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FRÉCHET-VALUED ANALYTIC FUNCTIONS AND LINEAR TOPOLOGICAL INVARIANTS

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Abstract: Let E, F be Fréchet spaces and D an open set in E. The main aim of this paper is to prove that every analytic function $f\colon D \to F$ (resp. $f\colon D \to \mathcal{H}(F')$ where F is Montel) which is weakly analytically extended to Ω is analytically extended to Ω when $\dim E < \infty$ and $F \in (\mathrm{DN})$ (resp. $F \in \overline{\mathrm{DN}}$). Moreover it also shows that every function on $D \times G$ which is holomorphic in $z \in D$ and weakly analytic in $x \in D$ is analytic.

Introduction

Let E, F be locally convex spaces and D an open set in E. Assume that $f\colon D \to F$ is an analytic function. We say that f is weakly analytically extended to Ω, a domain in E containing D, if $u \circ f$ is analytically extended to Ω for all $u \in F'$, the dual space of F. In [9] Ligocka and Siciak have been proved that if E is metrizable and Baire, F' is Baire and f is weakly analytically extended to Ω, then f can be analytically extended to Ω. A counterexample for the case where F' is not Baire was also given by them.

This is different with the holomorphic case, which was investigated in a collection of papers of Bogdanowicz [2], [3], [4], [5] and later of Nachbin [10], [11], Nguyen Thanh Van [12]. One of their main results is the equivalence of the weakly holomorphic and holomorphic extension in the case where E is metrizable.

The aim of the present note is to consider the result of Ligocka and Siciak for Fréchet-valued analytic functions and the recent one of Shiffman [14] about the analyticity of separately analytic functions in the case of infinite dimension.

The problem is formulated in terms of linear topological invariants which were introduced and investigated by Vogt in the 1980's and the proof of the other concerns the Nachbin topology of the spaces of the holomorphic functions.

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To formulate the results we recall the following.

Let E be a Fréchet space with an increasing fundamental system of semi-norms $\{\|\cdot\|_k\}_{k=1}^{\infty}$. We say that E has the property

$$\left.\begin{array}{ll}(\mathrm{DN}) & \exists\, p \geq 1 \;\; \forall\, q \geq 1 \;\; \forall\, d > 0 \;\; \exists\, k \geq q, \;\; C > 0 \\ (\overline{\mathrm{DN}}) & \exists\, p \geq 1 \;\; \partial q \geq 1 \;\; \exists\, k \geq q \;\; \forall\, d > 0 \;\; \exists\, C > 0\end{array}\right\} : \;\; \|\cdot\|_q^{1+d} \leq C \|\cdot\|_k \|\cdot\|_p^d \, .$$

The properties (DN) and $(\overline{\mathrm{DN}})$ and many other ones were introduced and investigated by Vogt ([15], [16], ...).

In this note we prove the following three Theorems

Theorem 1. *Let E, F be real Fréchet spaces and let $f\colon D \to F$ be analytic. If f is weakly analytically extended to Ω, then f is analytically extended to Ω when one of the following holds*

(**i**) $\dim E < \infty$ *and* $F \in (\mathrm{DN})$;

(**ii**) $F \in (\overline{\mathrm{DN}})$.

Theorem 2. *Let E be a real Fréchet space and F a complex Fréchet Montel space. Assume $f\colon D \to \mathcal{H}(F')$ is an analytic function, where F' is the strong dual of F. If $\delta_u f$ is analytically extended to Ω, where*

$$(\delta_u f)(x) = f(x)(u) \quad \text{for } \; x \in D \; \text{ and } \; u \in F'$$

then f is analytically extended to Ω when one of the following holds

(**i**) $\dim E < \infty$ *and* $F \in (\mathrm{DN})$;

(**ii**) $F \in (\overline{\mathrm{DN}})$.

Theorem 3. *Let E be a real Fréchet space and F a complex Fréchet space. Assume $f\colon D \times G \to \mathbb{C}$ is a function, where $D \times G$ is open in $E \times F$, satisfying the following*

$$f_x\colon z \mapsto f(x,z) \;\; \text{is holomorphic on } G \quad \text{for } \; x \in D$$

and

$$f^{\mu}\colon x \mapsto \mu(f(x,z)) \;\; \text{is analytic on } D \quad \text{for } \; \mu \in [\mathcal{H}(G)]' \, .$$

Then f is analytic.

The proofs of Theorems 1, 2 and 3 are given in §1, §2 and §3 respectively.

In §4 we show a necessary condition for a complex space X which satisfies that every weakly analytic function with values in $\mathcal{H}(X)$ is analytic.

Remark. In the case where D and G are open sets in $\mathbb{R}^m$ and $\mathbb{C}^n$ respectively with f_x is holomorphic for $x \in D$ and f^z is analytic for z belongs to a nonpluripolar set in G, Theorem 3 was proved by Shiffman [14].

1 – Proof of Theorem 1

To prove the Theorem 1, we first prove the following result.

Let F be a real Fréchet space with strong dual F' and F'_{bor} the bornological space associated to F'. We have

1.1 Lemma. *Let F be a real Fréchet space with $F \in$ (DN). Then $[F'_{\mathrm{bor}}]' \in$ (DN).*

Proof: It is known [16] that $F \in$ (DN) if and only if

$$\exists p \ \ \forall q \ \ \exists k, \ \ C > 0\colon \ \ \|\ \|_q \le Cr\|\ \|_k + \frac{1}{r}\|\ \|_p \ \ \forall r > 0$$

if and only if

$$\exists p \ \ \forall q \ \ \exists k, \ \ C > 0\colon \ \ U_q^0 \subseteq Cr\, U_k^0 + \frac{1}{r}\, U_p^0 \ \ \forall r > 0 \ .$$

Thus

$$\|u\|''_q = \sup_{x' \in U_q^0} |u(x')| \le \sup_{x' \in Cr\, U_k^0 + \frac{1}{r} U_p^0} |u(x')| \le$$
$$\le Cr \sup_{x' \in U_k^0} |u(x')| + \frac{1}{r} \sup_{x' \in U_p^0} |u(x')| = Cr\|u\|''_k + \frac{1}{r}\|u\|''_p$$

for all $r > 0$ and $u \in [F'_{\mathrm{bor}}]'$.

This means $[F'_{\mathrm{bor}}]' \in$ (DN). ∎

Proof of Theorem: (**i**) One can assume that $E = \mathbb{R}^n$ and Ω is connected. It suffices to show that f is analytically extended at every $x^0 \in \partial D \cap \Omega$.

Choose a neighbourhood $W = I_1 \times I_2 \times ... \times I_n$ of x^0 in Ω, where

$$I_i = [a_i, b_i]\,, \quad a_i < b_i, \ \ i = 1, ..., n \ .$$

For each $0 < \varepsilon < 1$, consider the linear map

$$S_\varepsilon\colon\ F' \to A(\varepsilon W)\ ,$$

given by

$$S_\varepsilon(u)(x) = u\,f(x) \quad \text{ for } \ u \in F', \ \ x \in \varepsilon W\ ,$$

where $A(\varepsilon W)$ is the space of analytic functions on εW. By the uniqueness, S_ε has the closed graph.

On the other hand, since

$$A(\varepsilon W) := \lim_{\widetilde{W}\downarrow\varepsilon W} \operatorname{ind} \mathcal{H}^\infty(\widetilde{W}) = \mathcal{H}(\varepsilon W)\ ,$$

where $\mathcal{H}^\infty(\widetilde{W})$ denotes the Banach space of bounded holomorphic functions on a neighbourhood $\widetilde{W}$ of εW in $\mathbb{C}^n$, it follows that $S_\varepsilon\colon\ F'_{\text{bor}} \to A(\varepsilon W)$ is continuous.

From the relations:

$$\begin{aligned}[\mathcal{H}(\varepsilon W)]' &\cong \Big[\mathcal{H}(\varepsilon I_1)\,\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi\, \mathcal{H}(\varepsilon I_n)\Big]' \\ &\cong \mathcal{H}(\overline{\mathbb{C}}\backslash\varepsilon I_1)\,\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi\, \mathcal{H}(\overline{\mathbb{C}}\backslash\varepsilon I_n) \\ &\cong \mathcal{H}(\Delta)\,\widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi\, \mathcal{H}(\Delta) \cong \mathcal{H}(\Delta^n) \in (\widetilde{\Omega}) \quad [16]\end{aligned}$$

where

$$\Delta = \Big\{\lambda \in \mathbb{C}\colon\ |\lambda| < 1\Big\}$$

and, by Lemma 1.1,

$$[F'_{\text{bor}}]' \in (\text{DN})$$

by Vogt [16], we can find a neighbourhood $\widetilde{W}_\varepsilon$ of εW in $\mathbb{C}^n$ such that $S_\varepsilon\colon\ F'_{\text{bor}} \to \mathcal{H}^\infty(\widetilde{W}_\varepsilon)$ is continuous.

Define a holomorphic extension

$$\widehat{f}_\varepsilon\colon\ \widetilde{W}_\varepsilon \to [F'_{\text{bor}}]'$$

by

$$\widehat{f}_\varepsilon(z)(u) = S_\varepsilon(u)(z) \quad \text{ for } \ z \in \widetilde{W}_\varepsilon, \ \ u \in F'_{\text{bor}}\ .$$

By the uniqueness the family $\{\widehat{f}_\varepsilon\}$ defines a holomorphic extension $\widehat{f}$ of f to a neighbourhood $\widetilde{W}$ of W in $\mathbb{C}^n$.

Since $\widehat{f}(W \cap D) \subset F$ and F is a closed subspace of $[F'_{\text{bor}}]'$ we have $\widetilde{f}(\widetilde{W}) \subset F$. This means f can be analytically extended at x^0.

(ii) a) First assume that E is a Banach space. By (i) f is extended to a Gateaux analytic function $g\colon\ \Omega \to F$. It remains to check that g is analytic.

Given $x^0 \in \Omega$. Consider the Taylor expansion of g at x^0,

$$g(x) = \sum_{n\geq 0} P_n\, g(x) \ .$$

By Ligocka and Siciak [9],

$$g_q := \omega_q\, g\colon\ \Omega \to F_q \quad \text{ is analytic for all } \ q \geq 1 \ ,$$

where F_q is the Banach space associated to the semi-norm $\|\cdot\|_q$ and $\omega_q\colon\ F \to F_q$ is the canonical map.

This yields the continuity of all $P_n\, g$.

For each $q \geq 1$, put

$$s_q = \limsup_n \sqrt[n]{\|P_n\, g\|_q} \quad \text{ and } \quad r_q = \frac{1}{s_q} \ ,$$

where

$$\|P_n\, g\|_q = \sup\Big\{\|P_n\, g(x)\|_q\colon\ \|x\| \leq 1\Big\} \ .$$

We have

$$r_q > 0 \quad \text{ for } \ q \geq 1 \ .$$

Since $F \in (\overline{\mathrm{DN}})$, we have

$$\exists\, p \ \ \forall\, q \geq p \ \ \exists\, k(q) \ \ \forall\, d > 0 \ \ \exists\, C_d > 0\colon\ \ \|\cdot\|_q \leq C_d^{\frac{1}{1+d}}\, \|\cdot\|_k^{\frac{1}{1+d}}\, \|\cdot\|_p^{\frac{1}{1+d}} \ .$$

Then

$$\begin{aligned} s_q &\leq \lim_n \Big(C_d^{\frac{1}{1+d}}\Big)^{1/n} \Big(\limsup_n \|P_n\, f\|_k^{1/n}\Big)^{\frac{1}{1+d}} \Big(\limsup_n \|P_n\, f\|_p^{1/n}\Big)^{\frac{d}{1+d}} \\ &= s_k^{\frac{1}{1+d}}\, s_p^{\frac{d}{1+d}} \ . \end{aligned}$$

As $d \to \infty$ we get

$$s_q \leq s_p \quad \text{ for } \ q \geq p$$

and hence

$$r_q \geq r_p \quad \text{ for } \ q \geq p \ .$$

This yields that g is analytic at x^0.

b) General case. By (i) f is extended to a Gateaux analytic function $g\colon \Omega \to F$.

Let $x^0 \in \Omega$. For $B \in \mathcal{B}(E)$, the family of all balanced convex compact sets in E, write

$$E_{\mathbb{C}} = E \oplus iE\,, \quad F_{\mathbb{C}} = F \oplus iF \quad \text{ and } \quad E(B)_{\mathbb{C}} = E(B) \oplus iE(B) \ ,$$

where $E(B)$ denotes the Banach space spanned by B.

By a) for each $B \in \mathcal{B}(E)$ with $x^0 \in E(B)$ there exists a convex neighbourhood W_B of $0 \in E(B)_{\mathbb{C}}$ and a holomorphic function $g_B\colon\ x^0 + W_B \to F_{\mathbb{C}}$ such that

$$g_B\Big|_{(x^0+W_B)\cap\Omega} = g\Big|_{(x^0+W_B)\cap\Omega} \ .$$

Put

$$W = \bigcup\Big\{x^0 + W_B\colon\ B \in \mathcal{B}(E)\Big\} \ .$$

By the uniqueness the family $\{g_B\}$ defines a function $\widehat{g}\colon\ W \to F_{\mathbb{C}}$ such that

$$\widehat{g}\Big|_{W\cap\Omega} = g\Big|_{W\cap\Omega} \ .$$

It remains to check that W is a neighbourhood of x^0 in $E_{\mathbb{C}}$.

Otherwise, there exists a sequence $\{z_n\}$, $z_n = x_n + iy_n \notin W$ for $n \geq 1$, converging to x^0. Let $B = \overline{\mathrm{conv}}\{x_n, y_n\}$. Choose $B_1 \in \mathcal{B}(E)$, $B \subset B_1$ such that canonical map $E(B)_{\mathbb{C}} \to E_{\mathbb{C}}(B_1)$ is compact. Such a set B_1 exists by [8].

Then $z_n \to x^0$ in $E(B_1)$ and hence

$$z_n \in x^0 + W_{B_1} \subset W \quad \text{for } n \text{ sufficiently large} \ .$$

This is impossible. ∎

2 – Proof of Theorem 2

For the proof of Theorem 2, we recall the following.

Let $\{\|\ \|_k\}_{k=1}^{\infty}$ be a fundamental system of continuous semi-norms of a Fréchet space E. For each subset B of E consider the general semi-norm

$$\|\ \|_B^*\colon\ E^* \to [0, +\infty]$$

given by

$$\|u\|_B^* = \Big\{\sup |u(x)|\colon\ x \in B\Big\} \ .$$

Write $\|\ \|_k^*$ for $B = U_k = \{x \in E\colon\ \|x\|_k \leq 1\}$.

We say that E has the property $(\widetilde{\Omega})$ if and only if

$(\widetilde{\Omega})$

$$\forall\, p \geq 1\ \ \exists\, q \geq 1\ \ \exists\, d > 0\ \ \forall\, k \geq 1,\ \ \exists\, C > 0\colon\ \ \|y\|_q^{*1+d} \leq C \|y\|_k^* \, \|y\|_p^{*d} \quad \text{for}\ \ y \in E^*.$$

The properties $(\widetilde{\Omega})$ and other properties were introduced and investigated by Vogt (see [15], [16], ...).

For the proof of Theorem 2 we need the following two Lemmas

2.1 Lemma. *Let B be a Banach space and E a Fréchet space with $E \in (\widetilde{\Omega})$. Assume that $f\colon B \to E^*$ is a holomorphic function of bounded type. Then there exists a neighbourhood V of 0 in E such that:*

$$\sup\Big\{\|f(x)\|_V^*\colon \|x\| < r\Big\} < \infty \quad \text{for all } r > 0 \,. \tag{1}$$

Proof: Let $\{\|\cdot\|_\gamma\}_{\gamma=1}^\infty$ be a fundamental system of semi-norms of E and let U be the unit ball of B. Since $f(U)$ is bounded in E^*, there exists $\alpha \geq 1$ such that

$$M(\alpha, 1) = \sup\Big\{\|f(x)\|_\alpha^*\colon x \in U\Big\} < \infty \,.$$

By the hypothesis $E \in (\widetilde{\Omega})$ we can find $\beta \geq \alpha$ and $d > 0$ such that

$$\forall\, \gamma \geq \beta \;\; \exists\, C > 0\colon \;\; \|\cdot\|_\beta^{*1+d} \leq C \|\cdot\|_\gamma^* \, \|\cdot\|_\alpha^{*d} \,. \tag{2}$$

We check that (1) is satisfied for $V = \{y \in E\colon \|y\|_\beta < 1\}$. Indeed, fix $r > 1$. Choose γ such that

$$M(\gamma, \rho) = \sup\Big\{\|f(x)\|_\gamma^*\colon \|x\| < \rho\Big\} < \infty \,,$$

where $\rho = (er)^{1+d} + 1$.

Writing the Taylor expansion of f at $0 \in B$,

$$f(x) = \sum_{n\geq 0} P_n f(x) \,, \qquad x \in B \,,$$

we have

$$\begin{aligned}
\|f(x)\|_\beta^* &\leq \sum_{n\geq 0} \|P_n f(x)\|_\beta^* = \sum_{n\geq 0} r^n \left\| P_n f\left(\frac{x}{r}\right)\right\|_\beta^* \\
&\leq C^{\frac{1}{1+d}} \sum_{n\geq 0} r^n \left\| P_n f\left(\frac{x}{r}\right)\right\|_\gamma^{*\frac{1}{1+d}} \left\| P_n f\left(\frac{x}{r}\right)\right\|_\alpha^{*\frac{d}{1+d}} \\
&\leq C^{\frac{1}{1+d}} \, M(\gamma,\rho)^{\frac{1}{1+d}} \, M(\alpha,1)^{\frac{d}{1+d}} \sum_{n\geq 0} \left(\frac{r}{\rho^{\frac{1}{1+d}}}\right)^n \frac{n^n}{n!} < \infty \,. \;\blacksquare
\end{aligned}$$

2.2 Lemma. *Let E and F be Fréchet space with $E \in (\widetilde{\Omega})$ and $F \in$ (DN). Assume that F is a Montel space. Then every holomorphic function from F^* into E^* can be factored through a Banach space.*

Proof: Given $f: F^* \to E^*$ a holomorphic function. By Vogt [15], F is isomorphic to a subspace of the space $B \widehat{\otimes}_\pi s$ for some Banach space B, where s is the space of rapidly decreasing sequences.

Since the restriction map R from $[B \widehat{\otimes}_\pi s]^* \cong B^* \widehat{\otimes}_\pi s^*$ into F^* is open, it suffices to prove the theorem for $g = fR$.

On the other hand, since every Banach space is isomorphic to a quotient space of the space $\ell^1(I)$ for some index set I, without loss of generality we may assume that $B^* \cong \ell^1(I)$.

For each $k \geq 1$ put

$$A_k = \left\{ u = (\eta_{ij})_{i \in I,\, j \geq 1} \subset \mathbb{C}\colon\ |||u|||_k = \sum_{i \in I,\, j \geq 1} |\eta_{ij}|\, j^{-k} < \infty \right\} .$$

It follows that:

$$\ell^1(I) \widehat{\otimes}_\pi s^* \cong \lim_k \operatorname{ind} A_k$$

and

$$\exists\, p \geq 1 \ \ \forall\, q \geq p, \ \ d > 0 \ \ \exists\, k \geq q, \ \ D > 0\colon\ |||\delta_{ij}|||_q^{1+d} \geq D |||\delta_{ij}|||_k \, |||\delta_{ij}|||_p^d \ , \tag{3}$$

for all $i \in I$ and all $j \geq 1$, where $\{\delta_{ij}\}$ is the canonical basis (which is not necessarily countable) of $\ell^1(I) \widehat{\otimes}_\pi s^*$.

By Lemma 2.1 for each $k \geq 1$ there exists $\gamma = \gamma(k)$ such that

$$M(k, \gamma, r) < \infty \quad \text{for all } \ r > 0 \ . \tag{4}$$

Put $\alpha = \gamma(p)$ and take $\beta \geq \alpha$, $d > 0$ such that (2) holds. We check that g is a holomorphic function from $\ell^1(I) \widehat{\otimes}_\pi s^*$ into E^*_β. Fix $q \geq p$. For q and d take $k \geq q$ and $D > 0$ such that (3) holds.

Then for every $u \in A_q$, $|||u|||_q < r$, we have

$$\begin{aligned}
\|g(u)\|^*_\beta &\leq \sum_{n \geq 0} \sum_{\substack{i_1, \dots, i_n \in I \\ j_1, \dots, j_n \geq 1}} |\eta_{i_1 j_1}| \cdots |\eta_{i_n j_n}| \, |||\delta_{i_1 j_1}|||_q \cdots |||\delta_{i_n j_n}|||_q \cdot \frac{\left\| P_n\, g(\delta_{i_1 j_1}, \dots, \delta_{i_n j_n}) \right\|^*_\beta}{|||\delta_{i_1 j_1}|||_q \cdots |||\delta_{i_n j_n}|||_q} \\
&\leq \sum_{n \geq 0} \sum_{\substack{i_1, \dots, i_n \in I \\ j_1, \dots, j_n \geq 1}} |\eta_{i_1 j_1}| \cdots |\eta_{i_n j_n}| \, |||\delta_{i_1 j_1}|||_q \cdots |||\delta_{i_n j_n}|||_q \cdot \frac{C^{\frac{1}{1+d}}}{D^{\frac{n}{1+d}}} \times \\
&\quad \times \frac{\left\| P_n\, g(\delta_{i_1 j_1}, \dots, \delta_{i_n j_n}) \right\|_\gamma^{*\frac{1}{1+d}} \left\| P_n\, g(\delta_{i_1 j_1}, \dots, \delta_{i_n j_n}) \right\|_\alpha^{*\frac{d}{1+d}}}{|||\delta_{i_1 j_1}|||_k^{\frac{1}{1+d}} \cdots |||\delta_{i_n j_n}|||_k^{\frac{1}{1+d}} \, |||\delta_{i_1 j_1}|||_p^{\frac{d}{1+d}} \cdots |||\delta_{i_n j_n}|||_p^{\frac{d}{1+d}}} \\
&\leq C^{\frac{1}{1+d}} \, M(k, \gamma, \rho)^{\frac{1}{1+d}} \, M(p, \alpha, \rho)^{\frac{d}{1+d}} \sum_{n \geq 0} \frac{n^n \, r^n}{D^{\frac{n}{1+d}} \rho^n \, n!} < \infty
\end{aligned}$$

for ρ sufficiently large. ∎

Now we are able to prove the Theorem.

As in Theorem 1 it suffices to prove (i).

Given $f : D \to \mathcal{H}(F')$ an analytic function such that $\delta_u f$ is analytically extended to Ω for every $u \in F'$, where D is an open set in $\mathbb{R}^n$ and F is a complex Fréchet–Montel space with $F \in (\mathrm{DN})$.

Fix $x^0 \in \partial D \cap \Omega$ and choose a cube neighbourhood W of x^0 in Ω.

Consider the function $\widehat{f}\colon F' \to \mathcal{H}(W)$ induced by f.

a) First we show that $\widehat{f}$ is holomorphic.

It suffices to show that $\widehat{f}_k\colon F'_k \to \mathcal{H}(W)$ is holomorphic for $k \geq 1$.

Let $\{U_m\}$ be a neighbourhood basis of W in $\mathbb{C}^n$. For each $m \geq 1$, put

$$A_m = \Big\{u \in F'_k\colon \|\delta_u f\|_{U_m} \leq m\Big\} .$$

It follows that A_m are closed in F'_k and $F'_k = \bigcup_{m=1} A_m$. From Baire's Theorem there exists m_0 such that $V = \operatorname{int} A_{m_0} \neq \emptyset$.

Consider the function $g\colon (W \times F'_k) \cup (U_{m_0} \times V) \to \mathbb{C}$ defined by

$$g(x,u) = \widehat{f}_k(u)(x) .$$

Then g is separately holomorphic. Since W is non-pluripolar and V is non-empty open, by [13] for every finite dimensional subspace $L \subset F'_k$, there exists a unique holomorphic function g_L, on $U_{m_0} \times L$ extending $g|_{(W\times L)\cup(U_{m_0}\times V\cap L)}$. Then the family $\{g_L\}$, by the Zorn Theorem [18], defines a holomorphic function $\widehat{g}\colon U_{m_0} \times F'_k \to \mathbb{C}$. This yields the holomorphicity of $\widehat{f}_k\colon F'_k \to \mathcal{H}(W)$.

b) Since $[\mathcal{H}(W)]' \in (\widetilde{\Omega})$ [17], by Lemma 2.2, we find a continuous semi-norm ρ on F' and a holomorphic function $\widehat{f}_\rho : F'_\rho \to \mathcal{H}(W)$ such that $f = \widehat{f}_\rho \, \omega_\rho$. Lemma 2.1 yields a neighbourhood U of W in $\mathbb{C}^n$ such that $\widehat{f}_\rho : F'_\rho \to \mathcal{H}(U)$ is holomorphic. Hence $f : D \to \mathcal{H}(F')$ is holomorphically extended to U, a neighbourhood of x^0 in $\mathbb{C}^n$. ∎

3 – Proof of Theorem 3

Without loss of generality we may assume that G is balanced.

(**i**) First consider the case where $\mathcal{H}(G) \cong [\mathcal{H}(G), \tau_\omega]$, where τ_ω denotes the Nachbin topology.

Consider the function $\widehat{f}\colon D \to \mathcal{H}(G)$ associated to f.

Since $\mathcal{H}(G) \cong \lim\operatorname{proj}_{K\Subset G}\mathcal{H}(K)$ [1], it follows that $R_K\widehat{f}$ is analytic for all $K \Subset G$, where $R_K\colon \mathcal{H}(G) \to \mathcal{H}(K)$ denotes the restriction map.

Now for each $(x, z) \in D \times G$ we find neighbourhoods $\widetilde{V}$ and W of x and z in $E_{\mathbb{C}}$ and G respectively and a holomorphic function $\widehat{g}\colon \widetilde{V} \to \mathcal{H}^\infty(W)$ such that

$$\widehat{g}\Big|_{\widetilde{V}\cap D} = R_W\,\widehat{f}\Big|_{\widetilde{V}\cap D} .$$

This shows that the function $g : \widetilde{V} \times X \to \mathbb{C}$ which is associated to $\widehat{g}$ is holomorphic. Since

$$g\Big|_{(\widetilde{V}\times W)\cap(D\times G)} = f\Big|_{(\widetilde{V}\times W)\cap(D\times G)}$$

we infer that f is analytic at (x, z).

(**ii**) The case where F is separable. Then there exists a continuous linear map from a Fréchet–Montel Köthe space $\lambda(A)$ onto F [6].

Since $\mathcal{H}(R^{-1}(G)) \cong [\mathcal{H}(R^{-1}(G), \tau_\omega)]$ [1], by (i) the function $f(\mathrm{id} \times R)\colon D \times R^{-1}(G) \to \mathbb{C}$ and hence f is analytic.

(**iii**) General case. By $\mathcal{F}(F)$ we denote the family of all closed separable subspaces of F.

Given $(x, z) \in D \times G$. By (ii) for each $P \in \mathcal{F}(F)$ with $z \in P$ there exists a neighbourhood W_P of (x, z) in $E_{\mathbb{C}} \times G$ and a holomorphic function $g_P\colon W_P \to \mathbb{C}$ such that

$$g_P\Big|_{W_P\cap(D\times G)} = f\Big|_{W_P\cap(D\times G)} .$$

The uniqueness implies that the family $\{g_P\}$ defines a function $g\colon W := \bigcup_P W_P \to \mathbb{C}$ such that

$$g_P\Big|_{W\cap(D\times G)} = f\Big|_{W\cap(D\times G)} .$$

It remains to check that W is a neighbourhood of (x, z) in $E_{\mathbb{C}} \times G$.

Otherwise there exists $\{(x_n, z_n)\} \subset E_{\mathbb{C}} \times F$ converging to (x, z) such that

$$(x_n + x,\ z_n + z) \notin W \quad \text{for all } \ n \geq 1 .$$

Put $P = \overline{\operatorname{span}}\{z_n\} \in \mathcal{F}(F)$.

Then

$$(x_n + x,\ z_n + z) \in W_P \subset W \ \text{ for } n \text{ sufficiently large} .$$

This is impossible. ∎

4 – Remark

Let X be a Stein connected manifold. In [7] L.M. Hai has proved that $\mathcal{H}(X) \in$ (DN) if and only if every $\mathcal{H}(X)$-valued weakly holomorphic function on and any L-regular compact set K in $\mathbb{C}^n$ is holomorphic on K. However for the analytic case we only proved the following

Proposition. *Let X be a connected complex space such that every weakly analytic function on an open set in $\mathbb{R}^n$ with values in $\mathcal{H}(X)$ is analytic. Then every bounded holomorphic function on X is constant.*

Proof: Otherwise, let $\varphi \in \mathcal{H}(X)$ such that $\varphi \neq$ const and

$$\sup_X |\varphi| = 1 \ .$$

Consider the function $f\colon (-1,1) \times X \to \mathbb{C}$ given by

$$f(t,z) = \frac{1}{1 + \dfrac{t^2}{1-\varphi(z)}} \ .$$

It follows that f is analytic.

We check that the function $\widehat{f}\colon (-1,1) \to \mathcal{H}(X)$ which is associated to f is weakly analytic.

Indeed, given $\mu \in [\mathcal{H}(X)]'$ and $t_0 \in (-1,1)$. Choose a compact set K in X such that $\operatorname{supp}\mu \subset K$. By the compactness of K we can find a neighbourhood $U \times V$ of $\{t_0\} \times K$ in $\mathbb{C}^n \times X$ and a holomorphic function $g\colon U \times V \to \mathbb{C}$ for which

$$g\Big|_{(U\times V)\cap((-1,1)\times X)} = f\Big|_{(U\times V)\cap((-1,1)\times X)} \ .$$

Since $\widehat{g}\colon U \to \mathcal{H}(V)$ is holomorphic and μ can be considered as an element of $[\mathcal{H}(V)]'$ it follows that $\mu\,\widehat{f}$ is extended holomorphically to $\mu\,\widehat{g}$ on U.

By the hypothesis $\widehat{f}$ is analytic. However this is impossible since the radius of the convergence $r(z)$ of the series

$$1 - \frac{t^2}{1-\varphi(z)} + \frac{t^4}{(1-\varphi(z))^2} - \frac{t^6}{(1-\varphi(z))^3} + \dots$$

is $\sqrt{|1-\varphi(z)|} \to 0$ as $z \to \partial X$. ∎

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