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# EXPONENTIAL STABILITY OF POSITIVE SOLUTIONS TO SOME NONLINEAR HEAT EQUATIONS

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**Abstract:** Following a recent work of A. Haraux in which he proves exponential stability of positive solutions of a heat equation with strictly convex nonlinearity, the same property is shown for a suitable perturbation of the nonlinearity which can, in particular, be non convex.

## 1 – Introduction and main results

Let $\Omega$ be a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary and let us consider the semilinear heat equation

$$(1.1)\qquad \begin{gathered} u_t - \Delta u + f(u) = k(t,x) \quad \text{in } \mathbb{R}^+ \times \Omega \ , \\ u(t,\cdot) = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega \ , \\ u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega \ , \end{gathered}$$

and the elliptic equation

$$(1.2)\qquad \begin{aligned} -\Delta u + f(u) &= 0 \quad \text{in } \Omega \ , \\ u &= 0 \quad \text{on } \partial\Omega \ , \end{aligned}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that

$$(1.3)\qquad f(0) = 0 \quad \text{and} \quad f(s) \to +\infty \ \text{ as } s \to +\infty$$

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and $k\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}$ satisfies the conditions

$$(1.4) \qquad k \in L^\infty(\mathbb{R}^+ \times \Omega) \quad \text{and} \quad k(t,x) \geq 0 \ \text{ a.e. on } \mathbb{R}^+ \times \Omega \ .$$

By using standard techniques from the theory of evolution equations, cf. e.g. [5], we know that for all $u_0 \in L^\infty(\Omega)$ with $u_0(x) \geq 0$ a.e. on $\Omega$, there exists a unique solution $u \in C((0,+\infty); H_0^1(\Omega) \cap L^\infty(\Omega)) \cap C([0,+\infty); L^2(\Omega))$ of (1.1) such that $u(0,\cdot) = u_0(\cdot)$. In addition we have

$$u(t,x) \geq 0 \quad \text{a.e. on } \ \mathbb{R}^+ \times \Omega \ .$$

As a consequence of (1.3) and the maximum principle, $u$ is uniformly bounded on $\Omega \times \mathbb{R}^+$. Then by the method of [11], it follows easily that

$$\bigcup_{t \geq 1} \{u(t,\cdot)\} \quad \text{is bounded in } \ C^{1+\alpha}(\overline{\Omega}) \ \text{ for every } \alpha \in [0,1) \ .$$

In particular the curve $t \mapsto u(t,\cdot)$ has a precompact range in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for $t \geq 1$ and it is natural to ask about the *asymptotic behavior* of $u(t,\cdot)$ as $t \to \infty$.

A. Haraux [8] has proved exponential convergence of nonnegative solutions of (1.1) when $f$ satisfies the additional hypotheses

$$(1.5) \qquad f \quad \text{strictly convex on } \ [0,+\infty) \quad \text{and} \quad f_d'(0) < -\lambda_1(-\Delta)$$

where $\lambda_1(-\Delta)$ is the smallest eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. The proof of this result is based on the uniqueness of positive solution of the equation (1.2) and the fact that $\lambda_1(-\Delta + f'(\varphi)) > 0$ ($\varphi$ is the unique positive solution of (1.2)).

The typical example of nonlinearities which verifies these hypotheses is the following

$$(1.6) \qquad f(s) = s^p - \lambda\, s\,, \quad \lambda > \lambda_1(-\Delta), \ \ p > 1 \ .$$

The question which we study in this paper is the following: What happens if we perturb the nonlinearity in such a way that convexity of $f$ is lost? In the special case of example (1.6) a question of interest is the following: Can we find $\varepsilon > 0$ such that the result of [8] persists for the new nonlinearity

$$h(s) = s^p - \lambda\, s - \varepsilon\, s^q$$

with $p$, $\lambda$ as in (1.6) and $1 < q < p$?

We are able to give a positive answer to this question. We use the same method as in [8]: At first we prove the uniqueness of positive solution of (1.2) with this new type of nonlinearity. We assume the following hypotheses: Let $f$ satisfying (1.3), (1.5), and let $g\colon \mathbb{R}^+ \to \mathbb{R}$ be a function of class $C^1$ such that

$$\begin{aligned} &g'(0) \geq 0\,, \qquad \lim_{s\to\infty} f(s) - C\,g(s) = \infty\ ,\\ &g(0) = 0\,, \qquad g(s) \geq 0 \ \ \forall\, s \geq 0\ , \end{aligned} \tag{1.7}$$

with $C > 0$ and we consider the nonlinear heat equation

$$\begin{aligned} u_t - \Delta u + f(u) = \varepsilon\, g(u) + k(t,x) \quad &\text{in}\ \ \mathbb{R}^+ \times \Omega\ ,\\ u(t,\cdot) = 0 \quad &\text{on}\ \ \mathbb{R}^+ \times \partial\Omega\ ,\\ u(0,\cdot) = u_0(\cdot) \quad &\text{in}\ \ \Omega \end{aligned} \tag{1.8}$$

The main results of this paper are the following

**Theorem 1.1.** *Let $f$, $g$ satisfy the hypotheses (1.3), (1.5), (1.7). Then there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in [0, \varepsilon_1)$ the equation*

$$\Psi \in H_0^1(\Omega)\,, \qquad -\Delta\Psi + f(\Psi) = \varepsilon\, g(\Psi)\ , \tag{1.9}$$

*has one and only one solution $\Psi \geq 0$ other than 0. In addition we have $\Psi > 0$ everywhere in $\Omega$ and*

$$\lambda_1\Big(-\Delta + f'(\Psi) - \varepsilon\, g'(\Psi)\Big) > 0 \qquad \forall\, \varepsilon \in [0, \varepsilon_1)\ . \tag{1.10}$$

**Theorem 1.2.** *Let $f$, $g$ and $k$ satisfy the hypotheses (1.3), (1.4), (1.5), (1.7). Then if $u_0, v_0 \in L^\infty$ with $u_0(x) \geq 0$ and $v_0(x) \geq 0$ a.e. on $\Omega$, consider the solution $u$, $v$ of (1.1) with respective initial data $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$. Assuming either that both $u_0$, $v_0$ are not identically 0 or that $k(t,x) > 0$ on a subset of positive measure of $\mathbb{R}^+ \times \Omega$. Then there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in [0, \varepsilon_2)$, there is $\gamma > 0$ independent of $k$ and $(u_0, v_0)$:*

$$\forall\, t \geq 0 \quad \|u(t,\cdot) - v(t,\cdot)\|_\infty \leq C(u_0, v_0, \varepsilon) \exp(-\gamma\, t)\ . \tag{1.11}$$

The paper is organized as follows: in Section 2 we prove Theorem 1.1, in Section 3 we establish Theorem 1.2 when $k = 0$. In Section 4, we establish Theorem 1.2 in the general case. In each section some remarks are presented.

## 2 – The stationnary problem

The object of this section is to prove Theorem 1.1.

**Proof of Theorem 1.1.** First we prove the existence of a positive solution for the equation (1.9). In fact, if $\varepsilon = 0$ then by a theorem of Berestycki [1] (Theorem 4, page 14, cf. also [2], [3]), there exists a unique positive solution $\varphi$ of (1.2) which verifies

$$\lambda_1(-\Delta + f'(\varphi)\, id) > 0 \ . \tag{2.1}$$

Since $g \geq 0$ then $\varphi$ is a subsolution of (1.9).

Now we assume that $\varepsilon < C$, then by (1.7) there exists $M > 0$ such that

$$f(M) - \varepsilon\, g(M) > 0 \ . \tag{2.2}$$

So $M$ is a supersolution of (1.9). We claim that $\|\varphi\|_\infty < M$. Indeed, let $x_0 \in \Omega$ such that $\varphi(x_0) = \|\varphi\|_\infty$, we have $\Delta\varphi(x_0) \leq 0$. Now if $\|\varphi\|_\infty \geq M$ then we have $f(M) \leq 0$. Hence $f(M) - \varepsilon\, g(M) \leq 0$, and this contradicts (2.2). Then there exist a solution $\Psi$ for (1.9) which verifies $\varphi \leq \Psi \leq M$. By using again the maximum principle, we have for all $\xi$ positive solution of (1.9) $\xi < M$. Then the problem (1.9) has a "maximal" solution $\Psi$ in the sense: any solution $\xi \neq \Psi$ of (1.9) is less than $\Psi$. (This solution can be constructed by a standard iterative scheme.)

Now we have to use the following lemma due to Haraux [9, 10].

**Lemma 2.1.** *Let $f$ satisfy the hypotheses (1.3), (1.5) and let $\varphi$ be the positive solution of the equation*

$$\varphi \in C(\Omega) \cap H_0^1(\Omega)\,, \quad -\Delta\varphi + f(\varphi) = 0 \ .$$

*Let on the other hand $\xi \geq 0$ be a solution of*

$$\xi \in C(\Omega) \cap H_0^1(\Omega)\,, \quad -\Delta\xi + f(\xi) \geq_0 \ .$$

*Then either $\xi = 0$ or $\xi \geq \varphi$.*

**Proof of Theorem 1.1** (continued). Let $\xi$ be a positive solution of (1.9), then by using Lemma 2.1 we have

$$\varphi \leq \xi < M \ .$$

Now we prove uniqueness. In fact we assume that we have a solution $\xi$ of (1.9) other than the "maximal" solution $\Psi$. Then we have:

$$-\Delta(\Psi-\xi)+f(\Psi)-f(\xi)=\varepsilon[g(\Psi)-g(\xi)]\ . \tag{2.3}$$

Multiplying (2.3) by $(\Psi-\xi)$ and integrating over $\Omega$ we find

$$\int_\Omega |\nabla(\Psi-\xi)|^2+[f(\Psi)-f(\xi)]\,(\Psi-\xi)\,dx = \\ = \varepsilon\int_\Omega [g(\Psi)-g(\xi)]\,(\Psi-\xi)\,dx\ . \tag{2.4}$$

Since $\varphi\le\xi\le\Psi<M$, then by using (1.5), (1.7) and (2.4) we find

$$\int_\Omega |\nabla(\Psi-\xi)|^2+f'(\varphi)\,|\Psi-\xi|^2\,dx\le\varepsilon\int_\Omega C_1\,|\Psi-\xi|^2\,dx \tag{2.5}$$

with $C_1=\sup\{|g'(s)|,\ s\in[0,M]\}>0$. So

$$\Big[\lambda_1(-\Delta+f'(\varphi))-\varepsilon\,C_1\Big]\int_\Omega|\Psi-\xi|^2\,dx\le 0\ . \tag{2.6}$$

Thank's to (2.1) $\lambda_1(-\Delta+f'(\varphi)\,id)>0$. Now let $\varepsilon'$ such that $\lambda_1(-\Delta+f'(\varphi)\,id)=\varepsilon'\,C_1$ and $\varepsilon_1=\inf(\varepsilon',C)$, with $C$ as in (1.7). Then for all $\varepsilon\in[0,\varepsilon_1)$, we have

$$\lambda_1(-\Delta+f'(\varphi))-\varepsilon\,C_1>0\ . \tag{2.7}$$

The uniqueness follows from (2.7), we note this solution by $\Psi$. By using (1.5), (1.7) and (2.7) we deduce

$$\lambda_1\Big(-\Delta+[f'(\Psi)-\varepsilon\,g'(\Psi)]\,id\Big)>0\quad\forall\,\varepsilon\in[0,\varepsilon_1)\ .\ \blacksquare \tag{2.8}$$

## 3 – The autonomous case

The object of this section is to prove Theorem 1.2 in the case $k=0$. We use the method of [8].

**Proof of Theorem 1.2.** Let $Z=\{z\in C(\overline{\Omega})\cap H_0^1(\Omega)\ /\ z\ge 0\}$. Subsequently $h=f-\varepsilon\,g$ with $\varepsilon\in[0,\varepsilon_1)$ and $\varepsilon_1$ is as in Theorem 1.1.

The equation (1.1) generates a dynamical system $\{S(t)\}_{t\ge 0}$ which assigns to each element $z\in Z$ the value $v(t)=S(t)\,z$ where $v$ is the solution of (1.8) such that $v(0)=z$. Now let $E$ be the functional defined by

$$\forall\,\varphi\in Z\quad E(\varphi)=\frac{1}{2}\int_\Omega|\nabla\varphi|^2\,dx+\int_\Omega H(\varphi)\,dx\ \ \text{where } H(u):=\int_0^u h(s)\,ds\ .$$

$E$ is a strict Liapunov functional on $Z$ relative to $S(t)$ and we refer to [9] for a simple proof.

Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then by using the maximum principle (cf. for example [5]) we have $u(t,x) \geq 0$ a.e. $(t,x) \in \mathbb{R}^+ \times \Omega$. By the standard invariance principle (cf. [9]), we conclude that the solution $u(t,\cdot)$ asymptotes the set of nonnegative solutions of (1.9) as $t \to \infty$. We now show that if $u_0 \neq 0$, $u(t,\cdot)$ cannot tend to 0 as $t \to \infty$.

In fact assuming that $\lim_{t\to\infty} \|u(t,\cdot)\|_\infty = 0$, then for each $\alpha > 0$, there is $T(\alpha)$ such that

$$\forall\, t \geq T(\alpha) \qquad h(u(t,x)) \leq \{h_d'(0) + \alpha\}\, u(t,x) \qquad \text{on} \;\; \Omega\ .$$

Choosing $\alpha > 0$ small enough such that $-h_d'(0) - \alpha - \lambda_1(\Omega) > 0$, multiplying the equation by the positive eigenfunction $\varphi_1$ corresponding to the first eigenvalue $\lambda_1(-\Delta)$ of $-\Delta$ in $H_0^1(\Omega)$ and integrating over $\Omega$ we find

$$\frac{d}{dt}\int_\Omega u(t,x)\,\varphi_1\,dx \geq 0 \qquad \forall\, t \geq T(\alpha)\ .$$

Since the function $t \mapsto \int_\Omega u(t,x)\,\varphi_1\,dx$ is nondecreasing on $[T(\alpha),\infty]$ and tends to 0 as $t \to \infty$, it must vanish identically on $[T(\alpha),\infty]$. Because $\varphi_1$ is positive in $\Omega$, this imply that $u(t,\cdot) = 0\ \forall\, t \geq T(\alpha)$. Then a classical connectedness argument shows that $u_0 = 0$. Therefore if $u_0 \neq 0$, the $\omega$-limit set of $u_0$ under $S(t)$ is reduced to a single point: $\omega(u_0) = \{\Psi\}$. Since $u(t,\cdot)$ remain bounded in $C^1(\Omega)$ for all $t \geq 1$ we deduce that

$$\lim_{t\to\infty} \|u(t,\cdot) - \Psi(\cdot)\|_{1,\infty} = 0\ .$$

For the end of the proof, we just need to use (2.8).

**Remark 3.1.** It is clear that $\lim_{t\to\infty} \|u(t,\cdot) - \Psi(\cdot)\|_{1,\infty} \exp(c\,t) = 0$, $\forall\, c < \lambda_1(-\Delta + h'(\psi)\,id)$. In [19], Wiegner has proved that in such a case the difference of two solutions tend to 0 as $\exp(-c_1\,t)$ with $c_1 = \lambda_1(-\Delta + h'(\psi)\,id)$. For related works in the asymptotic of autonomous parabolic equation we refer to [7–9, 12–19].

## 4 – The nonautonomous case

The object of this section is to prove Theorem 1.2 in the general case. Subsequently $\varepsilon \in [0,\varepsilon_1)$ and $\varepsilon_1$ as in Theorem 1.1. In the proof we can use the following lemmas from [8] which are also valid for the modified equation (1.8):

**Lemma 4.1.** *Let $\psi$ be the unique positive solution of (1.9) and let us consider the solution $z$ of (1.8) with initial condition $z(0)=\psi$. Then we have:*

$$\forall\, t \geq 0 \quad z(t,x) \geq \psi(x) \;\text{ on } \Omega \;.$$

**Lemma 4.2.** *Let $u_0 \in L^\infty(\Omega)$ with $u_0(x) \geq 0$ a.e. on $\Omega$ and consider the solution $u$ of (1.8) with initial datum $u(0,x)=u_0(x)$. Assuming either that $u_0$ is not identically 0 or that $k(t,x)>0$ on a subset of positive measure of $\mathbb{R}^+ \times \Omega$, we have*

$$\lim_{t\to\infty} \Big\| \Big(u(t,\cdot)-\psi(\cdot)\Big)^- \Big\|_\infty = 0 \;. \tag{3.1}$$

**Proof of Theorem 1.2:** Obviously, it is sufficient to prove the result when $v_0=\psi$. Then $v(t)=z(t)$ and

$$\forall\, t>0 \quad \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u(t,x)-z(t,x)|^2\,dx\Big) =$$
$$= -\int_\Omega |\nabla(u-z)|^2\,dx - \int_\Omega [f(u)-f(z)]\,(u-z)\,dx + \varepsilon \int_\Omega [g(u)-g(z)]\,(u-z)\,dx \;.$$

By convexity of $f$, since $z(t) \geq \psi$ for all $t$, we have $f(z)/z \geq f(\psi)/\psi$. Moreover from (3.1) it follows in particular that fixing some nonempty open set $\omega$ contained in a compact subset of $\Omega$, we have for $t \geq T$ depending on the solution $u$ that

$$\forall\, t \geq T \quad u(t,x) \geq \frac{1}{2}\,\psi(x) \;\text{ on } \omega \;. \tag{3.2}$$

Now from (3.2) we deduce easily the inequality

$$\forall\, t \geq T \quad \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u(t,x)-z(t,x)|^2\,dx\Big) =$$
$$= -\int_\Omega |\nabla(u-z)|^2\,dx - \int_\Omega c(x)\,|u-z|^2\,dx + \varepsilon \int_\Omega [g(u)-g(z)]\,(u-z)\,dx$$

with

$$c(x) = \begin{cases} \dfrac{f(\psi)}{\psi} & \text{outside } \omega, \\[2mm] 2\,\dfrac{f(\psi)-f(\psi/2)}{\psi} & \text{in } \omega \;. \end{cases}$$

Let

$$\delta = \inf\Big\{ \int_\Omega \Big(|\nabla w|^2 + c(x)\,w^2\Big)\,dx,\;\; w \in H^1_0(\Omega),\; \int_\Omega w^2\,dx = 1 \Big\} \;.$$

We can prove as in [8] that $\delta > 0$. In the other hand, there exists $C_1 > 0$ such that

$$\int_\Omega [g(u) - g(z)]\,(u-z)\,dx \le C_1 \int_\Omega |u-z|^2\,dx\ .$$

Set $\varepsilon'' = \frac{\delta}{C_1}$ and let $\varepsilon_2 = \inf(\varepsilon_1, \varepsilon'')$, then we obtain for all $t \ge T$

$$\frac{d}{dt}\Big(\int_\Omega |u(t,x) - z(t,x)|^2\,dx\Big) \le -(\delta - \varepsilon\,C_1) \int_\Omega |u-z|^2\,dx\ .$$

The end of the proof is the same as in [8].

**Remark 4.3.** It is instructive to compare the result of Theorem 1.2 with the result of Chen and Matano [6], recently completed with a simple proof by Brunovsky et al. [4]. The result of [4, 6] are proved for any nonlinearity but only in one space dimension and for time-periodic forcing terms. On the other hand Theorem 1.2 is valid for any space dimension, but it is restricted to positive solution and a special type of nonlinearities.

**Remark 4.4.** This result can be viewed as a "structural stability" property for the result of [8]. However our method of proof is constructive since given $\lambda_1(-\Delta + f'(\psi)\,id) = \gamma > 0$, we can specify explicitely $\varepsilon_1$ and $\varepsilon_2$ in terms of the function $g$.

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