathcal{D}_X)$) the derived category of the category of sheaves of left $\mathcal{D}_X$-modules with bounded (resp. bounded and coherent) cohomology. We define analogously the categories $D^b(\mathcal{E}_X)$ and $D^b_{coh}(\mathcal{E}_X)$. We will also use the category $D^b(\mathcal{E}_X|_\Omega)$. Suppose $\mathcal{M} \in \mathrm{Ob}(D^b_{coh}(\mathcal{D}_X))$. We denote by $\mathrm{char}(\mathcal{M})$ the characteristic variety of $\mathcal{M}$.

To a map $f\colon Y \to X$ of complex manifolds one associates the $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-bimodule $\mathcal{D}_{Y\to X} \equiv \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$ and the $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-bimodule $\mathcal{D}_{X\leftarrow Y} := f^{-1}\mathcal{D}_X \otimes_{f^{-1}\mathcal{O}_X} \Omega_{Y|X}$ (see [S-K-K] for more details).

The functors $\mathcal{T}hom(\cdot,\mathcal{D}b)$ and $\mathcal{T}\mu\mathrm{hom}(\cdot,\mathcal{D}b)$. (See [K 1], [A].)

We write $\mathcal{D}b$ for the sheaf of Schwartz distributions. Let U be an open subset of X. Let $x \in \partial U$. The distribution $u \in \Gamma(U;\mathcal{D}b_X)$ is said to be *tempered at* x if there exists an open neighborhood V of x and $v \in \Gamma(X;\mathcal{D}b_X)$ such that $v|_{V\cap U} = u|_{V\cap U}$; u is said to be *tempered* if it is tempered at any point in ∂U. We shall denote by $\mathcal{S}'_X(U)$ the subspace of $\mathcal{D}b_X(U)$ of tempered distributions.

Let F be a $\mathbb{R}$-constructible sheaf. One defines $\mathcal{T}hom(F,\mathcal{D}b_X)(U)$ as the space of sections $\phi \in \Gamma(U;\mathcal{H}\mathrm{om}(F,\mathcal{D}b_X))$ such that for all subanalytic relatively compact open sets $V \subset U$ and for all sections $s \in \Gamma(V;F)$, we have $\phi(s) \in \mathcal{S}'_X(V)$.

The correspondence $U \mapsto \mathcal{T}hom(F,\mathcal{D}b_X)(U)$ defines a sheaf that has the following properties.

1) $\mathcal{T}hom(F,\mathcal{D}b_X)$ is a $\mathcal{C}^\infty$-module, in particular it is a soft sheaf.

2) For all subanalytic open subsets V of X and for all subanalytic closed subsets Z of X we have

$$\Gamma(U;\mathcal{T}hom(\mathbb{C}_V)) = \mathcal{S}'_U(U\cap V)\,, \quad \mathcal{T}hom(F_Z,\mathcal{D}b_X) = \Gamma_Z\,\mathcal{T}hom(F,\mathcal{D}b_X)\,. \tag{7}$$

3) The functor $F \mapsto \mathcal{T}hom(F,\mathcal{D}b_X)$ is an exact functor from $\mathbb{R}-c(X)^{op}$ to the category $Mod(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules.

One also defines:

$$\mathcal{T}hom(\cdot, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\Big(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(\cdot, \mathcal{D}b_{X_{\mathbb{R}}})\Big) \ . \tag{8}$$

We recall that Andronikof [A] performed the construction of the functor

$$\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{D}b)\colon \mathrm{D}^b{}_{\mathbb{R}-c}(\mathbb{C}_X)^{op} \to \mathrm{D}^b(\pi^{-1}\mathcal{D}_X) \ . \tag{9}$$

The construction is the analog of the construction leading to the functor $\mu\mathrm{hom}(\cdot,\cdot)$. Recall that one has a morphism of functors

$$\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{D}b_X) \to \mu\mathrm{hom}(\cdot, \mathcal{D}b_X) \ . \tag{10}$$

Moreover, there are isomorphisms

$$\mathrm{R}\pi_* \, \mathcal{T}\mu\mathrm{hom}(F, \mathcal{D}b_X) \simeq \mathcal{T}hom_X(F, \mathcal{D}b_X)$$
$$\mathrm{R}\pi_! \, \mathcal{T}\mu\mathrm{hom}(F, \mathcal{D}b_X) \simeq R\mathcal{H}om(F, \mathbb{C}_X) \otimes \mathcal{D}b_X \ .$$

We also define:

$$\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\Big(\mathcal{O}_{\overline{X}}, \mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{D}b)\Big) \ . \tag{11}$$

For a more recent construction of the functors $\mathcal{T}hom(\cdot, \mathcal{D}_X)$ and $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ we refer the reader to [K-S 2].

$\mathcal{D}$-modules with regular singularities.

We review in this section some notions and results from Kashiwara–Oshima [K-O].

The ring $\mathcal{E}_X$ is naturally endowed with a $\mathbb{Z}$-filtration by the degree, and we denote by $\mathcal{E}_X(k)$ the sheaf of operators of degree at most k. Denote by $\mathcal{O}_{T^*X}(k)$ the sheaf of holomorphic functions on T^*X, homogeneous of degree k.

Let $V \subset T^*X$ be a conic regular involutive submanifold. Denote by $\mathcal{I}_V(k)$ the sheaf ideal of sections of $\mathcal{O}_{T^*X}(k)$ vanishing on V. Let $\mathcal{E}_V$ be the subalgebra of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the sections P of $\mathcal{E}_X(1)$ such that $\sigma_1(P)$ belongs to $\mathcal{I}_V(1)$ (here $\sigma_1(\cdot)$ denotes the symbol of order 1). For example, if $X = X' \times X''$, and $V = T^*_{X'}X' \times U$ for an open subset $U \subset \dot{T}^*X''$, then $\mathcal{E}_V$ is the subalgebra of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the differential operators of X'.

Let $\mathcal{P}$ be a coherent $\mathcal{E}_X$-module. One says that $\mathcal{P}$ has *regular singularities along V* if locally there exists a coherent sub-$\mathcal{E}_X(0)$-module $\mathcal{P}_0$ of $\mathcal{P}$ which generates it over $\mathcal{E}_X$, and such that $\mathcal{E}_V\mathcal{P}_0 \subset \mathcal{P}_0$. One says that $\mathcal{P}$ is *simple along V* if locally there exists an $\mathcal{E}_X(0)$-module $\mathcal{P}_0$ as above such that $\mathcal{P}_0/\mathcal{E}_X(-1)\,\mathcal{P}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank one.

One says that $\mathcal{P}$ has *regular singularities* if $\mathcal{P}$ has regular singularities along $\dot{T}^*X \cap \mathrm{char}(\mathcal{P})$.

To $\mathcal{M} \in \mathrm{D}^b{}_{coh}(\mathcal{D}_X)$ we associate its microlocalization

$$\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M} \ .$$

We say that $\mathcal{M}$ has regular singularities along V if $\mathcal{E}\mathcal{M}$ has regular singularities along V.

One of the main results of Sato–Kawai–Kashiwara [S-K-K] asserts that contact transformations can be quantized to give an equivalence of categories at the level of $\mathcal{E}$-modules. Remark that the notion of module with regular singularities is invariant by quantized contact transformations, and that a module with regular singularities along V is supported by V (cf. [K-O, Lemma 1.13]).

Theorem 2.3 (cf. [K-O, Theorem 1.9])**.** *Let $V = T^*_{X'}X' \times U \subset T^*(X' \times X'')$, where $U \subset \dot{T}^*X''$ is an open subset. If $\mathcal{P}$ has regular singularities along V, then it is a quotient of a multiple partial de Rham system:*

$$(\mathcal{O}_{X'} \underline{\boxtimes}\, \mathcal{E}_{X''})^N \to \mathcal{P} \to 0 \ .$$

Here, $\underline{\boxtimes}$ denotes the exterior tensor product for $\mathcal{E}$-modules.

3 – An inverse image theorem for $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$

The aim of this chapter is to prove the following result:

Proposition 3.1. *Let $p\colon X \to Y$ be a smooth morphism of real analytic manifolds and let $L \in D^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Then one has a canonical isomorphism:*

$$\text{(12)}\quad \mathrm{R}^t\, p'_*\, p_\pi^{-1}\, \mathcal{T}\mu\mathrm{hom}(L, \mathcal{D}b_Y) \xrightarrow{\sim} R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}\Big(\pi^{-1}\mathcal{D}_{X\to Y}, \mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{D}b_X)\Big) \,.$$

When restricting (12) to the zero section, one recovers the isomorphism (see [K 1], [A, Prop. 1.1.3], [K-S 2, Th. 4.5 (i)]):

$$\text{(13)}\qquad R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y}, \mathcal{T}hom(p^{-1}L, \mathcal{D}b_X)\Big) \xrightarrow{\sim} p^{-1}\, \mathcal{T}hom(L, \mathcal{D}b_Y) \ .$$

Corollary 3.2. *Let $p\colon X \to Y$ be a smooth morphism of complex analytic manifolds and let $L \in D^b_{\mathbb{R}-c}(\mathbb{C}_Y)$. Suppose that $\mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{O}_X) \in D^b(\mathcal{E}_X)$. Then one has a canonical isomorphism:*

$$\text{(14)}\qquad \mathrm{R}^t\, p'_*\, p_\pi^{-1}\, \mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_{X\to Y},\, \mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{O}_X)\Big) \ .$$

Proof: We apply the isomorphism (see [K-S 2, (5.5)]):

$$R\mathcal{H}om_{p^{-1}\mathcal{D}_{\overline{Y}}}\Big(p^{-1}\mathcal{O}_{\overline{Y}}, R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{D}_{X_{\mathbb{R}}\to Y_{\mathbb{R}}}, N)\Big) \simeq \simeq R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y}, R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, N)\Big)\ , \tag{15}$$

with N replaced by $\mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{D}b_X)$ and then we apply Proposition 3.1. Moreover, due to the fact that p is smooth we have $\mathcal{E}_{X\to Y} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{D}_{X\to Y}$. ∎

Proof of Proposition 3.1: First of all, we construct the morphism in (12). The construction of the morphism

$$\mathrm{R}^t p'_!\Big(\pi^{-1}\mathcal{D}_{X\to Y} \otimes^L_{\pi^{-1}p^{-1}\mathcal{D}_Y} p_\pi^{-1}\mathcal{T}\mu\mathrm{hom}(L, \mathcal{D}b_Y)\Big) \to \to \mathcal{T}\mu\mathrm{hom}(p^! L, \mathcal{D}b_X)[\dim(X) - \dim(Y)] \tag{16}$$

follows the same lines as that of $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O}_X)$ (see [A, Th. 3.3.6]). We just have to consider [A, Lemma 2.4.6], and to use the morphism [A, (1.1.10)]

$$\mathcal{D}_{Y\to X} \otimes_{p^{-1}\mathcal{D}_X} p^{-1}\mathcal{T}hom_X(F, \mathcal{D}b_X) \to \mathcal{T}hom_X(p^{-1}F, \mathcal{D}b_X) \tag{17}$$

to adapt [A, Prop. 3.3.3].

Since p is smooth, ${}^t p'$ is a closed embedding, and $p^! L[\dim(Y) - \dim(X)] \simeq p^{-1}L$. Hence, using the projection formula, (16) reads:

$$\pi_X^{-1}\mathcal{D}_{X\to Y} \otimes^L_{\pi_X^{-1}p^{-1}\mathcal{D}_Y} \mathrm{R}^t p'_* \, p_\pi^{-1}\, \mathcal{T}\mu\mathrm{hom}(L, \mathcal{D}b_Y) \to \mathcal{T}\mu\mathrm{hom}(p^{-1}L, \mathcal{D}b_X)\ . \tag{18}$$

The morphism in (12) is obtained by adjunction from the morphism above.

We have to show that the morphism in (12) is an isomorphism in $\mathrm{D}^b(\mathbb{C}_{T^*X})$. When restricted to the zero section of T^*X, (12) is nothing but the inverse of the isomorphism (13). We are then left to prove that the stalk at ξ of (12) is an isomorphism, for every $\xi \in \dot{T}^*X$.

If $\xi \in \dot{T}^*X \setminus X \times_Y \dot{T}^*Y$ both terms are zero. Let then $\xi \in X \times_Y \dot{T}^*Y$, and set $\eta = p_\pi(\xi)$.

As we have to examine the stalk of the morphism at ξ, the claim is of local nature. Then we can suppose, until the end of the present chapter, $X := Y \times Z$ with Y and Z finite dimensional real vector spaces, $\dim(X) = n$, $\dim(Y) = m$. Let p and q be the projections $p\colon X \to Y$, $q\colon X \to Z$, and $\pi_X(\xi) = 0$. Also recall the maps $q_1\colon X \times X \to X$; $(x, x') \mapsto x$ and $q_2\colon X \times X \to X$; $(x, x') \mapsto x'$. Let us begin by stating two lemmas.

Lemma 3.3. *Let γ be a closed cone in X with p proper on γ. Take $W \subset W'$ open neighborhoods of 0 in Z, $V \subset V'$ open neighborhoods of 0 in Y. Suppose that:*

$$q\Big(\gamma \cap p^{-1}(V'+V^a)\Big) + W \subset W' \ , \tag{19}$$

where $V^a := -V$. Then:

$$Z(\gamma) \cap q_2^{-1}(V' \times W') \cap q_1^{-1}(V \times W) = Z(\gamma) \cap q_2^{-1}(V' \times Z) \cap q_1^{-1}(V \times W) \ . \tag{20}$$

Refer to §2, Geometry, for notations.

Proof: Consider the following diagram:

$$\begin{array}{ccccccccc} Y & \xleftarrow{p} & Y \times Z & \xleftarrow{q_1} & X \times X & \xrightarrow{q_2} & Y \times Z & \xrightarrow{p} & Y \\ & & {\scriptstyle q}\downarrow & & & & {\scriptstyle q}\downarrow & & \\ & & Z & & & & Z & & \end{array} \tag{21}$$

Let $(x,x') \in Z(\gamma)$ with $x \in V \times W$ and $x' \in V' \times Z$. Then $p(x'-x) \in V'-V$ implies $x'-x \in p^{-1}(V'-V)$. Also, $x'-x \in \gamma$, so:

$$x' - x \ \in \ \gamma \cap p^{-1}(V' - V) \tag{22}$$

hence

$$q(x' - x) \ \in \ q\Big(\gamma \cap p^{-1}(V' - V)\Big) \ . \tag{23}$$

This toghether with $q(x) \in W$ give us:

$$q(x') = q(x' - x) + q(x) \ \in \ q\Big(\gamma \cap p^{-1}(V' - V)\Big) + W \tag{24}$$

and, by (19), $q(x') \in W'$; in other words $(x,x') \in Z(\gamma) \cap q_2^{-1}(V' \times W') \cap q_1^{-1}(V \times W)$. ∎

Lemma 3.4. *Let γ be a closed cone in X. Let $p|_\gamma$ be proper with convex fibers. Let $\gamma' := p(\gamma)$ and let $G \in \mathrm{D}^b(\mathbb{C}_Y)$. Then:*

$$p^{-1}(\mathbb{C}_{Z(\gamma')} \circ G) \ \simeq \ \mathbb{C}_{Z(\gamma)} \circ p^{-1} G \ . \tag{25}$$

Refer to §2, Sheaves, for notations.

Proof: Let $\Delta_p := \{(x,y) \in X \times Y;\ y = p(x)\}$. Then one has:

$$\begin{aligned} \mathbb{C}_{Z(\gamma)} \circ \mathbb{C}_{\Delta_p} &\simeq q_{13!}\Big(\mathbb{C}_{(Z(\gamma)\times Y)\cap(X\times\Delta_p)}\Big) \\ &\simeq \mathbb{C}_{\big\{(x,y)\,\in\, X\times Y;\ (x+\gamma)\,\cap\, p^{-1}(y)\,\neq\,\emptyset\big\}} \ , \end{aligned} \tag{26}$$

where $q_{13}: X\times X\times Y \to X\times Y$ is the projection on the first and third factors. Analogously:

$$\begin{aligned} \mathbb{C}_{\Delta_p}\circ\mathbb{C}_{Z(\gamma')} &\simeq q_{13!}\Big(\mathbb{C}_{(\Delta_p\times Y)\cap(X\times Z(\gamma'))}\Big) \\ &\simeq \mathbb{C}_{\big\{(x,y)\in X\times Y;\ y\in p(x)+\gamma'\big\}}\ . \end{aligned} \tag{27}$$

So we have $\mathbb{C}_{\Delta_p}\circ\mathbb{C}_{Z(\gamma')}\simeq\mathbb{C}_{Z(\gamma)}\circ\mathbb{C}_{\Delta_p}$. ∎

End of proof of Proposition 3.1.

We recall the stalk formula for $\mathcal{T}\mu\mathrm{hom}(\cdot,\mathcal{D}b_Y)$ at η (see [A, Prop. 2.3.3] and also [K-S 1, Prop. 3.5.4] and proof of [K-S 1, Prop. 4.4.4]): $\forall\, j$

$$H^j\Big(\mathcal{T}\mu\mathrm{hom}(L,\mathcal{D}b_Y)\Big)_\eta \simeq \varinjlim_{V,\gamma'} H^j\mathrm{R}\Gamma\Big(V;\mathcal{T}hom(\mathbb{C}_{Z(\gamma')}\circ L_V,\mathcal{D}b_Y)\Big)\ , \tag{28}$$

where γ' is a convex proper closed subanalytic cone in $\mathbb{R}^m$ such that $\gamma'\subset\mathrm{Int}(\{\eta\}^{\circ a})\cup\{0\}$, and V is a subanalytic open neighborhood of $\pi_Y(\eta)=0$ in Y.

Then we use the stalk formula for $\mathcal{T}\mu\mathrm{hom}(\cdot,\mathcal{D}b_X)$. Let γ be a convex proper closed subanalytic cone in $\mathbb{R}^n$ such that $\gamma\subset\mathrm{Int}(\{\xi\}^{\circ a})\cup\{0\}$, and let U, U' be subanalytic open neighborhoods of $\pi(\xi)=0$ in X, with $U\subset U'$. It is not restrictive to take $U=V\times W$, $U'=V'\times W'$ with V,V' subanalytic open sets in Y, W,W' subanalytic open sets in Z. Also, take V,V', W,W' as to satisfy (19). The stalk formula reads: $\forall\, j$

$$\begin{aligned} &H^j R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\mathcal{T}\mu\mathrm{hom}(p^{-1}L,\mathcal{D}b_X)\Big)_\xi \simeq \\ &\simeq \varinjlim_{U',U,\gamma} H^j\mathrm{R}\Gamma\bigg(U;\ R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\ \mathcal{T}hom_X\Big(\mathbb{C}_{Z(\gamma)}\circ((p^{-1}L)_{U'}),\mathcal{D}b_X\Big)\Big)\bigg). \end{aligned} \tag{29}$$

Let us make use of the following morphisms:

$$\begin{aligned} \mathbb{C}_{Z(\gamma)}\circ((p^{-1}L)_{U'}) &\simeq \mathbb{C}_{Z(\gamma)}\circ((p^{-1}L)_{U'})_U \\ &\simeq \mathrm{R}q_{1!}\Big(q_2^{-1}p^{-1}L\otimes\mathbb{C}_{Z(\gamma)\cap q_2^{-1}U'\cap q_1^{-1}U}\Big) \\ &\simeq \mathrm{R}q_{1!}\Big(q_2^{-1}p^{-1}L\otimes\mathbb{C}_{Z(\gamma)\cap q_2^{-1}p^{-1}V'\cap q_1^{-1}U}\Big) \\ &\simeq (\mathbb{C}_{Z(\gamma)}\circ p^{-1}L_{V'})_U \\ &\simeq \mathbb{C}_{Z(\gamma)}\circ p^{-1}L_{V'} \\ &\simeq p^{-1}(\mathbb{C}_{Z(\gamma')}\circ L_{V'})\ , \end{aligned}$$

where the third isomorphism is due to Lemma 3.3, whereas the last one follows from Lemma 3.4. We can rephrase (29) taking into account the following:

$$
\begin{aligned}
&\mathrm{R}\Gamma\Big(U; R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\, \mathcal{T}hom_X\big(p^{-1}(\mathbb{C}_{Z(\gamma')}\circ L_{V'}),\, \mathcal{D}b_X\big)\Big)\Big) \simeq \\
&\qquad \simeq \mathrm{R}\Gamma\Big(U;\, p^{-1}\,\mathcal{T}hom_Y(\mathbb{C}_{Z(\gamma')}\circ L_{V'},\, \mathcal{D}b_Y)\Big) \\
&\qquad \simeq \mathrm{R}\Gamma\Big(V\times Z;\, p^{-1}\,\mathcal{T}hom_Y(\mathbb{C}_{Z(\gamma')}\circ L_{V'},\, \mathcal{D}b_Y)\Big) \\
&\qquad \simeq \mathrm{R}\Gamma\Big(V;\, \mathrm{R}\,p_*\,p^{-1}\,\mathcal{T}hom_Y(\mathbb{C}_{Z(\gamma')}\circ L_{V'},\, \mathcal{D}b_Y)\Big) \\
&\qquad \simeq \mathrm{R}\Gamma\Big(V;\, \mathcal{T}hom_Y(\mathbb{C}_{Z(\gamma')}\circ L_{V'},\, \mathcal{D}b_Y)\Big)\ .
\end{aligned} \tag{30}
$$

The first isomorphism is due to (13); the last one to p having contractible fibers, so that $\mathrm{R}\,p_*\,p^{-1}(\cdot) = \mathrm{id}_Y$. By this and and (28) we get: $\forall\, j$

$$
\begin{aligned}
&H^j\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{D}_{X\to Y},\, \mathcal{T}\mu\mathrm{hom}(p^{-1}L,\, \mathcal{D}b_X)\Big)_\xi \simeq \\
&\qquad \simeq \varinjlim_{V',V,\gamma'} H^j\,\mathrm{R}\Gamma\Big(V;\, \mathcal{T}hom_Y(\mathbb{C}_{Z(\gamma')}\circ L_{V'},\, \mathcal{D}b_Y)\Big) \\
&\qquad \simeq H^j\,\mathrm{R}^t\,p'_*\,p_\pi^{-1}\,\mathcal{T}\mu\mathrm{hom}(L, \mathcal{D}b_Y)_\xi\ ,
\end{aligned} \tag{31}
$$

which proves (12). *This completes the proof of Proposition 3.1.* ∎

4 – Applications to the Cauchy problem

4.1. A result for $\mathcal{T}hom(\cdot, \mathcal{O})$

We present here, as an application of Proposition 3.1, an inverse image theorem for $\mathcal{T}hom(\cdot, \mathcal{O})$ which will be useful in §4.2, §4.3, §4.4. It can also be regarded as a tempered version of [D'A-S 1, Th. 2.1.1].

Theorem 4.1. *Let X be a complex manifold, Y a closed submanifold of X, $f\colon Y \hookrightarrow X$ the embedding. Let Z be a subset of Y. Let V be a smooth, conic, involutive, regular submanifold of $\dot{T}^*X$. Let K be an object of $\mathrm{D}^b{}_{\mathbb{R}-c}(\mathbb{C}_X)$, L an object of $\mathrm{D}^b{}_{\mathbb{R}-c}(\mathbb{C}_Y)$, $\mathcal{M}$ an object of $\mathrm{D}^b{}_{coh}(\mathcal{D}_X)$. Assume to be given a morphism $\psi\colon L \to f^{-1}K$. Assume that:*

i) *$\mathcal{M}$ has regular singularities along V;*

ii) *V is non glancing with respect to Y in a neighborhood of $\dot{\pi}_X^{-1}(Z)$;*

iii) *$\mathrm{SS}(K) \subset V$ out of ${}^t f'^{-1} T^*_Z Y$;*

iv) *$\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X)$ (respectively $\mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y)$) is locally concentrated in one degree outside of the zero section $T^*_X X$ (resp. $T^*_Y Y$);*

v) *the morphism induced by ψ, $L \to f^{-1}_{\mu,\xi}K$ is an isomorphism in $\mathrm{D}^b(\mathbb{C}_Y; \xi_Y)$ $\forall\, \xi \in \dot\pi_X^{-1}(Z) \cap f_\pi^{-1}(\mathrm{char}(\mathcal{M}))$;*

vi) *the morphism induced by ψ, $\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \to \mathrm{R}\Gamma_{\{x\}}(K \otimes \omega_X)$, is an isomorphism for every $y \in Z$, $x = f(y)$.*

Then the natural morphism induced by ψ:

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mathcal{T}hom(K, \mathcal{O}_X)\Big)\Big|_Z \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathcal{T}hom(L, \mathcal{O}_Y)\Big)\Big|_Z , \tag{32}$$

is an isomorphism.

Remark 4.2. Let us explain how the morphism (32) is obtained. We have a morphism induced by ψ, $\mathcal{T}hom(f^{-1}K, \mathcal{O}_Y) \to \mathcal{T}hom(L, \mathcal{O}_Y)$. We have the chain of morphisms:

$$\begin{aligned} f^{-1}\, & R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mathcal{T}hom(K, \mathcal{O}_X)\Big) \to \\ & \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(f^{-1}\mathcal{M},\, f^{-1}\mathcal{T}hom(K, \mathcal{O}_X)\Big) \\ & \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},\, \mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}(\mathcal{T}hom(K, \mathcal{O}_X))\Big) \\ & \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathcal{T}hom(f^{-1}K, \mathcal{O}_Y)\Big) \\ & \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathcal{T}hom(L, \mathcal{O}_Y)\Big) . \end{aligned}$$

The third morphism here is induced by the canonical morphism (see [A, Prop. 1.2.3(i)])

$$\mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{T}hom(K, \mathcal{O}_X) \to \mathcal{T}hom(f^{-1}K, \mathcal{O}_Y) . \tag{33}$$

Remark 4.3. Before going into the proof, let us explain how the morphism in (vi) is constructed. From ψ and the natural morphism $f^{-1}K \otimes \omega_{Y/X} \to f^!K$, we obtain the arrow $L \otimes \omega_{Y/X} \to f^!K$ and hence the arrow $L \otimes \omega_Y \to f^!K \otimes f^{-1}\omega_X \simeq f^!(K \otimes \omega_X)$. Applying the functor $\mathrm{R}\Gamma_{\{y\}}(\cdot)$, we obtain

$$\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \to \mathrm{R}\Gamma_{\{y\}} f^!(K \otimes \omega_X) \simeq \mathrm{R}\Gamma_{\{y\}}(K \otimes \omega_X) . \tag{34}$$

Remark 4.4. We are dealing with the issue that $\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X)$ and other sheaves obtained by applying the functor $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O})$ be concentrated in one degree. This is a sufficient condition in order to make $\mathcal{T}\mu\text{hom}(K,\mathcal{O}_X)$ into an object of the category $\mathrm{D}^b(\mathcal{E}_X)$ (see also [A, 5.6.1]). Alternatively, condition (iv) in Theorem 4.1 may be substituted by the requirement that locally out of $T^*_X X$ (respectively $T^*_Y Y$) $\mathcal{T}\mu\text{hom}(K,\mathcal{O}_X)$ (resp. $\mathcal{T}\mu\text{hom}(L,\mathcal{O}_Y)$) be a well defined object in $\mathrm{D}^b(\mathcal{E}_X)$ (resp. $\mathrm{D}^b(\mathcal{E}_Y)$); for this other condition refer to §4.4 and the Appendix. We remark that it is conjectured that $\mathcal{T}\mu\text{hom}(K,\mathcal{O}_X)$ be naturally an object of $\mathrm{D}^b(\mathcal{E}_X)$, but we are not taking it for granted in the present paper.

Remark 4.5. Identify $Y\times_X T^*X$ to a subset of T^*X. Indeed, (ii) reads:

$$\forall\,\xi\in(Y\times_X T^*X)\cap V\,,\qquad T_\xi(Y\times_X T^*X)^\perp\cap T_\xi V=\{0\}\ . \tag{35}$$

Since $T_\xi(Y\times_X T^*X)^\perp = T_\xi({}^t f'^{-1}\,{}^t f'(\xi))$, this implies:

$$T_\xi({}^t f'^{-1}\,{}^t f'(\xi))\cap T_\xi(\mathrm{char}(\mathcal{M}))=\{0\}\ . \tag{36}$$

This implies that ${}^t f'^{-1}\,{}^t f'(\xi)\cap\mathrm{char}\,\mathcal{M}$ is a finite set. We proved that, thanks to assumption (ii), $\forall\,\eta\in\dot\pi_Y^{-1}(Z)$ there exist $\xi^1,\dots,\xi^r\in{}^t f'^{-1}(\eta)$ with: ${}^t f'^{-1}(\eta)\cap f_\pi^{-1}(\mathrm{char}(\mathcal{M}))\subset\{\xi^1,\dots,\xi^r\}$. Then f is non-characteristic for $\mathcal{M}$, that is, ${}^t f'$ is finite on $f_\pi^{-1}(\mathrm{char}(\mathcal{M}))$.

Lemma 4.6.

(**a**) *There is a canonical isomorphism:*

$$\mathrm{R}\pi_{X*}\,R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}\mu\text{hom}(K,\mathcal{O}_X)\Big)\;\simeq\;R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}\text{hom}(K,\mathcal{O}_X)\Big)\ . \tag{37}$$

(**b**) *(See [D'A-S 1, Proof of Th. 2.1.1]) Suppose hypothesis (i) of Theorem 4.1 holds, then:*

$$\begin{aligned}\mathrm{R}\pi_{Y*}\,\mathrm{R}^t f'_!\,f_\pi^{-1}\,R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}\mu\text{hom}(K,\mathcal{O}_X)\Big)\;\simeq\\ \simeq\;f^{-1}R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}hom(K,\mathcal{O}_X)\Big)\ .\end{aligned} \tag{38}$$

Proof: (**a**) The formula follows from [A, Prop. 3.1.4].

(**b**) By our hypothesis f is non-characteristic for $\mathcal{M}$. We consider the following commutative diagram:

$$\begin{array}{ccccc} T^*Y & \xleftarrow{\;{}^t f'\;} & Y\times_X T^*X & \xrightarrow{\;f_\pi\;} & T^*X\\ {\scriptstyle\pi_Y}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_X}\downarrow\\ Y & \xleftarrow{\;\mathrm{id}_Y\;} & Y & \xrightarrow{\;f\;} & X \end{array} \tag{39}$$

We have the chain of isomorphisms:

$$\begin{aligned}(40)\qquad & \mathrm{R}\pi_{Y*}\,\mathrm{R}^t f'_!\, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \simeq \\ & \simeq \mathrm{R}\pi_{Y*}\,\mathrm{R}^t f'_*\, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \\ & \simeq \mathrm{R}\pi_*\, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \\ & \simeq f^{-1}\,\mathrm{R}\pi_{X*}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \\ & \simeq f^{-1} R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mathcal{H}om(K,\mathcal{O}_X)\Big)\ ,\end{aligned}$$

where the first isomorphism is due to the fact (being f non-characteristic for $\mathcal{M}$) that ${}^t f'$ is proper on $f_\pi^{-1}\,\mathrm{char}(\mathcal{M})$ and hence on

$$(41)\qquad f_\pi^{-1}\,\mathrm{supp}\Big(R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big)\Big)$$

and the third to the fact that, being $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X))$ conic,

$$\begin{aligned}(42)\qquad & \mathrm{R}\pi_{X*}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \simeq \\ & \simeq i^{-1} R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big)\ ,\end{aligned}$$

i denoting the immersion of the zero-section X in T^*X. The last isomorphism follows from (a). ∎

Set

$$\begin{aligned}(43)\qquad A &= \mathrm{R}^t f'_!\, f_\pi^{-1} R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big)\ , \\ B &= R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, T\mu\mathrm{hom}(L,\mathcal{O}_Y)\Big)\ .\end{aligned}$$

These two sheaves are objects in the full subcategory of $\mathrm{D}^b(\mathbb{C}_X)$ whose objects have locally constant cohomology along the orbits of the action of $\mathbb{R}^+$ on T^*X.

We remark that there exists a natural morphism $A \to B$. In fact, we have the following morphisms:

$$\begin{aligned} & \mathrm{R}^t f'_!\, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \to \\ \to\ & \mathrm{R}^t f'_!\, R\mathcal{H}om_{f^{-1}\mathcal{D}_X}\Big(f^{-1}\mathcal{M}, f_\pi^{-1} T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \\ \to\ & \mathrm{R}^t f'_!\, R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},\ \mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f_\pi^{-1}\, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big) \\ \to\ & R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},\ \mathrm{R}^t f'_!\Big(\mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f_\pi^{-1}\, T\mu\mathrm{hom}(K,\mathcal{O}_X)\Big)\Big) \\ \to\ & R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{D}_{Y\to X}\otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},\ T\mu\mathrm{hom}(f^{-1}K,\mathcal{O}_X)\Big) \\ \to\ & R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}_Y, T\mu\mathrm{hom}(L,\mathcal{O}_X)\Big)\ ,\end{aligned}$$

where the fourth morphism is due to [A, (3.3.4)], and the fifth is induced by ψ. We apply the functor $\mathrm{R}\pi_*(\cdot)$ to the distinguished triangle

$$\mathrm{R}\Gamma_{T^*_X X} \to \mathrm{R}\Gamma_{T^*X} \to \mathrm{R}\Gamma_{\dot{T}^*X} \overset{+1}{\to} \ . \tag{44}$$

Keeping in mind that for conic sheaves $\mathrm{R}\pi_* \, \mathrm{R}\Gamma_{T^*_X X}(\cdot) \simeq \mathrm{R}\pi_!(\cdot)$, we obtain the following distinguished triangle

$$\mathrm{R}\pi_{Y!} \to \pi_{Y*} \to \dot{\pi}_{Y*} \overset{+1}{\to} \ . \tag{45}$$

We apply this triangle to the natural morphism $A \to B$ of the preceding lemma:

$$\begin{array}{ccccccc} \mathrm{R}\pi_{Y!}\, A & \longrightarrow & \mathrm{R}\pi_{Y*}\, A & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}\, A & \overset{+1}{\longrightarrow} \\ {}^{1}\!\downarrow & & {}^{2}\!\downarrow & & {}^{3}\!\downarrow & \\ \mathrm{R}\pi_{Y!}\, B & \longrightarrow & \mathrm{R}\pi_{Y*}\, B & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}\, B & \overset{+1}{\longrightarrow} \end{array} \tag{46}$$

Taking Lemma 4.6 into account, our theorem will be proven if we get to prove that 2 is an isomorphism. In order to achieve that, we will prove that 1 and 3 are isomorphisms.

First vertical arrow. We have to prove that:

$$\mathrm{R}\pi_{Y!}\, A \simeq \mathrm{R}\pi_{Y!}\, B \ . \tag{47}$$

We have $\mathrm{R}\pi_{Y!}\, \mathrm{R}^t f'_! \, f_\pi^{-1} \simeq \mathrm{R}\pi_! \, f_\pi^{-1} \simeq f^{-1}\, \pi_{X!}$. We have the following formula (see [A, Prop. 3.1.4(ii)]):

$$\mathrm{R}\pi_! \, T\mu\mathrm{hom}(F, \mathcal{O}_X) \simeq \mathrm{R}\pi_! \, \mu\mathrm{hom}(F, \mathcal{O}_X) \ . \tag{48}$$

By this and by $\mathcal{M}$ being coherent we have

$$\begin{aligned} \mathrm{R}\pi_{Y!}\, A &\simeq f^{-1}\pi_{X!}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mu\mathrm{hom}(K, \mathcal{O}_X)\Big) \\ &\simeq f^{-1}\Big(R\mathcal{H}om(K, \mathbb{C}_X) \otimes R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\Big) \ . \end{aligned}$$

On the other hand, thanks again to [A, Prop. 3.1.4(ii)], we have:

$$\begin{aligned} \mathrm{R}\pi_{Y!}\, B &\simeq \mathrm{R}\pi_{Y!}\, R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, T\mu\mathrm{hom}(L, \mathcal{O}_Y)\Big) \\ &\simeq \mathrm{R}\pi_{Y!}\, R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mu\mathrm{hom}(L, \mathcal{O}_Y)\Big) \\ &\simeq R\mathcal{H}om(L, \mathbb{C}_Y) \otimes R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \ . \end{aligned} \tag{49}$$

But $\mathcal{M}$ is a coherent module non-characteristic with respect to f, so also $\mathcal{M}_Y$ is a coherent $\mathcal{D}_Y$-module, so by the Cauchy–Kowalevski theorem, we have

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{O}_Y) \ . \tag{50}$$

We are thus reduced to proving:

$$f^{-1}\, R\mathcal{H}om(K,\mathbb{C}_X) \simeq R\mathcal{H}om(L,\mathbb{C}_Y) \ . \tag{51}$$

Then it is possible to make use of hypothesis (vi) in order to argue exactly as in [D'A-S 1, §2A] and conclude that the first vertical arrow is an isomorphism.

So we are left with the third vertical arrow.

Third vertical arrow. Let $H \subset X$ be a submanifold such that $Y \subset H$. Remark that all our hypotheses will hold also for H in place of Y with respect to $\mathcal{M}$. Thus, the morphism (32) is an isomorphism if and only if for any such H we have:

$$R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}hom(K,\mathcal{O}_X)\Big)\Big|_Z \to R\mathcal{H}om_{\mathcal{D}_H}\Big(\mathcal{M}_H,\mathcal{T}hom(L,\mathcal{O}_H)\Big)\Big|_Z \ . \tag{52}$$

By considering a chain of submanifolds:

$$Y = H_0 \subset H_1 \subset \cdots \subset H_{\mathrm{cod}_X Y} = H \ ,$$

we easily reduce to the case $\mathrm{cod}_X Y = 1$.

Our thesis is now that the natural morphism

$$\mathrm{R}\dot{\pi}_{Y*}\, A \to \mathrm{R}\dot{\pi}_{Y*}\, B \tag{53}$$

is an isomorphism. In other words that, for every $\eta \in \dot{\pi}_Y^{-1}(Z)$,

$$\begin{aligned} &\mathrm{R}^t f'_!\, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M},\mathcal{T}\mu\mathrm{hom}(K,\mathcal{O}_X)\Big)_\eta \to \\ &\qquad \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y,\mathcal{T}\mu\mathrm{hom}(L,\mathcal{O}_Y)\Big)_\eta \end{aligned} \tag{54}$$

is an isomorphism. As we noticed in Remark 4.4, there exist $\xi^1,...,\xi^r \in {}^t f'^{-1}(\eta)$ such that:

$${}^t f'^{-1}(\eta) \cap f_\pi^{-1}(V) \subset \{\xi^1,...,\xi^r\} \tag{55}$$

and thanks to hypotheses (ii), (iii) we then have that f is non-characteristic for K and $\xi^1,...,\xi^r$ are isolated in ${}^t f'^{-1}(\eta) \cap f_\pi^{-1}(\mathrm{SS}(K))$. Then (see [D'A-S 1, Lemma 2.1.3]) we can find sheaves $K_1,...,K_r$ on X such that:

(**1**) there exists a distinguished triangle $\bigoplus_{i=1}^{r} K_i \to K \to K_0 \overset{+1}{\to}$;

(**2**) $f_\pi^{-1}\,\mathrm{SS}(K_i) \cap {}^t f'^{-1}(\eta) \subset \{\xi^i\}$ and $\xi^i \notin \mathrm{SS}(K_0)$ for $i = 1, ..., r$;

(**3**) $f_{\mu,\xi^i}^{-1} K = f^{-1}K_i$ at η for $i = 1, ..., r$.

Also (see [P-S, Lemma 4.2], [S-K-K, Chapter II, Th. 2.2.2]) we can find $\mathcal{E}_X$-modules $\mathcal{M}_1, ..., \mathcal{M}_r$ such that:

(**4**) $\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{M} \simeq \bigoplus_{i=1}^{r} \mathcal{M}_i$ in $\mathrm{D}^b(\mathcal{E}_X|_{f_\pi {}^t f'^{-1}(\eta)})$;

(**5**) $f_\pi^{-1}\,\mathrm{SS}(\mathcal{M}_i) \cap {}^t f'^{-1}(\eta) \subset \{\xi^i\}$;

(**6**) $f^{-1}(\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{M}) \simeq \bigoplus_{i=1}^{r} f^{-1}\mathcal{M}_i$ in $\mathrm{D}^b(\mathcal{E}_Y|_\eta)$.

Remark that the K_i's are $\mathbb{R}$-constructible. In fact in [D'A-S 1, Lemma 2.1.3] the K_i's are obtained by a "microlocal cut-off". See [A, §A.1] for the "microlocal cut-off lemma" [K-S 1, Prop. 6.1.4] in the framework of $\mathbb{R}$-constructible sheaves.

Due to $\xi^i \notin \mathrm{SS}(K_0)$ and the distinguished triangle $\bigoplus_{i=1}^{r} K_i \to K \to K_0 \overset{+1}{\to}$, the $T\mu\mathrm{hom}(K_i, \mathcal{O}_X)$'s are concentrated in one degree. We then get the chain of isomorphisms:

$$\begin{aligned}
&\mathrm{R}^t f'_! \, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, T\mu\mathrm{hom}(K, \mathcal{O}_X)\Big)_\eta \simeq \\
&\simeq \mathrm{R}^t f'_! \, f_\pi^{-1}\, R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_X \otimes^L_{\mathcal{D}_X} \mathcal{M},\, T\mu\mathrm{hom}(K, \mathcal{O}_X)\Big)_\eta \\
&\simeq \bigoplus_{i=1}^{r} R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_X \otimes^L_{\mathcal{D}_X} \mathcal{M},\, T\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i} \\
&\simeq \bigoplus_{i=1}^{r} R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_X \otimes^L_{\mathcal{D}_X} \mathcal{M},\, T\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i} \\
&\simeq \bigoplus_{i=1}^{r} R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{M}_i, T\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i},
\end{aligned} \tag{56}$$

where the second and fourth isomorphisms are due to conditions 2 and 5 above. Then we are reduced to prove the following isomorphism:

$$R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{M}_i, T\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i} \simeq R\mathcal{H}om_{\mathcal{E}_Y}\Big(\mathcal{M}_{Y,\xi^i}, T\mu\mathrm{hom}(L, \mathcal{O}_Y)\Big)_\eta, \tag{57}$$

where $\mathcal{M}_{Y,\xi^i} = (\mathcal{M}_i)_{Y,\eta}$ denotes the $\mathcal{E}_X$-module inverse image of $\mathcal{M}$ at p. Remark that we have $\bigoplus_{i=1}^{r} \mathcal{M}_{Y,\xi^i} = \mathcal{M}_{Y,\eta}$.

Thanks to the assumption (ii) that V be non glancing with respect to Y, we can apply [S 1, Cor. A.4.5] with $W = Y \times_X T^*X$, that is:

Lemma 4.7 ([S 1, Cor. A.4.5], [S 1, Cor. I.6.2.3]). *Let V and W be two conic involutive manifolds in a neighborhood of $p \in \dot{T}^*X$, V being regular. Assume that V is non glancing with respect to W. Then there exists a system of local homogeneous symplectic coordinates $(x;\xi)$ such that:*

$$\begin{cases} p = (0; \mathrm{d}\, x_n) \\ V = \Big\{(x;\xi);\ \xi_1 = ... = \xi_r = 0\Big\}, \quad r < n\ , \\ W = \Big\{(x;\xi);\ x_1 = ... = x_d = 0\Big\}, \quad d \le r\ . \end{cases} \tag{58}$$

We may quantize this contact transformation using the theory of contact transforms for the functor $\mathcal{T}\mu\mathrm{hom}(\cdot, \mathcal{O}_X)$ as developed in [A, Chapter 5].

Then we use an argument similar to [K 2, page 32]. By Theorem 2.3, $\mathcal{M}$ is a quotient of a multiple de Rham system. Hence, we have an exact sequence

$$0 \to \mathcal{Q} \to (\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''})^N \xrightarrow{\alpha} \mathcal{M} \to 0, \tag{59}$$

where $\mathcal{Q} = \ker \alpha$ has regular singularities along V.

Let us suppose that we have proved the claim for $\mathcal{M} = (\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''})^N$. Considering the long exact cohomology sequence obtained by applying the functor

$$R\mathcal{H}om_{\mathcal{E}_X}\Big(\cdot, \mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i} \tag{60}$$

to (59), we would have isomorphisms:

$$\mathcal{E}xt^{j+1}_{\mathcal{E}_X}\Big(\mathcal{M}, \mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i} \simeq \mathcal{E}xt^{j}_{\mathcal{E}_X}\Big(\mathcal{Q}, \mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X)\Big)_{\xi^i}\ . \tag{61}$$

Starting from $j = -1$, we would thus get the claim by induction on j. We are then reduced to prove our claim in case

$$\mathcal{M}_i = \left(\frac{\mathcal{E}_X}{\mathcal{E}_X D_1 + \cdots + \mathcal{E}_X D_r}\right)^N \tag{62}$$

where $r := \mathrm{codim}(V)$ and N is an integer; of course we may now assume $N = 1$. We just treat the case $\mathrm{codim}\, V = 1$, the other cases being analogous. So suppose $\mathcal{M}_i = \frac{\mathcal{E}_X}{\mathcal{E}_X D_1}$. We remark that in the present situation $\frac{\mathcal{E}_X}{\mathcal{E}_X D_1} = \mathcal{E}_{X \to Y}$ and $\mathcal{M}_{Y,\xi^i} \simeq \mathcal{E}_Y$, where $\mathcal{M}_{Y,\xi^i}$ denotes the $\mathcal{E}_X$-module inverse image of $\mathcal{M}$ at ξ^i. Remark that locally $X = Y \times Z$, with Y and Z being local charts, with a projection $p\colon X \to Y$ and $p \circ f = \mathrm{id}_Y$. Also, due to assumption (iii), $\mathrm{SS}(K_i) \subset V$. Thanks to [K-S 1, Prop. 6.6.2] we have $K_i = p^{-1}L'\ \exists L' \in \mathrm{D}^b(\mathbb{C}_X)$. But $p \circ f = 1_Y$, so that $K_i = p^{-1}L$.

By these reductions and taking into account (57), we are left to prove the following isomorphism:

$$\mathrm{R}^t f'_! \, f_\pi^{-1} \, R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_{X\to Y}, T\mu\mathrm{hom}(p^{-1}L, \mathcal{O}_X)\Big)_{\xi^i} \simeq T\mu\mathrm{hom}(L, \mathcal{O}_Y)_{\xi^i} \, . \tag{63}$$

If we apply the functor $\mathrm{R}^t f'_! \, f_\pi^{-1}(\cdot)$ to the isomorphism given by Corollary 3.2 we get

$$\begin{aligned} &\mathrm{R}^t f'_! \, f_\pi^{-1} \, \mathrm{R}^t p'_* \, p_\pi^{-1}\Big(T\mu\mathrm{hom}(L, \mathcal{O}_Y)\Big)_{\xi^i} \simeq \\ &\qquad \simeq \mathrm{R}^t f'_! \, f_\pi^{-1} \, R\mathcal{H}om_{\mathcal{E}_X}\Big(\mathcal{E}_{X\to Y}, T\mu\mathrm{hom}(p^{-1}L, \mathcal{O}_X)\Big)_{\xi^i} \, . \end{aligned} \tag{64}$$

Remark that $\mathrm{R}^t p'_* = \mathrm{R}^t p'_!$. Since $p \circ f = \mathrm{id}_Y$, we have $\mathrm{R}^t f'_! \, f_\pi^{-1} \, \mathrm{R}^t p'_! \, p_\pi^{-1} = 1_{T^*Y}$, by which we obtain (63).

This completes the proof of Theorem 4.1. ∎

4.2. Logarithmic ramifications

Theorem 4.1 allows us to recover in particular a result of Y. Laurent (see [La, Th. 3.2.6]). Our proof doesn't make use of second microlocalization.

Let $z \in \mathbb{C}$ be a coordinate and set $D = \frac{\partial}{\partial z}$. Define $\mathcal{N} := \frac{\mathcal{D}_\mathbb{C}}{\mathcal{D}_\mathbb{C} DzD}$; $\mathcal{N}$ is a coherent $\mathcal{D}_\mathbb{C}$-module which represents the complex of holomorphic tempered functions on $\mathbb{C}$ with logarithmic ramification at 0. Set $L^1_{\{0\}|\mathbb{C}} := R\mathcal{H}om_{\mathcal{D}_\mathbb{C}}(\mathcal{N}, \mathcal{O}_\mathbb{C})$. Remark that, by [K 1], $T\mathrm{hom}(L^1_{\{0\}|\mathbb{C}}, \mathcal{O}_\mathbb{C}) = \mathcal{N}$. Let X be a complex analytic manifold and Y a smooth complex hypersurface in X, and Z a smooth complex hypersurface in Y. Let $f\colon Y \to X$ be the embedding.

Let us see how this geometrical setting can be applied to the Cauchy problem. Take $\mathcal{M}$ a left coherent $\mathcal{D}_X$-module. Let $V := \mathrm{char}(\mathcal{M})$. Let $\mathcal{M}$ have regular singularities along V, and V be smooth and non glancing with respect to $Y \times_X T^*X$. We remark that in the present situation $Y \times_X T^*X$ is transversal to $\mathrm{char}(\mathcal{M})$.

Lemma 4.8 ([S 1, Prop. III.2.2.2], [D'A-S 1, Prop. 3.1.3])**.** *Under the previous hypotheses there exist smooth hypersurfaces $Z_1, ..., Z_r$ of X pairwise transversal, transversal to Y, and such that $Z_i \cap Y = Z$ for every i. Moreover, for a neighborhood W of $\dot{T}^*_Z Y$,*

$$\mathrm{char}(\mathcal{M}) \cap {}^t f'^{-1}(T^*_Z Y) \subset \bigcup_{i=1}^r T^*_{Z_i} X \cup T^*_X X \, . \tag{65}$$

By possibly shrinking X, we have that there exist complex analytic functions $g\colon Y \to \mathbb{C}$ with $\mathrm{d}g \neq 0$ and $g^{-1}(0) = Z$, $g_1, ..., g_r\colon X \to \mathbb{C}$ with $\mathrm{d}g_i \neq 0$ and $g_i^{-1}(0) = Z_i$, such that $g_i \circ f = g$.

Moreover our hypotheses also imply that f is non-characteristic for $\mathcal{M}$. Set $L\colon= g^{-1}L^1_{\{0\}|\mathbb{C}}$ and $K_i\colon= g_i^{-1}L^1_{\{0\}|\mathbb{C}}$.

We have the technical lemma:

Lemma 4.9 ([D'A-S 1, Lemma 3.2.1])**.** *There exists a natural map* $\tau\colon L \to \mathbb{C}_X$*. Moreover, applying the functor* $\mathrm{R}\Gamma_{\{0\}}(\cdot)$ *to this map we get an isomorphism:*

$$\mathrm{R}\Gamma_{\{0\}}\, L \xrightarrow{\sim} \mathrm{R}\Gamma_{\{0\}}\, \mathbb{C}_X \ . \tag{66}$$

Proof: We have the morphisms

$$\mathcal{O}_{\mathbb{C}} \hookrightarrow \mathcal{D}_{\mathbb{C}} \to \mathcal{N} \simeq \frac{\mathcal{D}_{\mathbb{C}}}{\mathcal{D}_{\mathbb{C}}\, Dz\, D} \ . \tag{67}$$

This gives us a morphism

$$L^1_{\{0\}|\mathbb{C}} \simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{N}, \mathcal{O}_{\mathbb{C}}) \to \mathbb{C}_{\mathbb{C}} \simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{O}_{\mathbb{C}}, \mathcal{O}_{\mathbb{C}}) \ , \tag{68}$$

which induces the morphism

$$L \simeq g^{-1}R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{N}, \mathcal{O}_{\mathbb{C}}) \xrightarrow{\tau} g^{-1}\mathbb{C}_{\mathbb{C}} \simeq \mathbb{C}_X \ . \tag{69}$$

We omit the proof of the isomorphism. ∎

Define K to be the first term of a distinguished triangle

$$K \to \bigoplus_{i=1}^{r} K_i \xrightarrow{h} \bigoplus_{i=1}^{r-1} \mathbb{C}_X \xrightarrow{+1} \tag{70}$$

where h is the composite of the natural morphism $\bigoplus_{i=1}^r \tau_i\colon \bigoplus_{i=1}^r K_i \to \bigoplus_{i=1}^r \mathbb{C}_X$ and the map $\bigoplus_{i=1}^r \mathbb{C}_X \to \bigoplus_{i=1}^{r-1} \mathbb{C}_X$ given by $(a_1, ..., a_r) \mapsto (a_2 - a_1, ..., a_r - a_{r-1})$. Due to f being non characteristic for K_i the isomorphism

$$\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \xrightarrow{\sim} \mathrm{R}\Gamma_{\{y\}}(K \otimes \omega_X) \tag{71}$$

reads as

$$\mathrm{R}\Gamma_{\{y\}}(L) \xrightarrow{\sim} \mathrm{R}\Gamma_{\{y\}}(f^{-1}K) \ . \tag{72}$$

By Lemma 4.9 hypothesis (vi) of Theorem 32 is satisfied.

We write $\mathcal{O}^1_{[Z|Y]} := \mathcal{T}hom(L, \mathcal{O}_Y)$ the complex of tempered holomorphic functions on Y with ramification along Z of logarithmic type. We write $\sum_i \mathcal{O}^1_{[Z_i|X]} := \mathcal{T}hom(K, \mathcal{O}_X)$ the complex of holomorphic tempered functions on X with ramification of logarithmic type along the Z_i's.

Thanks to [K-S 1, Chapter 10], and g being finite, L is a perverse sheaf and thanks to [A, Cor. 5.6.1] $\mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y)$ is concentrated in one degree. The same argument holds for $\mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X)$.

We have that, out of the zero section $T^*_X X$:

$$\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X) \overset{\sim}{\to} \bigoplus_{i=1}^{r} \mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X) \tag{73}$$

so that, out of the zero section, $\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X)$ is concentrated in one degree. Applying Theorem 4.1 we get the following theorem.

Theorem 4.10. *Keeping the same notation as above, the natural morphism:*

$$f^{-1}\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathcal{O}^1_{[Z_i|X]}\Big)\Big|_Z \simeq R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathcal{O}^1_{[Z|Y]}\Big)\Big|_Z \tag{74}$$

is an isomorphism.

This theorem states existence and uniqueness of the solution for the Cauchy problem with tempered holomorphic data logarithmically ramified along the Z_i's and tempered holomorphic traces logarithmically ramified along Z.

4.3. Swallow's tail

This case of the swallow's tail appeared first in [Le 2]. Here we adapt the method of [D'A-S 2] to the tempered case. Let X be an open subset of $\mathbb{C}^{n+1}$ containing 0 and endowed with coordinates $x = (x_0, x')$. Set $Y = \{x \in X;\ x_0 = 0\}$. Let

$$T = \Big\{ x' \in Y;\ \text{ the polynomial in } z,\ \ A(z, x') = z^{n+1} - x_n z^{n-1} - \ldots \\ \ldots - x_2 z - x_1 \ \text{ has at least one double root} \Big\}.$$

Define $\tilde{Y} \subset \mathbb{C}_z \times Y$ the variety given by the equation $A(z, x') = 0$. If we consider the projection $\eta\colon \tilde{Y} \to Y$, we find that the "swallow's tail" T is nothing but the image by η of the point in $\tilde{Y}$ where η is not smooth. T is thus a singular variety, but nevertheless its conormals $\Lambda = \overline{T^*_{T_{reg}} Y}$ give a Lagrangian manifold in $\dot{T}^* X$, featuring just one codirection above 0. Let $L = \eta_!\, \mathbb{C}_{\tilde{Y}}$; there is a canonically induced morphism $\tau\colon L \to \mathbb{C}_Y$. As in the previous section, we get that $\mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y)$ is concentrated in one degree. Define $\mathcal{O}^{ram}_{[T|Y]} := \mathcal{T}hom(L, \mathcal{O}_Y)$.

Moreover, suppose there exist $T_1, ..., T_r$ "swallow's tails" in X such that the T_i are mutually transversal and transversal to Y (i.e. $\Lambda_i \cap \Lambda_j \subset T^*_X X$ for $i \neq j$ and $\Lambda_i \cap T^*_Y X \subset T^*_X X$ for every i) with $T_i \cap Y = T$. Define $K_i{:} = \eta_{i!}\mathbb{C}_{\tilde{X}_i}$, $i = 1, ..., r$ where $\eta_i{:}\, \tilde{X}_i \to X$ are defined analogously to above. There are canonical morphisms $\tau_i{:}\, K_i \to \mathbb{C}_X$. As in the previous section, we get that $\mathcal{T}\mu\mathrm{hom}(K_i, \mathcal{O}_X)$ is concentrated in one degree. Let as above K be the complex defined by (70) with this choice of the K_i's. As in the previous section, out of $T^*_X X$, $\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X)$ is concentrated in one degree. Define $\sum_i \mathcal{O}^{ram}_{[T_i|X]}{:} = \mathcal{T}hom(K, \mathcal{O}_X)$.

Lemma 4.11. *The canonically induced morphism:* $\mathrm{R}\Gamma_{\{0\}}\eta_!\mathbb{C}_{\tilde{Y}} \to \mathrm{R}\Gamma_{\{0\}}\mathbb{C}_Y$ *is an isomorphism.*

This follows from the fact that $\eta^{-1}(0) = \{0\}$, and that $\tilde{Y}$ is a complex manifold of the same dimension as Y.

Due to f being non characteristic for K_i the isomorphism

$$\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \overset{\sim}{\to} \mathrm{R}\Gamma_{\{y\}}(K \otimes \omega_X) \tag{75}$$

reads as

$$\mathrm{R}\Gamma_{\{y\}}(L) \overset{\sim}{\to} \mathrm{R}\Gamma_{\{y\}}(f^{-1}K)\ . \tag{76}$$

By Lemma 4.11 hypothesis (vi) of Theorem 32 is satisfied.

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module. Let $V{:} = \mathrm{char}(\mathcal{M})$. Let $\mathcal{M}$ have regular singularities, and let V be smooth and non glancing with respect to $Y \times_X T^*X$. We denote by f the immersion of Y into X. Suppose also that:

$$\mathrm{char}(\mathcal{M}) \cap {}^t f'^{-1}(T^*_T Y) \subset \bigcup_{i=1}^{r} T^*_{T_i} X \cup T^*_X X\ . \tag{77}$$

Applying Theorem 4.1 we get the following theorem.

Theorem 4.12. *Keeping the same notation as above, the natural morphism*

$$f^{-1} R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathcal{O}^{ram}_{[T_i|X]}\Big)\Big|_T \simeq R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathcal{O}^{ram}_{[T|Y]}\Big)\Big|_T \tag{78}$$

is an isomorphism.

We skip the details (see §4.2) and just remark that this theorem proves existence and uniqueness for the solution of the Cauchy problem, when holomorphic tempered traces and data ramified along T and the T_i's are considered; the solution itself being a holomorphic tempered function ramified along the T_i's.

4.4. Decomposition at the boundary

Let Y be a smooth complex analytic hypersurface of a complex analytic manifold X and let $f\colon Y \to X$ be the embedding. Let ω be an open subset of Y with smooth boundary Q. Consider a linear partial differential operator with holomorphic coefficients $P = P(z, D)$. Let $\mathcal{M} = \frac{\mathcal{D}_X}{\mathcal{D}_X P}$. Suppose $\mathcal{M}$ has regular singularities. Assume that $V := \mathrm{char}(\mathcal{M})$ be smooth and non glancing with respect to $Y \times_X T^*X$. Then, by [D'A-S 1, Prop. 3.3.7] there exist Ω_i $(i = 1, ..., r)$ open subsets of X with smooth boundaries Q_i such that $\Omega_i \cap Y = \omega$ and the Q_i's are pairwise transversal and transversal to Y. Remark that $T^*_\omega Y = \mathrm{SS}(\mathbb{C}_\omega) \subset T^*Y$. Moreover condition (65) holds with Z, Z_i replaced by Q, Q_i.

Let $K_i = \mathbb{C}_{\Omega_i}$ $(i = 1, ..., r)$; they are objects of $\mathrm{D}^b{}_{\mathbb{R}-c}(\mathbb{C}_X)$. Let $L = \mathbb{C}_\omega$; it is an object of $\mathrm{D}^b{}_{\mathbb{R}-c}(\mathbb{C}_Y)$. Of course, $f^{-1}K_i = L$ and one has canonical morphisms $\tau_i\colon K_i \to \mathbb{C}_X$. Let K be the first term of a distinguished triangle

$$K \to \bigoplus_{i=1}^{r} K_i \xrightarrow{h} \bigoplus_{i=1}^{r-1} \mathbb{C}_X \xrightarrow{+1} \tag{79}$$

constructed like in §4.2. Of course $\mathrm{SS}(K_i) \subset T^*_{Q_i}X \cup T^*_X X$.

Set for short $\sum_i \mathrm{R}\Gamma_{[\Omega_i]}\mathcal{O}_X = \mathcal{T}hom(K, \mathcal{O}_X)$, $\mathrm{R}\Gamma_{[\omega]}\mathcal{O}_Y = \mathcal{T}hom(L, \mathcal{O}_Y)$.

Theorem 4.13. *The canonical morphism*

$$R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathrm{R}\Gamma_{[\Omega_i]}\mathcal{O}_X\Big)\Big|_Y \to R\mathcal{H}om_{\mathcal{D}_Y}\Big(\mathcal{M}_Y, \mathrm{R}\Gamma_{[\omega]}\mathcal{O}_Y\Big) \tag{80}$$

is an isomorphism.

Proof: See [D'A-S 1, Th. 3.3.6]. When considering the stalk at $y \in Y$, the only non trivial case is when $y \in \partial\omega$. We have to check that $\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X)$ and $\mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y)$ are well defined objects in $\mathrm{D}^b(\mathcal{E}_X)$ and $\mathrm{D}^b(\mathcal{E}_Y)$. As said in Remark 4.4 of §4, this is what we need in order to be able to employ Theorem 4.1. Alternatively, a sufficient condition is given by assumption (iv) in Theorem 4.1. We can make use of Lemma A.1 with $U = \Omega_i$ keeping in mind that, out of the zero section $T^*_X X$,

$$\mathcal{T}\mu\mathrm{hom}(K, \mathcal{O}_X) \simeq \bigoplus_{i=1}^{r} \mathcal{T}\mu\mathrm{hom}(\mathbb{C}_{\Omega_i}, \mathcal{O}_X)\ . \tag{81}$$

As far as $\mathcal{T}\mu\mathrm{hom}(L, \mathcal{O}_Y)$ is concerned, we can make use of Lemma A.1 again, with $U = \omega$. So all hypotheses are satisfied and we may apply Theorem 4.1. ∎

Theorem 4.13 gives a result of decomposition at the boundary for the Cauchy problem with tempered data (refer also to [Sc]). In fact, let us apply the functor $\mathrm{R}\Gamma(Y, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\cdot, \mathcal{O}_X))|_Y)$ to the distinguished triangle (79). By the Cauchy–Kowalevski theorem we find that, as Y is locally Stein, the sequence

$$0 \to \bigoplus_{i=1}^{r-1} \Gamma\Big(Y;\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mathcal{T}hom(\mathbb{C}_X, \mathcal{O}_X)|_Y\Big)\Big) \to$$
$$\to \bigoplus_{i=1}^{r} \Gamma\Big(Y;\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \mathcal{T}hom(K_i, \mathcal{O}_X)|_Y\Big)\Big) \to$$
$$\to \Gamma\Big(Y;\, R\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathrm{R}\Gamma_{[\Omega_i]}\mathcal{O}_X\Big)\Big|_Y\Big) \to 0$$

is exact. Hence by (80) we get that the local holomorphic solution of the Cauchy problem

$$\begin{cases} P(z, D)\, u(z) = 0 \ , \\ D_1^h u(z)|_Y \in \Gamma(\omega; \mathcal{T}hom(L, \mathcal{O}_Y)), \quad 0 \le h < m \ , \end{cases} \tag{82}$$

may be written as a sum $u = \sum_{i=1}^{r} u_i$, where u_i is a tempered holomorphic function on $\Omega_i \cap W$ which satisfies the equation $Pu_i = 0$, W being an open neighborhood of Y in X.

A Appendix

As usual, we suppose that X is a complex manifold. Let $X \times \overline{X}$ be a complexification of X and $\delta\colon X \to X \times \overline{X}$ the diagonal immersion. The following result was communicated to the author by P. Schapira.

Lemma A.1. *Suppose $U \subset X$ is open with real analytic boundary of codimension one and U is locally on one side of ∂U. Then $\mu\mathrm{hom}(\mathbb{C}_U, \mathcal{O}_X)$ is a well defined object in $\mathrm{D}^b(\mathcal{E}_X)$.*

Proof: By [S 2] we have that

$$H^j\Big(\mu\mathrm{hom}(\delta_* \mathbb{C}_{\overline{U}}, \mathcal{O}_{X \times \overline{X}})\Big) = 0 \quad \forall\, j \neq 2 \dim_{\mathbb{R}} X \ . \tag{83}$$

Hence by [K-S 1, Chapter 11] $\mu\mathrm{hom}(\delta_* \mathbb{C}_{\overline{U}}, \mathcal{O}_{X \times \overline{X}})$ is a well defined object of $\mathrm{D}^b(\mathcal{E}_{X \times \overline{X}})|_{T^*_\Delta X \times \overline{X}}$. Since

$$\mu\mathrm{hom}(\mathbb{C}_{\overline{U}}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}\Big(\mathcal{O}_{\overline{X}},\, \mu\mathrm{hom}(\delta_* \mathbb{C}_{\overline{U}}, \mathcal{O}_{X \times \overline{X}})\Big) \ , \tag{84}$$

we get $\mu\mathrm{hom}(\mathbb{C}_{\overline{U}}, \mathcal{O}_X) \in \mathrm{D}^b(\mathcal{E}_X)$. The same result holds with $\mu\mathrm{hom}(\mathbb{C}_{\partial U}, \mathcal{O}_X)$, so that by using the distinguished triangle $\mathbb{C}_U \to \mathbb{C}_{\overline{U}} \to \mathbb{C}_{\partial U} \xrightarrow{+1}$ it holds also for $\mu\mathrm{hom}(\mathbb{C}_U, \mathcal{O}_X)$. ∎

Remark that Lemma A.1 holds true replacing μhom by $\mathcal{T}\mu$hom. We make use of this lemma in §4.4.

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