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CHEBYSHEV POLYNOMIALS AND SOME METHODS OF APPROXIMATION

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1 – Chebyshev polynomials $(T_n(x))_{n\geq 0}$ and $(U_n(x))_{n\geq 0}$ of the first and the second kind, respectively, are defined by the recurrence relations:

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)\,, \quad x \in \mathbb{C}, \quad n \in \mathbb{N}^*\,, \tag{1.1}$$

with $T_0(x) = 1$, $T_1(x) = x$;

$$U_{n+1}(x) = 2x \cdot U_n(x) - U_{n-1}(x)\,, \quad x \in \mathbb{C}, \quad n \in \mathbb{N}^*\,, \tag{1.2}$$

where $U_0(x) = 1$, $U_1(x) = 2x$.

On the other hand there are the sequences $(\widetilde{T}_n(x))_{n\geq 0}$ and $(\widetilde{U}_n(x))_{n\geq 0}$ — "associated" of the Chebyshev polynomials $(T_n(x))_{n\geq 0}$ and $(U_n(x))_{n\geq 0}$, respectively — defined by

$$\widetilde{T}_{n+1}(x) = 2x \cdot \widetilde{T}_n(x) + \widetilde{T}_{n+1}(x)\,, \quad x \in \mathbb{C}, \quad n \in \mathbb{N}^*\,, \tag{1.3}$$

with $\widetilde{T}_0(x) = 1$ and $\widetilde{T}_1(x) = x$;

$$\widetilde{U}_{n+1}(x) = 2x \cdot \widetilde{U}_n(x) + \widetilde{U}_{n-1}(x)\,, \quad x \in \mathbb{C}, \quad n \in \mathbb{N}^*\,, \tag{1.4}$$

where $\widetilde{U}_0(x) = 1$, $\widetilde{U}_1(x) = 2x$.

There is a simple connection between the sequences $(\widetilde{T}_n)_{n\geq 0}$, $(\widetilde{U}_n)_{n\geq 0}$ and the sequences $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$, respectively:

$$\begin{cases} \widetilde{T}_k(x) = \dfrac{T_k(i\cdot x)}{i^k}\,, \\[2ex] T_k(x) = \dfrac{\widetilde{T}_k(i\cdot x)}{i^k}, \quad x \in \mathbb{C}, \quad k \in \mathbb{N}\,, \end{cases} \tag{1.5}$$

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$$(1.6)\qquad \begin{cases} \widetilde{U}_k(x) = \dfrac{U_k(i\cdot x)}{i^k}\ , \\[2mm] U_k(x) = \dfrac{\widetilde{U}_k(i\cdot x)}{i^k}, \quad x\in\mathbb{C},\ \ k\in\mathbb{N}\ , \end{cases}$$

where $i^2 = -1$.

Also, there is an interesting connection between the sequence $(F_n)_{n\geq 0}$ of Fibonacci numbers

$$(1.7)\qquad F_{n+1} = F_n + F_{n-1},\ \ n\in\mathbb{N}^*\ , \qquad F_0 = 0,\ \ F_1 = 1\ ,$$

the sequence $(L_n)_{n\geq 0}$ of Lucas numbers

$$(1.8)\qquad L_{n+1} = L_n + L_{n-1},\ \ n\in\mathbb{N}^*\ , \qquad L_0 = 2,\ \ L_1 = 1\ ,$$

and, on the other hand, the sequences $(T_n)_{n\geq 0}$, $(\widetilde{T}_n)_{n\geq 0}$, $(U_n)_{n\geq 0}$, $(\widetilde{U}_n)_{n\geq 0}$:

$$(1.9)\qquad T_n\left(\frac{i}{2}\right) = \frac{1}{2}\cdot i^n\cdot L_n\ , \qquad n\in\mathbb{N}\ ,$$

$$(1.9')\qquad \widetilde{T}_n\left(\frac{1}{2}\right) = \frac{1}{2}\cdot L_n\ , \qquad n\in\mathbb{N}\ ,$$

$$(1.10)\qquad U_n\left(\frac{i}{2}\right) = i^n\cdot F_{n+1}\ , \qquad n\in\mathbb{N}\ ,$$

$$(1.10')\qquad \widetilde{U}_n\left(\frac{1}{2}\right) = F_{n+1}\ , \qquad n\in\mathbb{N},\ \ i^2 = -1\ .$$

One has, also, the remarkable formulas:

$$(1.11)\qquad T_k(\cos\varphi) = \cos k\,\varphi\ , \qquad \varphi\in\mathbb{C},\ \ k\in\mathbb{N}\ ,$$

$$(1.11')\qquad \widetilde{T}_k(i\cdot\cos\varphi) = i^k\cdot\cos k\varphi\ , \qquad \varphi\in\mathbb{C},\ \ k\in\mathbb{N}\ ,$$

and

$$(1.12)\qquad U_{k-1}(\cos\varphi) = \frac{\sin k\,\varphi}{\sin\varphi}\ , \qquad \varphi\in\mathbb{C},\ \ \sin\varphi\neq 0,\ \ k\in\mathbb{N}^*\ ,$$

$$(1.12')\qquad \widetilde{U}_{k-1}(i\cdot\cos\varphi) = i^{k-1}\cdot\frac{\sin k\,\varphi}{\sin\varphi}\ , \qquad \varphi\in\mathbb{C},\ \ \sin\varphi\neq 0,\ \ k\in\mathbb{N}^*\ ,$$

(see (1.5)–(1.6)).

2 – Let us consider the sequence $(\widetilde{\lambda}_m(a))_{m\geq 1}$ defined by

$$\widetilde{\lambda}_m(a) = \frac{\widetilde{U}_{p(m+1)-1}(a)}{\widetilde{U}_{pm-1}(a)}\,, \qquad m \in \mathbb{N}^*\,, \tag{2.1}$$

where $a \in \mathbb{R}^*$ and $p \in \mathbb{N}^*$.

We wish now to find a quadratic equation to be satisfied by the limit of the sequence $(\widetilde{\lambda}_m(a))_{m\geq 1}$. In order to obtain this equation we need to prove the following

Lemma. *If $(T_n(x))_{n\geq 0}$ and $(U_n(x))_{n\geq 0}$ are the sequences of Chebyshev polynomials of the first kind and the second kind, respectively, then one has*

$$U_{m+p-1}(a) = 2 \cdot T_p(a) \cdot U_{m-1}(a) - U_{m-p-1}(a)\,, \tag{2.2}$$

$\forall\, m \geq p+1$, $m, p \in \mathbb{N}^*$, $\forall\, a \in \mathbb{C}$.

Proof: Indeed, let a be an element of $\mathbb{C}$; then $\exists\, \varphi \in \mathbb{C}$ such that $a = \cos\varphi$. We have

$$\begin{aligned}
&U_{m+p-1}(a) + U_{m-p-1}(a) = U_{m+p-1}(\cos\varphi) + U_{m-p-1}(\cos\varphi) = \\
&\quad = \frac{\sin(m+p)\,\varphi}{\sin\varphi} + \frac{\sin(m-p)\,\varphi}{\sin\varphi} = \frac{\sin(m+p)\,\varphi + \sin(m-p)\,\varphi}{\sin\varphi} \\
&\quad = \frac{2\cdot \sin m\,\varphi \cdot \cos p\,\varphi}{\sin\varphi} = 2\cdot\cos p\,\varphi \cdot \frac{\sin m\,\varphi}{\sin\varphi} = 2\cdot T_p(\cos\varphi)\cdot U_{m-1}(\cos\varphi) \\
&\quad = 2\cdot T_p(a)\cdot U_{m-1}(a)\,, \qquad \text{q.e.d.}\,.
\end{aligned}$$

Now, from (2.2) and (1.5)–(1.6) we obtain

$$\widetilde{U}_{m+p-1}(a) = 2 \cdot \widetilde{T}_p(a) \cdot \widetilde{U}_{m-1}(a) - (-1)^p \cdot \widetilde{U}_{m-p-1}(a)\,, \tag{2.3}$$

$\forall\, m \geq p+1$, $m, p \in \mathbb{N}^*$, $\forall\, a \in \mathbb{C}$.

Dividing equation (2.3) through by $\widetilde{U}_{m-1}(a)$ and replacing m by mp, we find

$$\widetilde{\lambda}_m(a) = 2 \cdot \widetilde{T}_p(a) - \frac{(-1)^p}{\widetilde{\lambda}_{m-1}(a)}\,, \qquad m \geq 2\,. \tag{2.4}$$

The limit $\widetilde{\lambda}_\infty(a) = \lim_{m\to\infty} \widetilde{\lambda}_m(a)$, $a \in \mathbb{R}^*$, therefore satisfies the quadratic equation

$$(\widetilde{\lambda}_\infty(a))^2 - 2 \cdot \widetilde{T}_p(a) \cdot \widetilde{\lambda}_\infty(a) + (-1)^p = 0\,. \;\blacksquare \tag{2.5}$$

Remarks.

a) The limit $\widetilde{\lambda}_\infty(a) = \lim_{m\to\infty} \widetilde{\lambda}_m(a)$ can be obtained from (2.1) and

$$\widetilde{U}_n(a) = \frac{\left(a+\sqrt{a^2+1}\right)^n - \left(a-\sqrt{a^2+1}\right)^n}{2\cdot\sqrt{a^2+1}}\,, \qquad a \in \mathbb{R}\,. \tag{2.6}$$

One obtains

$$\widetilde{\lambda}_\infty(a) = \begin{cases} \left(a-\sqrt{a^2+1}\right)^p, & a<0\,, \\ \left(a+\sqrt{a^2+1}\right)^p, & a>0\,. \end{cases} \tag{2.7}$$

b) From (2.5), (1.11′) and (1.12′), we have the same result:

$$\begin{aligned}
\widetilde{\lambda}_\infty(a) &= \widetilde{T}_p(a) \pm \sqrt{a^2+1}\cdot \widetilde{U}_{p-1}(a) = \widetilde{T}_p(i\cdot b) \pm \sqrt{(i\cdot b)^2+1}\cdot \widetilde{U}_{p-1}(i\cdot b) = \\
&= i^p\cdot T_p(b) \pm i\cdot\sqrt{b^2-1}\cdot i^{p-1}\cdot U_{p-1}(b) = i^p\cdot\left(T_p(b)\pm\sqrt{b^2-1}\cdot U_{p-1}(b)\right) \\
&= i^p\cdot\left(T_p(\cos\varphi)\pm\sqrt{\cos^2\varphi-1}\cdot U_{p-1}(\cos\varphi)\right) = i^p\cdot\left(\cos p\,\varphi \pm i\sin\varphi\,\frac{\sin p\,\varphi}{\sin\varphi}\right) \\
&= i^p\cdot(\cos\varphi\pm i\sin\varphi)^p = \left(i\cdot\left(b\pm\sqrt{b^2-1}\right)\right)^p = \left(i\cdot b\pm\sqrt{(i\cdot b)^2+1}\right)^p \\
&= \left(a\pm\sqrt{a^2+1}\right)^p, \qquad \text{q.e.d.}\ .
\end{aligned}$$

Hence

$$\widetilde{\lambda}_\infty(a) = \begin{cases} \widetilde{T}_p(a) - \sqrt{a^2+1}\cdot\widetilde{U}_{p-1}(a), & a<0\,, \\ \widetilde{T}_p(a) + \sqrt{a^2+1}\cdot\widetilde{U}_{p-1}(a), & a>0\,. \end{cases} \tag{2.7′}$$

Clearly we utilized, in b), the identity

$$(\widetilde{T}_m(x))^2 - (x^2+1)\cdot(\widetilde{U}_{m-1}(x))^2 = (-1)^m\,, \qquad \forall\, x\in\mathbb{C},\ \ \forall\, m\in\mathbb{N}^*\,. \tag{2.8}$$

From (2.5), we obtain, for $a=\frac{1}{2}$, that the number $\theta^p = \frac{1}{2}\cdot(L_p+\sqrt{5}\cdot F_p)$ ($\theta = \frac{1+\sqrt{5}}{2}$ — the "golden ratio") satisfies the quadratic equation

$$x^2 - L_p\cdot x + (-1)^p = 0\,, \tag{2.9}$$

(see [1]).

3 – The Aitken sequence $(x_n^*)_{n\geq 2}$ is the sequence defined by

$$x_n^* = \frac{x_{n+1}\cdot x_{n-1} - x_n^2}{x_{n+1} - 2\cdot x_n + x_{n-1}}\,, \qquad n\geq 2\,, \tag{3.1}$$

where $(x_n)_{n\geq 1}$ is a convergent sequence.

In this section we establish the result

$$\frac{\widetilde{\lambda}_{n+r}(a)\cdot\widetilde{\lambda}_{n-r}(a) - (\widetilde{\lambda}_n(a))^2}{\widetilde{\lambda}_{n+r}(a) - 2\cdot\widetilde{\lambda}_n(a) + \widetilde{\lambda}_{n-r}(a)} = \widetilde{\lambda}_{2n}(a)\,, \tag{3.2}$$

$1\leq r<n$, where $(\widetilde{\lambda}_n(a))_{n\geq 1}$ is the sequence defined by (2.1) and $a\in\mathbb{R}^*$.

Proof: Let a be an element of $\mathbb{C}$; then $\exists\,\varphi\in\mathbb{C}$ such that $a=i\cdot\cos\varphi$, $i^2=-1$. One has

a) $\widetilde{\lambda}_{n+r}(a)\cdot\widetilde{\lambda}_{n-r}(a) - (\widetilde{\lambda}_n(a))^2 = \widetilde{\lambda}_{n+r}(i\cdot\cos\varphi)\cdot\widetilde{\lambda}_{n-r}(i\cdot\cos\varphi) - (\widetilde{\lambda}_n(i\cdot\cos\varphi))^2 =$

$$= \frac{\widetilde{U}_{p(n+r+1)-1}(i\cdot\cos\varphi)}{\widetilde{U}_{p(n+r)-1}(i\cdot\cos\varphi)}\cdot\frac{\widetilde{U}_{p(n-r+1)-1}(i\cdot\cos\varphi)}{\widetilde{U}_{p(n-r)-1}(i\cdot\cos\varphi)} - \left(\frac{\widetilde{U}_{p(n+1)-1}(i\cdot\cos\varphi)}{\widetilde{U}_{p\,n-1}(i\cdot\cos\varphi)}\right)^2$$

$$= i^p\cdot\frac{\sin p(n+r+1)\,\varphi}{\sin p(n+r)\,\varphi}\cdot i^p\cdot\frac{\sin p(n-r+1)\,\varphi}{\sin p(n-r)\,\varphi} - \left(i^p\cdot\frac{\sin p(n+1)\,\varphi}{\sin p\,n\,\varphi}\right)^2$$

$$= \dots = (-1)^p\cdot\frac{\sin p\,\varphi\cdot\sin^2 p\,r\,\varphi\cdot\sin p(2n+1)\,\varphi}{\sin^2 p\,n\,\varphi\cdot\sin p(n+1)\,\varphi\cdot\sin p(n-r)\,\varphi}\;;$$

b) $\widetilde{\lambda}_{n+r}(a) - \widetilde{\lambda}_n(a) = \widetilde{\lambda}_{n+r}(i\cdot\cos\varphi) - \widetilde{\lambda}_n(i\cdot\cos\varphi)$

$$= \dots = i^{p+1}\cdot\frac{\sin p\,\varphi\cdot\sin p\,r\,\varphi}{\sin p\,n\,\varphi\cdot\sin p(n+r)\,\varphi}\;;$$

c) $\widetilde{\lambda}_{n+r}(a) - 2\cdot\widetilde{\lambda}_n(a) + \widetilde{\lambda}_{n-r}(a) = \Big(\widetilde{\lambda}_{n+r}(a) - \widetilde{\lambda}_n(a)\Big) - \Big(\widetilde{\lambda}_n(a) - \widetilde{\lambda}_{n-r}(a)\Big)$

$$= \dots = i^p\cdot\frac{2\cdot\sin p\,\varphi\cdot\sin^2 p\,r\,\varphi\cdot\cos p\,n\,\varphi}{\sin p\,n\,\varphi\cdot\sin p(n-r)\,\varphi\cdot\sin p(n+r)\,\varphi}\;.$$

Finally, on combining a), b) and c), we derive the required result:

$$\frac{\widetilde{\lambda}_{n+r}(a)\cdot\widetilde{\lambda}_{n-r}(a) - (\widetilde{\lambda}_n(a))^2}{\widetilde{\lambda}_{n+r}(a) - 2\cdot\widetilde{\lambda}_n(a) + \widetilde{\lambda}_{n-r}(a)} = \widetilde{\lambda}_{2n}(a)\,, \qquad 1\leq r<n\;. \tag{$*$}$$

For $r=1$ in $(*)$ one obtains that the Aitken sequence $(\widetilde{\lambda}_n^*(a))_{n\geq 2}$ verifies the relation

$$\widetilde{\lambda}_n^*(a) = \widetilde{\lambda}_{2n}(a)\,, \qquad n\geq 2,\quad a\in\mathbb{R}^*\;. \blacksquare \tag{3.3}$$

4 – We began with a sequence $(\widetilde{\lambda}_n(a))_{n\geq 1}$, $a \in \mathbb{R}^*$, and applied *Aitken acceleration* to give a sequence $(\widetilde{\lambda}^*_n(a))_{n\geq 2}$. We now investigate the application of Aitken acceleration to $(\widetilde{\lambda}^*_n(a))_{n\geq 2}$, $a \in \mathbb{R}^*$, and so on, repeatedly. It is helpful to use a different notation. Let us write

$$\widetilde{\lambda}^{(k+1)}_n(a) = \frac{\widetilde{\lambda}^{(k)}_{n+1}(a) \cdot \widetilde{\lambda}^{(k)}_{n-1}(a) - \Big(\widetilde{\lambda}^{(k)}_n(a)\Big)^2}{\widetilde{\lambda}^{(k)}_{n+1}(a) - 2 \cdot \widetilde{\lambda}^{(k)}_n(a) + \widetilde{\lambda}^{(k)}_{n-1}(a)} , \tag{4.1}$$

for $k = 0, 1, 2, 3, ...$, where $\widetilde{\lambda}^{(0)}_n(a) = \widetilde{\lambda}_n(a)$, $a \in \mathbb{R}^*$, $n \in \mathbb{N}^*$.

Thus $(\widetilde{\lambda}^{(0)}_n(a))_{n\geq 1}$ is our original sequence and $(\widetilde{\lambda}^{(1)}_n(a))_{n\geq 2}$ is the sequence which we have hitherto called $(\widetilde{\lambda}^*_n(a))_{n\geq 2}$. For $k \geq 0$, the $(k+1)^{\text{th}}$ sequence $(\widetilde{\lambda}^{(k+1)}_n(a))_{n\geq k+1}$ is obtained by using Aitken acceleration on the sequence $(\widetilde{\lambda}^{(k)}_n(a))_{n\geq k}$.

We have already seen from (3.3) that

$$\widetilde{\lambda}^{(1)}_n(a) = \widetilde{\lambda}_{2n}(a) , \quad n \geq 2, \quad a \in \mathbb{R}^* . \tag{4.2}$$

It follows from this and (4.1) that

$$\widetilde{\lambda}^{(2)}_n(a) = \frac{\lambda^1_{n+1}(a) \cdot \widetilde{\lambda}^{(1)}_{n-1}(a) - (\widetilde{\lambda}^{(1)}_n(a))^2}{\widetilde{\lambda}^{(1)}_{n+1}(a) - 2 \cdot \widetilde{\lambda}^{(1)}_n(a) + \widetilde{\lambda}^1_{n-1}(a)} = \frac{\widetilde{\lambda}_{2n+1}(a) \cdot \widetilde{\lambda}_{2n-1}(a) - (\widetilde{\lambda}_{2n}(a))^2}{\widetilde{\lambda}_{2n+2}(a) - 2\,\widetilde{\lambda}_{2n}(a) + \widetilde{\lambda}_{2n-2}(a)} .$$

Using (3.2) with n replaced by $2n$ and with $r = 2$, we deduce that

$$\lambda^{(2)}_n(a) = \widetilde{\lambda}_{4n}(a) , \quad n \geq 3 . \tag{4.3}$$

Finally, it follows by induction that

$$\widetilde{\lambda}^{(k)}_n(a) = \widetilde{\lambda}_{n\cdot 2^k}(a) \underset{k}{\longrightarrow} \widetilde{\lambda}_\infty(a) , \tag{4.4}$$

which holds for each $k \geq 0$ and all $n \geq k+1$.

5 – Let us consider now *Newton's method*: given an initial approximation a_0 to the number $\widetilde{\lambda}_\infty(a) = (a+\sqrt{a^2+1})^p$, $a \in \mathbb{R}^*$, we compute the Newton sequence $(a_n)_{n\geq 0}$ from

$$\begin{aligned} a_{n+1} &= a_n - \frac{a_n^2 - 2 \cdot \widetilde{T}_p(a) \cdot a_n + (-1)^p}{2 \cdot \Big(a_n - \widetilde{T}_p(a)\Big)} \\ &= \frac{a_n^2 - (-1)^p}{2 \cdot \Big(a_n - \widetilde{T}_p(a)\Big)} , \quad n \geq 0 . \end{aligned} \tag{5.1}$$

If $a_n = \widetilde{\lambda}_k(a)$ for some values of $k \in \mathbb{N}^*$, we find that

$$\begin{aligned} a_{n+1} &= \frac{a_n^2 - (-1)^p}{2 \cdot \left(a_n - \widetilde{T}_p(a)\right)} = \frac{(\widetilde{\lambda}_k(a))^2 - (-1)^p}{2 \cdot \left(\widetilde{\lambda}_k(a) - \widetilde{T}_p(a)\right)} = \ldots \\ &= \frac{(\widetilde{U}_{p(k+1)-1}(a))^2 - (-1)^p \cdot (\widetilde{U}_{pk-1}(a))^2}{2 \cdot \widetilde{U}_{pk-1}(a) \cdot \left(\widetilde{U}_{p(k+1)-1}(a) - \widetilde{T}_p(a) \cdot \widetilde{U}_{pk-1}(a)\right)} \\ &= \frac{\widetilde{U}_{p\cdot(2k+1)-1}(a)}{\widetilde{U}_{p\cdot 2k-1}(a)} = \widetilde{\lambda}_{2k}(a) \ , \end{aligned}$$

i.e.,

$$a_{n+1} = \widetilde{\lambda}_{2k}(a) \, , \qquad a \in \mathbb{R}^* \ , \tag{5.2}$$

since:

1) $(\widetilde{U}_{p\cdot(k+1)-1}(a))^2 - (-1)^p \cdot (\widetilde{U}_{pk-1}(a))^2 = \widetilde{U}_{p-1}(a) \cdot \widetilde{U}_{p\cdot(2k+1)-1}(a)$,
$\forall\, a \in \mathbb{C}$, $\forall\, p \in \mathbb{N}^*$, $\forall\, k \in \mathbb{N}^*$;

2) $\widetilde{U}_{p\cdot(2k+1)-1}(a) - \widetilde{T}_p(a) \cdot \widetilde{U}_{pk-1}(a) = \widetilde{U}_{p-1}(a) \cdot \widetilde{T}_{pk}(a)$,
$\forall\, a \in \mathbb{C}$, $\forall\, p \in \mathbb{N}^*$, $\forall\, k \in \mathbb{N}^*$;

3) $\widetilde{U}_{2m-1}(a) = 2 \cdot \widetilde{T}_m(a) \cdot \widetilde{U}_{m-1}(a)$, $\forall\, a \in \mathbb{C}$, $\forall\, m \in \mathbb{N}^*$.

Thus, if we choose as the initial approximant $a_0 = \widetilde{\lambda}_1(a) = \frac{\widetilde{U}_{2p-1}(a)}{\widetilde{U}_{p-1}(a)} = 2 \cdot \widetilde{T}_p(a)$, $a \in \mathbb{R}^*$, we see by induction that

$$\begin{aligned} a_1 = a_{0+1} &= \widetilde{\lambda}_{2\cdot 1}(a) = \widetilde{\lambda}_2(a) \ , \\ a_1 &= \widetilde{\lambda}_2(a) \ ; \end{aligned} \tag{5.2$'$}$$

$$\begin{aligned} a_2 = a_{1+1} &= \widetilde{\lambda}_{2\cdot 2}(a) \ , \\ a_2 &= \widetilde{\lambda}_{2^2}(a) \ ; \end{aligned} \tag{5.2$''$}$$

$$\begin{aligned} a_3 = a_{2+1} &= \widetilde{\lambda}_{2\cdot 2^2}(a) \ , \\ a_3 &= \widetilde{\lambda}_{2^3}(a) \ . \end{aligned} \tag{5.2$'''$}$$

Hence

$$a_n = \widetilde{\lambda}_{2^n}(a) \underset{n}{\longrightarrow} \widetilde{\lambda}_\infty(a) \ . \tag{5.3}$$

6 – Finally, we consider *the secant method*, in which we approximate to a root of the equation $f(x)=0$ as follows: we choose two initials approximants a_1 and a_2 and compute the sequence $(a_n)_{n\geq 2}$ from

$$a_{n+1} = a_n - \frac{f(a_n)\cdot(a_n - a_{n-1})}{f(a_n) - f(a_{n-1})}\,, \qquad n \geq 2\ . \tag{6.1}$$

In our case $f(x) = x^2 - 2\cdot \widetilde{T}_p(a)\cdot x + (-1)^p$ and, if we choose $a_1 = \widetilde{\lambda}_k(a)$ and $a_2 = \widetilde{\lambda}_m(a)$, $a\in\mathbb{R}^*$, for some $m,k\in\mathbb{N}^*$, $m\neq k$, we find that

$$a_3 = \frac{a_1\cdot f(a_2) - a_2\cdot f(a_1)}{f(a_2) - f(a_1)} = \frac{a_1\cdot a_2 - (-1)^p}{a_1 + a_2 - 2\cdot\widetilde{T}_p(a)} = \dots =$$

$$= \frac{\widetilde{U}_{p(m+1)-1}(a)\cdot\widetilde{U}_{p(k+1)-1}(a) - (-1)^p\cdot\widetilde{U}_{pm-1}(a)\cdot\widetilde{U}_{pk-1}(a)}{\widetilde{U}_{pk-1}(a)\cdot\Big(\widetilde{U}_{p(m+1)-1}(a) - \widetilde{T}_p(a)\cdot\widetilde{U}_{pm-1}(a)\Big) + \widetilde{U}_{pm-1}(a)\cdot\Big(\widetilde{U}_{p(k+1)-1}(a) - \widetilde{T}_p(a)\cdot\widetilde{U}_{pk-1}(a)\Big)}$$

$$= \frac{\widetilde{U}_{p(m+k+1)-1}(a)}{\widetilde{U}_{p(m+k)-1}(a)} = \widetilde{\lambda}_{m+k}(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

since:

1′) $\widetilde{U}_{p(m+1)-1}(a)\cdot\widetilde{U}_{p(k+1)-1}(a) - (-1)^p\cdot\widetilde{U}_{pm-1}(a)\cdot\widetilde{U}_{pk-1}(a) = \widetilde{U}_{p-1}(a)\cdot\widetilde{U}_{p(m+k+1)-1}(a)$, $\forall\, a\in\mathbb{C}$, $\forall\, p,m,k\in\mathbb{N}^*$;

2′) $\widetilde{U}_{p(n+1)-1}(a) - \widetilde{T}_p(a)\cdot\widetilde{U}_{pn-1}(a) = \widetilde{U}_{p-1}(a)\cdot\widetilde{T}_{pn}(a)$, $\forall\, a\in\mathbb{C}$, $\forall\, p,n\in\mathbb{N}^*$;

3′) $\widetilde{U}_{pk-1}(a)\cdot\widetilde{T}_{pm}(a) + \widetilde{U}_{pm-1}(a)\cdot\widetilde{T}_{pk}(a) = \widetilde{U}_{p(m+k)-1}(a)$, $\forall\, a\in\mathbb{C}$, $\forall\, p,m,k\in\mathbb{N}^*$.

Hence

$$a_3 = \widetilde{\lambda}_{m+k}(a)\,, \qquad m,k\in\mathbb{N}^*,\quad m\neq k\ . \tag{6.2}$$

An induction argument shows that, if we choose as initial values $a_1 = \widetilde{\lambda}_1(a) = 2\cdot\widetilde{T}_p(a)$ and $a_2 = \widetilde{\lambda}_2(a) = 2\cdot\widetilde{T}_p(a) - \frac{(-1)^p}{\widetilde{\lambda}_1(a)}$ (see (2.4)), $a\in\mathbb{R}^*$, *the secant method* gives:

$$a_1 = \widetilde{\lambda}_1(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

$$a_2 = \widetilde{\lambda}_2(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

$$a_3 = \widetilde{\lambda}_{m+k}(a) = \widetilde{\lambda}_{1+2}(a) = \widetilde{\lambda}_3(a) = \widetilde{\lambda}_{F_4}(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

$$a_4 = \widetilde{\lambda}_{m+k}(a) = \widetilde{\lambda}_{2+3}(a) = \widetilde{\lambda}_{F_5}(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

$$a_5 = \widetilde{\lambda}_{m+k}(a) = \widetilde{\lambda}_{3+5}(a) = \widetilde{\lambda}_8(a) = \widetilde{\lambda}_{F_6}(a)\,, \qquad a\in\mathbb{R}^*\ ,$$

i.e.,

$$a_n = \widetilde{\lambda}_{F_{n+1}}(a) \underset{n}{\longrightarrow} \widetilde{\lambda}_\infty(a)\ , \tag{6.3}$$

as the n^{th} approximant to the number $\widetilde{\lambda}_\infty(a) = (a \pm \sqrt{a^2+1})^p = \widetilde{T}_p(a) \pm \sqrt{a^2+1} \cdot \widetilde{U}_{p-1}(a)$, $a \in \mathbb{R}^*$, where $(F_n)_{n\geq 0}$ is the sequence of Fibonacci numbers.

In conclusion, we see that for $a = \frac{1}{2}$ and $p \in \mathbb{N}^*$, one obtains the Jamieson's results (see (1)) and for $a = \frac{1}{2}$ and $p = 1$, we derive the Phillips's results (see (2)).

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