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PERIODIC SOLUTIONS FOR A THIRD ORDER DIFFERENTIAL EQUATION UNDER CONDITIONS ON THE POTENTIAL

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Abstract: We prove an existence result to the nonlinear periodic problem

$$\begin{cases} x''' + a\,x'' + g(x') + c\,x = p(t) \ , \\ x(0) = x(2\pi)\,, \quad x'(0) = x'(2\pi)\,, \quad x''(0) = x''(2\pi) \ , \end{cases}$$

where $g\colon \mathbb{R} \mapsto \mathbb{R}$ is continuous, $p\colon [0,2\pi] \mapsto \mathbb{R}$ belongs to $\mathbb{L}^1(0,2\pi)$, $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$, under conditions on the asymptotic behaviour of the primitive of the nonlinearity g. This work uses the Leray–Schauder degree theory and improves a result contained in [EO], weakening the condition on the oscillation of g. The arguments used were suggested by [GO], [HOZ] and [SO].

1 – Introduction and statements

Consider the third order differential equation

$$x''' + a\,x'' + g(x') + c\,x = p(t) \tag{1.1}$$

for $t \in [0,2\pi]$, with periodic boundary conditions

$$x(0) = x(2\pi), \quad x'(0) = x'(2\pi), \quad x''(0) = x''(2\pi) \ , \tag{1.2}$$

where $g\colon \mathbb{R} \mapsto \mathbb{R}$ is continuous, $p\colon [0,2\pi] \mapsto \mathbb{R}$ belongs to $\mathbb{L}^1(0,2\pi)$, $a, c \in \mathbb{R}$ and $c \neq 0$. In [EO] Ezeilo and Omari studied problem (1.1)–(1.2) assuming that g satisfies the following condition

$$m^2 + h^-(|s|) \le \frac{g(s)}{s} \le (m+1)^2 - h^+(|s|) \tag{1.3}$$

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for $|s| \geq r > 0$, where $m \in \mathbb{N}$ and $h^{\pm}\colon\ [0,+\infty[\ \mapsto \mathbb{R}$ are two functions such that

$$\lim_{|s|\to+\infty} |s|\, h^{\pm}(|s|) = +\infty\ . \tag{1.4}$$

We observe that conditions (1.3) and (1.4) imply, for $|s|$ big enough, that $\dfrac{g(s)}{s}$ lies strictly between m^2 and $(m+1)^2$.

Moreover $\liminf\limits_{|s|\to+\infty} \dfrac{g(s)}{s}$ or $\limsup\limits_{|s|\to+\infty} \dfrac{g(s)}{s}$ may attain either m^2 or $(m+1)^2$ but "slowly" on account of condition (1.4).

In our work $\dfrac{g(s)}{s}$ is not obliged to stay in the interval $[m^2,(m+1)^2]$, although there is some "density" control given by a condition about the asymptotic behaviour of the potential of g, as used in [GO], [SO] and [OZ] (see conditions (g) and (G)).

We prove the existence of a periodic solution to the problem (1.1)–(1.2), using degree theory, spaces $\mathbb{L}^p(0,2\pi)$, with norms $\|\ \|_p$ $(1 \leq p \leq +\infty)$, $\mathcal{C}^k(0,2\pi)$, of k-times continuously differentiable functions, whose norms are denoted by $\|\ \|_{\mathcal{C}^k}$ $(k=0,1,2,...)$ and the Sobolev spaces $\mathbb{W}^{3,p}_{2\pi}(0,2\pi)$, that consist of functions u in $\mathbb{W}^{3,p}(0,2\pi)$ such that $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$, $u''(0)=u''(2\pi)$.

Consider the eigenvalue problem

$$x''' + a\,x'' + c\,x = -\lambda\, x' \tag{1.5}$$

with conditions (1.2), $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$ and λ a real parameter.

We recall [EO] that:

(**a**) Any $\lambda \neq m^2$ is not an eigenvalue, for each $m = 1,2,...\,$;

(**b**) $\lambda = m^2$ is an eigenvalue, for some $m=1,2,...$, if and only if $c = a\,m^2$.

Note that, from (a) and (b), the eigenvalue, when exists, is unique and the corresponding eigenspace, which we denote by $\mathcal{E}_m$, consists of elements x that can be written as

$$x = \frac{1}{\sqrt{2\pi}}\Big(A_m\, e^{imt} + A_{-m}\, e^{-imt}\Big)$$

with $m \in \mathbb{N}_1$, $A_m \in \mathbb{C}$ and $A_{-m} = \overline{A}_m$. For more details see [AOZ].

2 – Existence result

Let us consider the problem

$$(\mathcal{P})\qquad \begin{cases} x''' + a\,x'' + g(x') + c\,x = p(t)\ , \\ x(0) = x(2\pi)\,, \quad x'(0) = x'(2\pi)\,, \quad x''(0) = x''(2\pi)\ , \end{cases}$$

with $a, c \in \mathbb{R}$, $c \neq 0$, $g\colon \mathbb{R} \mapsto \mathbb{R}$ continuous and $p \in \mathbb{L}^1(0, 2\pi)$, a real function.

Denote by G the primitive of the nonlinear function g, that is, $G(u) = \int_0^u g(\tau)\,d\tau$.

Theorem 1. *For $m \in \mathbb{N}$, assume that g satisfies*

$$(g)\qquad m^2 \leq \liminf_{|u|\to\pm\infty} \frac{g(u)}{u} \leq \limsup_{|u|\to\pm\infty} \frac{g(u)}{u} \leq (m+1)^2$$

and

$$(G)\qquad m^2 < \limsup_{u\to+\infty} \frac{2\,G(u)}{u^2}\,, \qquad \liminf_{u\to+\infty} \frac{2\,G(u)}{u^2} < (m+1)^2\ .$$

Then problem $(\mathcal{P})$ has, at least, one solution for every $p \in \mathbb{L}^1(0, 2\pi)$.

To prove Theorem 1 we need some preliminar results.

Let us define an operator $\mathcal{A}\colon \mathbb{W}^{3,1}_{2\pi}(0, 2\pi) \mapsto \mathbb{L}^1(0, 2\pi)$ by

$$\mathcal{A}\,x = x''' + a\,x'' + c\,x$$

and denote the inner product in $\mathbb{L}^2(0, 2\pi)$ as $\langle\cdot,\cdot\rangle$.

Lemma 1. *For every $x \in \mathbb{W}^{3,2}_{2\pi}(0, 2\pi)$, we have*

$$\Big\langle \mathcal{A}\,x + m^2\,x',\ \ \mathcal{A}\,x + (m+1)^2\,x' \Big\rangle \ \geq\ 0\ ,$$

and the equality holds if and only if $x = 0$ or either m^2 or $(m+1)^2$ is an eigenvalue of (1.5) and $x \in \mathcal{E}_m$ or $x \in \mathcal{E}_{m+1}$, respectively.

Proof. Using the Fourier expansion of x, we can write

$$x(t) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} c_k\, e^{ikt}$$

and obtain

$$\Big\langle \mathcal{A}\,x + m^2\,x',\ \ \mathcal{A}\,x + (m+1)^2\,x' \Big\rangle \ \geq$$
$$\geq\ \sum_{k\in\mathbb{Z}} \Big[k^2\,(m^2 - k^2)\,\Big((m+1)^2 - k^2\Big) + (c - a\,k^2)^2 \Big]\,|c_k|^2 \ \geq\ 0\ .$$

Furthermore, the equality holds if and only if $c_k = 0$ unless

$$k^2 = m^2 \quad \text{or} \quad k^2 = (m+1)^2 \qquad \text{and} \qquad c = a\,k^2 \ ,$$

that means, if and only if $x = 0$ or either m^2 or $(m+1)^2$ is an eigenvalue of (1.5) and $x \in \mathcal{E}_m$ or $x \in \mathcal{E}_{m+1}$, respectively. ∎

For the sequel, let us fix a number θ such that $m^2 < \theta < (m+1)^2$ and define an operator $\mathcal{L}_\theta \colon \mathbb{W}^{3,1}_{2\pi}(0, 2\pi) \mapsto \mathbb{L}^1(0, 2\pi)$, by setting

$$\mathcal{L}_\theta\, x = x''' + a\,x'' + c\,x + \theta\,x' \ .$$

So, $\mathcal{L}_\theta$ is invertible with the inverse $\mathcal{K}_\theta \colon \mathbb{L}^1(0, 2\pi) \mapsto \mathbb{W}^{3,1}_{2\pi}(0, 2\pi)$. By the compact imbedding of $\mathbb{W}^{3,1}_{2\pi}(0, 2\pi)$ into $\mathcal{C}^1(0, 2\pi)$, problem $(\mathcal{P})$ can be reformulated as a compact fixed point problem in the form

$$x = \mathcal{K}_\theta \Big[\theta\,x' - g(x') + p(t)\Big] \tag{2.1}$$

in, say, $\mathcal{C}^1(0, 2\pi)$.

We consider the homotopy

$$x = \mu\,\mathcal{K}_\theta \Big[\theta\,x' - g(x') + p(t)\Big] \ , \tag{2.2}$$

with $\mu \in [0, 1]$ and the corresponding problem

$$(\mathcal{P}_\mu) \quad \begin{cases} x''' + a\,x'' + c\,x = (\mu - 1)\,\theta\,x' + \mu\,[p(t) - g(x')] \ , \\ x(0) = x(2\pi)\,, \quad x'(0) = x'(2\pi)\,, \quad x''(0) = x''(2\pi) \ . \end{cases}$$

In order to apply Leray–Schauder degree theory we prove the existence of a bounded set Ω in $\mathcal{C}^1([0, 2\pi])$, containing the origin, such that no solution of $(\mathcal{P}_\mu)$, or equivalently of (2.2), for any $\mu \in [0, 1]$, belongs to the boundary of Ω.

Next steps will guarantee the tools for building such set Ω.

Claim 1. *Let x be a solution of $(\mathcal{P}_\mu)$. Then there are constants $d_0 > 0$ and $K > 0$, independent of x, such that when $\|x\|_{\mathcal{C}^1} > d_0$ we have $\|x\|_\infty \le K\|x'\|_\infty$.*

Proof: Integrating the equation of $(\mathcal{P}_\mu)$ one obtains

$$c\int_0^{2\pi} x(t)\,dt \ = \ \mu \int_0^{2\pi} \Big[p(t) - g(x')\Big]\,dt \ .$$

By (g) there exist $a_1, a_2 \in \mathbb{R}^+$ such that $|g(x')| \le a_1|x'| + a_2$. So, using the Mean Value Theorem, for some $t_0 \in [0, 2\pi]$,

$$|x(t_0)| \ \le \ \frac{1}{2\pi\,|c|} \int_0^{2\pi} \Big|p(t) - g(x')\Big|\,dt \ \le \ \kappa_1\,\|x'\|_\infty + \kappa_2 \ .$$

By the Fundamental Theorem of Calculus and Hölder's inequality,

$$|x(t)| \;\leq\; \int_{t_0}^{t} |x'(t)|\,dt + |x(t_0)| \;\leq\; \kappa_3\,\|x'\|_\infty + \kappa_4 \;,$$

where the constants κ_1, κ_2, κ_3 and κ_4 are independent of x. But this inequality implies that if $\|x\|_{\mathcal{C}^1} \to +\infty$ then $\|x'\|_\infty \to +\infty$ and so the thesis follows easily. ∎

The above estimate on the solutions of $(\mathcal{P}_\mu)$ will be very useful in several steps of the proof of Theorem 1 and will play an important role in the construction of a set Ω, where the degree is well defined.

Claim 2. *Let (x_n) be a sequence of solutions of*

$$(\mathcal{P}_n)\quad \begin{cases} x_n''' + a\,x_n'' + c\,x_n = (\mu_n - 1)\,\theta\,x_n' + \mu_n\Big[p(t) - g(x_n')\Big]\;, \\ x_n(0) = x_n(2\pi)\,, \quad x_n'(0) = x_n'(2\pi)\,, \quad x_n''(0) = x_n''(2\pi)\;, \end{cases}$$

with $\mu_n \in [0,1]$, $m^2 < \theta < (m+1)^2$, *such that* $\|x_n'\|_\infty \to +\infty$.

Then, for a subsequence, $\dfrac{x_n}{\|x_n'\|_\infty}$ *converges in* $\mathbb{W}^{3,1}_{2\pi}(0,2\pi)$ *to some function* $v \not\equiv 0$, *when* $\mu_n \to 1$.

Moreover, either

$$m^2 \text{ is an eigenvalue of } \mathcal{A}\,,\quad v \in \mathcal{E}_m \quad\text{and}\quad \frac{\Big\|g(x_n') - m^2\,x_n'\Big\|_1}{\|x_n'\|_\infty} \longrightarrow 0\;,$$

or

$$(m{+}1)^2 \text{ is an eigenvalue of } \mathcal{A}\,,\quad v \in \mathcal{E}_{m+1} \quad\text{and}\quad \frac{\Big\|g(x_n') - (m{+}1)^2\,x_n'\Big\|_1}{\|x_n'\|_\infty} \longrightarrow 0\;.$$

Proof: Consider, as in [HOZ] (Prop. 2.1), $g(u) = q(u)\,u + r(u)$ with q and r continuous functions such that

$$m^2 \leq q(u) \leq (m+1)^2\,, \qquad \forall\, u \in \mathbb{R}\;, \tag{2.3}$$

and

$$\lim_{|u|\to+\infty} \frac{r(u)}{u} = 0\;.$$

Applying this decomposition and setting $v_n = \dfrac{x_n}{\|x_n'\|_\infty}$, then v_n satisfies

$$\begin{cases} v_n''' + a\,v_n'' + c\,v_n \;=\; (\mu_n - 1)\,\theta\,v_n' - \mu_n\,q(x_n')\,v_n' + \mu_n\,\dfrac{p(t) - r(x_n')}{\|x_n'\|_\infty}\;, \\ v_n(0) = v_n(2\pi)\,, \quad v_n'(0) = v_n'(2\pi)\,, \quad v_n''(0) = v_n''(2\pi)\;. \end{cases}$$

The second member of the equation is bounded in $\mathbb{L}^\infty(0,2\pi)$ and so, for a subsequence, it converges weakly in $\mathbb{L}^1(0,2\pi)$. By the continuity of the inverse operator, it follows that v_n converges weakly in $\mathbb{W}^{3,1}_{2\pi}(0,2\pi)$ and then strongly in $\mathcal{C}^1(0,2\pi)$ to a function $v \not\equiv 0$, since $\|v'\|_\infty = 1$. Furthermore, we can suppose that, for a subsequence, $\mu_n \to \mu_0 \in [0,1]$ and $q(x_n')$ converges in $\mathbb{L}^\infty(0,2\pi)$, with respect to the weak* topology, to a function $q_0(t) \in \mathbb{L}^\infty(0,2\pi)$, where

$$m^2 \le q_0(t) \le (m+1)^2 \ .$$

If we set

$$\tilde q(t) = (\mu_0 - 1)\,\theta - \mu_0\, q_0(t) \ , \tag{2.4}$$

the weak continuity of $\mathcal{L}_\theta$ implies that v verifies

$$\begin{cases} v''' + a\,v'' + c\,v \;=\; \tilde q(t)\,v' \\ v(0) = v(2\pi)\,, \quad v'(0) = v'(2\pi)\,, \quad v''(0) = v''(2\pi) \ , \end{cases} \tag{2.5}$$

with

$$-(m+1)^2 \le \tilde q \le -m^2 \ . \tag{2.6}$$

Using Lemma 1, (2.5) and (2.6) we obtain

$$\begin{aligned} 0 \;\le\; \Big\langle \mathcal{A}\,v + m^2\,v',\ \ \mathcal{A}\,v + (m+1)^2\,v' \Big\rangle \;=& \\ =\; \int_0^{2\pi} (\tilde q + m^2)\,\Big(\tilde q + (m+1)^2\Big)\,(v')^2\,dt \;\le\; 0 \ ,& \end{aligned}$$

which implies $\langle \mathcal{A}\,v + m^2\,v',\ \mathcal{A}\,v + (m+1)^2\,v'\rangle = 0$. Since $v \ne 0$, if $c \ne a\,m^2$ and $c \ne a\,(m+1)^2$, by Lemma 1, the above equality can not hold and then Claim 2 is trivially satisfied. So suppose that either $c = a\,m^2$ or $c = a\,(m+1)^2$. Then either

$$m^2 \text{ is an eigenvalue of } \mathcal{A}\,, \quad v \in \mathcal{E}_m \quad \text{and} \quad \tilde q = -m^2 \ , \tag{2.7}$$

or

$$(m+1)^2 \text{ is an eigenvalue of } \mathcal{A}\,, \quad v \in \mathcal{E}_{m+1} \quad \text{and} \quad \tilde q = -(m+1)^2 \ . \tag{2.8}$$

From (2.4), we also conclude that $\mu_0 = 1$ and $q(x_n') \to -\tilde q$ in $\mathbb{L}^\infty(0,2\pi)$, with respect to the weak* topology. Therefore if (2.7) holds, using (2.3) we have

$$\Big\| q(x_n') - m^2 \Big\|_1 \;=\; \int_0^{2\pi} \Big| q(x_n') - m^2 \Big|\,dt \;=\; \int_0^{2\pi} \Big(q(x_n') - m^2 \Big)\,dt \;\longrightarrow\; 0 \ .$$

Hence

$$\left\| \frac{g(x_n')}{\|x_n'\|_\infty} - m^2\, v' \right\|_1 = \left\| q(x_n')\, v_n' + \frac{r(x_n')}{\|x_n'\|_\infty} - m^2\, v' \right\|_1 \le$$

$$\le \|q(x_n')\|_\infty\, \|v_n' - v'\|_1 + \|q(x_n') - m^2\|_1\, \|v'\|_\infty + \left\| \frac{r(x_n')}{\|x_n'\|_\infty} \right\|_1 \longrightarrow 0 \;.$$

If (2.8) holds the proof is similar. ∎

Claim 3. *There are constants $d_1 > 0$ and $0 < \eta_1 < 1 < \eta_2$ such that if x is a solution of ($\mathcal{P}_\mu$), for some $\mu \in [0,1]$ and satisfying $\|x'\|_\infty \ge d_1$, then*

$$\max x_n' \cdot \min x_n' < 0 \quad \textit{and} \quad \eta_1 < \frac{\max x'}{-\min x'} < \eta_2 \;.$$

Proof: Assume, by contradiction, that the first part of the thesis does not hold. So, there is a sequence (x_n) of solutions of $(\mathcal{P}_n)$ such that $\|x_n'\|_\infty \to +\infty$ and $\max x_n' \cdot \min x_n' \ge 0$.

By Claim 2, $\dfrac{x_n}{\|x_n'\|_\infty} \to v$ in $\mathbb{W}^{3,1}_{2\pi}(0,2\pi)$ and, therefore, $\dfrac{x_n'}{\|x_n'\|_\infty} \to v'$ in $\mathcal{C}^0(0,2\pi)$ with either $v \in \mathcal{E}_m$ or $v \in \mathcal{E}_{m+1}$. Moreover, we can write

$$v'(t) = A_m \cos mt + B_m \sin mt$$

or

$$v'(t) = A_{m+1} \cos(m{+}1)\, t + B_{m+1} \sin(m{+}1)\, t$$

and, on both cases,

$$\max \frac{x_n'}{\|x_n'\|_\infty} \cdot \min \frac{x_n'}{\|x_n'\|_\infty} \longrightarrow \max v' \cdot \min v' < 0 \;.$$

For proving the second part, we suppose, again by contradiction, that, for every $n \in \mathbb{N}$ there is a (x_n) solution of some $(\mathcal{P}_n)$, with $\|x_n'\|_\infty \ge d_1$, such that $\dfrac{\max x_n'}{-\min x_n'} \le \dfrac{1}{n}$. Then $\dfrac{\max x_n'}{-\min x_n'} \to 0$, which contradicts

$$\frac{\max \dfrac{x_n'}{\|x_n'\|_\infty}}{-\min \dfrac{x_n'}{\|x_n'\|_\infty}} \longrightarrow \frac{\max v'}{-\min v'} > 0 \;.$$

The proof for η_2 is similar. ∎

In the proof of next claim we shall use the condition on the potential.

Claim 4. *Suppose that conditions (g) and (G) hold. Then there is a sequence (γ_n), with $\gamma_n \to +\infty$, such that, if x is a solution of $(\mathcal{P}_\mu)$, for some $\mu \in [0,1]$, we have $\max x' \neq \gamma_n$, for every n.*

Proof: By condition (G) we can take a sequence of real numbers (γ_n), with $\gamma_n \to +\infty$, such that

$$\lim_{\gamma_n \to +\infty} \frac{2\,G(\gamma_n)}{\gamma_n^2} = \lambda \;\in\;]m^2, (m+1)^2[\ . \tag{2.9}$$

Assume, by contradiction, that there is a subsequence of (γ_n), which we shall note by (γ_n) too, and a sequence $\mu_n \in [0,1]$ such that if (x_n) is a solution of $(\mathcal{P}_{\mu_n})$, one has $\max x_n' = \gamma_n$. Therefore, by (2.9), for $\varepsilon > 0$ small enough and large n, we can write

$$\frac{2\,G(\gamma_n)}{\gamma_n^2} > m^2 + \varepsilon\ ,$$

that is,

$$\frac{2\,G(\gamma_n) - m^2\,\gamma_n^2}{\|x_n'\|_\infty^2} > \varepsilon\,\frac{\gamma_n^2}{\|x_n'\|_\infty^2} > 0\ . \tag{2.10}$$

Due to the first part of Claim 3, there exist $t_{n_0}, t_{n_1} \in [0, 2\pi]$ such that

$$\gamma_n = \max(x_n'(t)) = x_n'(t_{n_1}) \qquad \text{and} \qquad x_n'(t_{n_0}) = 0\ .$$

Then

$$\begin{aligned} G(\gamma_n) - \frac{m^2}{2}\,\gamma_n^2 &= G(x_n'(t_{n_1})) - G(x_n'(t_{n_0})) - \frac{m^2}{2}\Big[{x_n'}^2(t_{n_1}) - {x_n'}^2(t_{n_0})\Big] \\ &= \int_{t_{n_0}}^{t_{n_1}} \Big[g(x_n'(t)) - m^2\,x_n'(t)\Big]\,x_n''(t)\,dt \\ &\le \int_0^{2\pi} \Big|g(x_n'(t)) - m^2\,x_n'(t)\Big|\,|x_n''(t)|\,dt\ . \end{aligned}$$

By Claim 2 and the continuous imbedding of $\mathbb{W}^{3,1}_{2\pi}(0,2\pi)$ into $\mathcal{C}^2([0,2\pi])$, one has

$$\begin{aligned} \frac{2\,G(\gamma_n) - m^2\,\gamma_n^2}{\|x_n'\|_\infty^2} &\le \int_0^{2\pi} \frac{\Big|g(x_n'(t)) - m^2\,x_n'(t)\Big|\,|x_n''(t)|}{\|x_n'\|_\infty^2}\,dt \\ &\le \left\|\frac{g(x_n'(t)) - m^2\,x_n'(t)}{\|x_n'\|_\infty}\right\|_1 \|v_n''(t)\|_\infty \;\to\; 0\ , \end{aligned}$$

since (v_n'') is bounded in $\mathbb{L}^\infty$.

This fact contradicts (2.10). ∎

Proof of Theorem 1: Let (γ_n) be a sequence given by Claim 4 and let n_0 be such that $\gamma_{n_0} > \max\{d_0, d_1\}$, where d_0 and d_1 are referred in Claims 1 and 3, respectively. Take also $K > 0$ and $0 < \eta_1 < 1$ as in Claims 1 and 3 and define the open set Ω in $\mathcal{C}^1([0,2\pi])$, containing the origin:

$$\Omega = \left\{x \in \mathcal{C}^1([0,2\pi])\colon \; -\frac{\gamma_{n_0}}{\eta_1} < x'(t) < \gamma_{n_0} \;\wedge\; \|x\|_\infty < K\,\frac{\gamma_{n_0}}{\eta_1}, \quad \forall\, t \in [0,2\pi]\right\} .$$

Let x be a solution of $(\mathcal{P}_\mu)$, for some $\mu \in [0,1]$, such that $x \in \bar{\Omega}$. From Claims 3, 4 and 1 we deduce that $x \in \Omega$. So, the degree is well defined and it is nonzero for every $\mu \in [0,1]$. Then, the homotopy invariance of the degree guarantees the existence of a solution of $(\mathcal{P}_\mu)$ for, say, $\mu = 1$, that is, a solution of $(\mathcal{P})$. ∎

Remark. The statement of Theorem 1 still holds if (G) is replaced by one of the following conditions

(G_1) $$m^2 < \limsup_{u\to-\infty} \frac{2\,G(u)}{u^2}\,, \qquad \liminf_{u\to+\infty} \frac{2\,G(u)}{u^2} < (m+1)^2 \,,$$

or

(G_2) $$m^2 < \limsup_{u\to+\infty} \frac{2\,G(u)}{u^2}\,, \qquad \liminf_{u\to-\infty} \frac{2\,G(u)}{u^2} < (m+1)^2 \,,$$

or

(G_3) $$m^2 < \limsup_{u\to-\infty} \frac{2\,G(u)}{u^2}\,, \qquad \liminf_{u\to-\infty} \frac{2\,G(u)}{u^2} < (m+1)^2 \,. \;\square$$

In fact, under condition (G_1), we can prove as Claim 4 that solutions of $(\mathcal{P}_\mu)$ are bounded in $\mathcal{C}^1$, by following similar lines. If (G_2) or (G_3) is assumed, the result can be easily derived from the previous ones by the change of variable $v := -u$.

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