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ON THE GEOMETRY OF $\mathcal{L}(l_2^p, l_2^q)$ AND $l_2^q \otimes_\varepsilon l_2^q$

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Abstract: In this paper the characterization of extreme, exposed (and smooth) points of the unit ball of the space of continuous linear operators acting from l_2^p, $p > 2$ to its conjugate space is obtained. The class of extreme contractions found here is different from those of the special cases, which have already been solved.

1 – Introduction

The aim of this paper is the continuation of investigation of extreme contractions. The case of the operators on $C(X)$ is evident. This fact together with the well-known isomorphism $l^\infty \to C(\beta\mathbb{N})$ gives characterization of extreme contractions on l^∞ (see e.g. Sharir [18], Kim [14], Gendler [2] and references there). From this, making use duality, the l^1-spaces case has been achieved (see Iwanik [10]). On the Hilbert space extreme contractions are isometries and coisometries (see Kadison [11], Grząślewicz [5]). More results have been achieved in finite dimensional case (see for instance Lindenstrauss and Perles [15]).

Let $1 < p < \infty$. By q the dual power coefficient is denoted, which is such a number that $1/p + 1/q = 1$. By l_2^p we denote $\mathbb{R}^2$ with the standard l^p-norm, i.e. $\|\mathbf{x}\| = \|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}$. For Banach spaces E, F by $\mathcal{L}(E,F)$ we denote the Banach space of all linear bounded operators from E into F, and by $E \otimes F$ their tensor product. Additionally we denote by $E \otimes_\varepsilon F$ the (complete) injective tensor product. Note that $l_2^p \otimes_\varepsilon l_2^p$ is norm isomorfic to $\mathcal{L}(l_2^q, l_2^p)$. Moreover $(l_2^p \otimes_\varepsilon l_2^p)^* \cong l_2^q \otimes_\pi l_2^q$ (cf. [1]). For any Banach space E by $B(E)$ we denote its closed unit ball and by $B_E(\mathbf{x}, r)$ the set $\{\mathbf{y} \in E\colon \|\mathbf{y} - \mathbf{x}\|_E \le r\}$. The characterization of extreme points of the unit ball $B(E \otimes_\pi F)$ is given by Ruess and Stegall [17].

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In particular they have proved that

$$\operatorname{ext}\Big(B((l_2^p \otimes_\varepsilon l_2^p)^*)\Big) = \Big(\operatorname{ext} B(l_2^q) \otimes \operatorname{ext} B(l_2^q)\Big) = S(l_2^q) \otimes S(l_2^q) ,$$

where $S(\cdot)$ denotes the unit sphere and $\operatorname{ext} Q$ — the set of extreme points of Q. The characterization of extreme points in $l_2^2 \otimes l_2^2 \otimes l_2^2$ is presented in [6]. In [3] a characterization of $B(\mathcal{L}(l_2^p, l_2^p))$ is given (for some generalizations for the infinite dimensional case see [4], [12], [13]). Furthemore, the consideration of the spaces $\mathcal{L}(l_m^p, l_n^2)$ and $\mathcal{L}(l_m^2, l_n^p)$ can be found in [7].

In this paper we continue the characterization in question for $\mathcal{L}(l_2^q, l_2^p)$ or equivalently for $l_2^p \otimes_\varepsilon l_2^p$.

2 – Extreme points

Let $\mathbf{x} = (x_1, x_2) \in S(l_2^p)$, $\mathbf{y} = (y_1, y_2) \in S(l_2^q)$; recall $1/p + 1/q = 1$. Put

$$J_{\mathbf{x},\mathbf{y}} = \Big\{T \in B(\mathcal{L}(l_2^p, l_2^q))\colon\ T\mathbf{x} = \mathbf{y}\Big\} ,$$

where B denotes the unit ball in $\mathcal{L}(l_2^p, l_2^q)$. We are going to prove that $J_{\mathbf{x},\mathbf{y}}$ is identical with the set of all the contractions of the form

$$(1) \quad T_\mu = (x_1^{p-1}, x_2^{p-1}) \otimes (y_1, y_2) + \mu \cdot (-x_2, x_1) \otimes (-y_2^{q-1}, y_1^{q-1}) =\colon \\ =\colon \mathbf{x}^{p-1} \otimes \mathbf{y} + \mu\, \mathbf{x}^\perp \otimes (\mathbf{y}^{q-1})^\perp , \qquad \mu \in \mathbb{R} ,$$

here $\mathbf{x} \otimes \mathbf{y}$ denotes one-dimensional operator for which $(\mathbf{x} \otimes \mathbf{y})(\mathbf{z}) = \langle \mathbf{z}, \mathbf{x}\rangle\, \mathbf{y}$; $\mathbf{a}^\perp = (a_1, a_2)^\perp = (-a_2, a_1)$, and $\mathbf{a}^s = (a_1, a_2)^s = (\operatorname{sgn}(a_1) \cdot |a_1|^s, \operatorname{sgn}(a_2) \cdot |a_2|^s)$ (note that for $\mathbf{x} \in S(l_2^p)$ we have $\langle \mathbf{x}, \mathbf{x}^\perp\rangle = 0$ and the vector $\mathbf{x}^{p-1} \in S(l_2^q)$ is the only possible functional for which $\langle \mathbf{x}, \mathbf{x}^{p-1}\rangle = 1$).

Indeed, $J_{\mathbf{x},\mathbf{y}}$ contains all operators of such a form. Conversely, for $S, T \in J_{\mathbf{x},\mathbf{y}}$ we have $(S - T)\,\mathbf{x} = \mathbf{0}$, hence $\dim(\operatorname{Im}(S - T)) \le 1$ and therefore $S - T = \mathbf{x}^\perp \otimes \mathbf{z}$ for some $\mathbf{z} \in l_2^q$. Since $S^*(\mathbf{y}^{q-1}) = T^*(\mathbf{y}^{q-1}) = \mathbf{x}^{p-1}$, we have also $(S - T)^*(\mathbf{y}^{q-1}) = \mathbf{0}$ and $(S - T)^* = (\mathbf{y}^{q-1})^\perp \otimes \mathbf{w}$ for some $\mathbf{w} \in l_2^p$, which implies that $S - T = \mathbf{w} \otimes (\mathbf{y}^{q-1})^\perp$ so $\mathbf{z} = \mu \cdot (\mathbf{y}^{q-1})^\perp$ for some $\mu \in \mathbb{R}$, thus completing the proof.

Note that if $\mu_1 > \mu_2 > 0$ (or if $0 > \mu_2 > \mu_1$), then $\|T_{\mu_1}\| \ge \|T_{\mu_2}\| \ge 1$. Indeed, if e.g.: $\mu_1 > \mu_2 > 0$, the vector $\mathbf{y}$ belongs to l_2^q and the functional $\mathbf{y}^*$ is equal to $\mathbf{y}^{q-1}$ (then $\mathbf{y}^*(\mathbf{y}) = 1 = \|\mathbf{y}^*\|$ and $\mathbf{y}^*((\mathbf{y}^{q-1})^\perp) = 0$) and if

$$Q = B_E\Big(\mathbf{0}, \|\mathbf{y} + \mu_1 (\mathbf{y}^{q-1})^\perp\|\Big) \cap \Big\{\mathbf{z}\colon\ \mathbf{y}^*(\mathbf{z}) \le 1\Big\} ,$$

then Q is a convex set contained in the ball $B_E(\mathbf{0}, \|\mathbf{y}+\mu_1(\mathbf{y}^{q-1})^\perp\|)$, so the whole interval $\mathbf{y} + \alpha\,\mu_1(\mathbf{y}^{q-1})^\perp$, $\alpha \in [0,1]$ lies in Q. This implies $\|\mathbf{y} + \mu_2(\mathbf{y}^{q-1})^\perp\| \le \|\mathbf{y} + \mu_1(\mathbf{y}^{q-1})^\perp\|$ and ends the proof. ∎

Consider a function

$$\begin{aligned} \Phi_\mu(\lambda) &= \Big\|\mathbf{x} + \lambda(\mathbf{x}^{p-1})^\perp\Big\|_p^{pq} - \Big\|T_\mu\Big(\mathbf{x} + \lambda(\mathbf{x}^{p-1})^\perp\Big)\Big\|_q^{pq} \\ &= \Big(|x_1 - \lambda\, x_2^{p-1}|^p + |x_2 + \lambda\, x_1^{p-1}|^p\Big)^q \\ &\quad - \Big(|y_1 - \lambda\,\mu\, y_2^{q-1}|^q + |y_2 + \lambda\,\mu\, y_1^{q-1}|^q\Big)^p . \end{aligned} \tag{2}$$

If $\|T_\mu\| \le 1$ then $\Phi_\mu(\lambda) \ge 0$ for all $\lambda \in \mathbb{R}$. By a standard calculation we obtain

$$\begin{aligned} \Phi_\mu(0) &= 0 , \\ \Phi'_\mu(0) &= 0 , \\ \Phi''_\mu(0) &= p\,q\Big[(p-1)\,|x_1x_2|^{p-2} - \mu^2(q-1)\,|y_1y_2|^{q-2}\Big] , \\ \Phi'''_\mu(0) &= p\,q\Big[(p-1)\,(p-2)\ \mathrm{sgn}(x_1x_2)\,|x_1x_2|^{p-3}\Big(|x_1|^p - |x_2|^p\Big) \\ &\qquad - \mu^3(q-1)\,(q-2)\ \mathrm{sgn}(y_1y_2)\,|y_1y_2|^{q-3}\Big(|y_1|^q - |y_2|^q\Big)\Big] \end{aligned}$$

for such x_1, x_2, y_1, y_2 that make sense for the above-mentioned expressions.

Let μ be the maximal (or minimal) number for which $\|T_\mu\| = 1$. Then T_μ is the extreme contraction, because the norm of operator $T_\mu \pm R$ cannot increase neither in direction $\mathbf{y}$ nor $(\mathbf{y}^{q-1})^\perp$, hence $R = \mathbf{0}$, and $\Phi''_\mu(0) \ge 0$.

Let $\Phi''_\mu(0) > 0$ and $\mu > 0$. Then for all $\varepsilon > 0$ we have $\|T_{\mu+\varepsilon}\| > 1$. The continuity of $\Phi''_\mu(0)$ as a function of μ gives that there exists $\varepsilon_0 > 0$ such that $\Phi''_{\mu+\varepsilon}(0) > 0$ for all $0 < \varepsilon < \varepsilon_0$. Hence there exists $\delta > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and for all

$$\mathbf{u} \in \Big\{\mathbf{u}\colon\ 0 < \|\mathbf{u} - \mathbf{x}\| < \delta\ \wedge\ \|\mathbf{u}\| = 1\Big\}$$

we have $\|T_{\mu+\varepsilon}\,\mathbf{u}\| < 1$. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence in which $\varepsilon_n < \varepsilon_0$ for all n and $\varepsilon_n \to 0$. Let $\mathbf{u}_n$ be such a vector for which $\|T_{\mu+\varepsilon_n}\mathbf{u}_n\| = 1$. The compactness of the unit ball implies the existence of such $\mathbf{u}_0$ that $\mathbf{u}_{\varepsilon_n} @ > n \to \infty >> \mathbf{u}_0$. Evidently $\|T_\mu(\mathbf{u}_0)\| = 1$. But $\|\mathbf{u}_{\varepsilon_n} - \mathbf{x}\| > \delta$, hence $\|\mathbf{u}_0 - \mathbf{x}\| \ge \delta$ and $\mathbf{u}_0$ and $\mathbf{x}$ are such two linearly independent vectors that T_μ attains its norm on them. Therefore, if μ is the maximal (or minimal) element then T_μ is such an extreme operator which attains its norm on two linearly independent vectors.

Let now $\Phi''_\mu(0) = 0$. Then $\Phi'''_\mu(0) = 0$ as well, hence in this case the following pair of equalities is true:

$$(p-1)\,|x_1x_2|^{p-2} = \mu^2\,(q-1)\,|y_1y_2|^{q-2}\ , \tag{3}$$

$$\begin{aligned}(p-1)\,(p-2)\ \mathrm{sgn}(x_1x_2)\,|x_1x_2|^{p-3}(|x_1|^p - |x_2|^p) &= \\ = \mu^3\,(q-1)\,(q-2)\ \mathrm{sgn}(y_1y_2)\,|y_1y_2|^{q-3}(|y_1|^q - |y_2|^q)&\ ,\end{aligned} \tag{4}$$

which is equivalent to

$$\begin{aligned}(p-2)\,(q-1)^{1/2}\,\mathrm{sgn}(x_1x_2)\,|y_1y_2|^{q/2}(|x_1|^p - |x_2|^p) &= \\ = (q-2)\,(p-1)^{1/2}\,\mathrm{sgn}(y_1y_2)\,|x_1x_2|^{p/2}(|y_1|^q - |y_2|^q)&\ .\end{aligned} \tag{5}$$

The set of solutions of (5) is contained in the set of solutions of

$$\begin{aligned}(p-2)^2\,(q-1)\,|y_1y_2|^q\,(|x_1|^{2p} - 2\,|x_1x_2|^p + |x_2|^{2p}) &= \\ = (q-2)^2\,(p-1)\,|x_1x_2|^p\,(|y_1|^{2q} - 2\,|y_1y_2|^q + |y_2|^{2q})&\ .\end{aligned} \tag{6}$$

Let $\alpha = |x_1|^p$, $\beta = |y_1|^q$. Then (6) is equivalent to

$$\frac{(2\alpha-1)^2}{\alpha(1-\alpha)}\,(p-2)^2\,(q-1) = \frac{(2\beta-1)^2}{\beta(1-\beta)}\,(q-2)^2\,(p-1)\ .$$

Relations between p and q imply that

$$(p-2)^2\,(q-1) = (q-2)^2\,(p-1)\ ,$$

so

$$\frac{(2\alpha-1)^2}{\alpha(1-\alpha)} = \frac{(2\beta-1)^2}{\beta(1-\beta)}\ .$$

This means that $\alpha = \beta$ or $\alpha = 1-\beta$ (we may consider only the first case, establishing a suitable base) and

$$\mu = \pm(p-1)\,\frac{y_1\,y_2}{x_1\,x_2}\ .$$

Analysing the signs of both sides of (5) we conclude, that "+" is possible only if $\alpha = \frac{1}{2}$.

Let us denote

$$\begin{aligned}\Psi(\lambda) = {}&\Big(\alpha\cdot\Big|1-\lambda(1-\alpha)\Big|^p + (1-\alpha)\cdot|1+\lambda\alpha|^p\Big)^{1/p} \\ &- \Big(\alpha\cdot\Big|1-\lambda(1-p)\,(1-\alpha)\Big|^{\frac{p}{p-1}} + (1-\alpha)\cdot\Big|1+\lambda(1-p)\,\alpha\Big|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\ .\end{aligned}$$

If T_μ is a contraction, then $\Psi(\lambda) \geq 0$ for all λ. But $\Psi'(0) = \Psi''(0) = \Psi'''(0) = 0$ and

$$\Psi^{(4)}(0) = 3\alpha^2(1-\alpha)^2\,(1-p)\,p(p-2) - p(p-1)\,(p-2)\,\alpha(1-\alpha)\,(\alpha^3 + (1-\alpha)^3) \\ < 0 \quad \text{for all } \alpha \in (0;1) \text{ and } p > 2\,.$$

Hence, for $p > 2$ and $\mathbf{x} \neq \mathbf{e}_i$, $i = 1, 2$ the operator T_μ is not a contraction.

Let us assume now that $x_1x_2\,y_1y_2 = 0$. We need to consider the following cases:

1) $x_1x_2 = 0$, $y_1y_2 \neq 0$, $p > 2$. Then $\Phi''_\mu(0) = -\mu^2(q-1)\,|y_1y_2|^{q-2} = 0$ iff $\mu = 0$, i.e.

$$T_\mu = [\,y_1 \quad 0y_2 \quad 0\,] \text{ or } [\,0 \quad y_10 \quad y_2\,]\,, \qquad \mathbf{y} \in l_2^q\,.$$

2) $x_1x_2 \neq 0$, $y_1y_2 = 0$, $p > 2$. Then $T_\mu(x_1 + \lambda x_2^{p-1}, x_2 - \lambda x_1^{p-1}) = (1, -\lambda\mu)$. Hence T_μ is a contraction iff

$$\Big(1 + |\lambda\mu|^q\Big)^{1/q} \leq \Big(|x_1 + \lambda x_2^{p-1}|^p + |x_2 - \lambda x_1^{p-1}|^p\Big)^{1/p}\,. \tag{7}$$

Let

$$f(\lambda) = \Big(|x_1 + \lambda x_2^{p-1}|^p + |x_2 - \lambda x_1^{p-1}|^p\Big)^{1/p} - \Big(1 + |\lambda\mu|^q\Big)^{1/q}\,.$$

Easy calculation shows that $f(0) = 0$, $f'(0) = 0$, $f''(\lambda)@ > \lambda \to 0 >> -\infty$ for $\mu \neq 0$, which in accordance with the Taylor formula with the second remainder, contradicts the inequality (7). Hence $\mu = 0$, so

$$T_\mu = [\,0 \quad 0x_1^{p-1} \quad x_2^{p-1}\,] \text{ or } T_\mu = [\,x_1^{p-1} \quad x_2^{p-1}0 \quad 0\,]\,, \qquad \mathbf{x} \in l_2^p\,.$$

3) $x_1x_2 = 0$, $y_1y_2 = 0$, $p > 2$. Then $\mathbf{x} = \pm\mathbf{e}_i$, $\mathbf{y} = \pm\mathbf{e}_j$, for $i, j \in \{1, 2\}$. Hence $T_\mu \in J_{\mathbf{x},\mathbf{y}}$ iff $\mu = 0$, because from the Taylor theorem, for all $\mu \neq 0$ it is not a contraction.

In this way we have proved the following:

Lemma 1. *Let $2 < p < \infty$, let q be such a number that $(1/p) + (1/q) = 1$, and $T \in L(l_2^p, l_2^q)$ be an extreme contraction of such form of (1) in which $x_1x_2\,y_1y_2 = 0$. Then T assumes one of the following forms:*

a) $T = [\,y_1 \quad 0y_2 \quad 0\,]$ *or* $T = [\,0 \quad y_10 \quad y_2\,]$, $\quad \mathbf{y} \in S(l_2^q)$.

b) $T = [\,0 \quad 0x_1^{p-1} \quad x_2^{p-1}\,]$ *or* $T = [\,x_1^{p-1} \quad x_2^{p-1}0 \quad 0\,]$, $\quad \mathbf{x} \in S(l_2^p)$.

Lemma 2. *Let $p > 2$, let $\mathbf{y} \in S(l_2^q)$ and let $\mathbf{y}_i \neq 0$ for all $i \in \mathbb{N}$. Then the operator $T = \mathbf{e}_i \otimes \mathbf{y}$ is an extreme contraction in $\mathcal{L}(l_2^p, l_2^q)$.*

Proof: Let $T = \mathbf{e}_i \otimes \mathbf{y}$, i.e. $T\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \, \mathbf{y}$ and $\|T \pm R\| \le 1$ for some $R \in \mathcal{L}(l_2^p, l_2^q)$. Without the loss of generality we can assume that $T = \mathbf{e}_1 \otimes \mathbf{y}$. Then $T\mathbf{e}_1 = \mathbf{y}$, hence from the strict convexity of l_2^p we have $R\,\mathbf{e}_1 = \mathbf{0}$. Evidently $T\,\mathbf{e}_2 = \mathbf{0}$. Let $\mathbf{z} = R\,\mathbf{e}_2$. Therefore we have:

$$(T \pm R)\,(\mathbf{e}_1 + \lambda \mathbf{e}_2) = \mathbf{y} \pm \lambda \mathbf{z} \ .$$

Because of $\|\mathbf{e}_1 + \lambda \mathbf{e}_2\|_p = (1 + |\lambda|^p)^{1/p}$ the following inequality should be fulfilled:

$$L(\lambda)\colon \Big(|y_1 \pm \lambda z_1|^q + |y_2 \pm \lambda z_2|^q\Big)^{p/q} \le 1 + |\lambda|^p =\colon P(\lambda) \ . \tag{8}$$

By differentiating $L(\lambda)$ and $P(\lambda)$ by λ we obtain:

$$P'(0) = 0 \quad \text{and} \quad L'(0) = p\Big(|y_1|^{q-1} \cdot \operatorname{sgn} y_1 \cdot z_1 + |y_2|^{q-1} \cdot \operatorname{sgn} y_2 \cdot z_2\Big) \ .$$

The inequality (8) can be satisfied only if $L'(0) = 0$. Differentiating again, we obtain $P''(0) = 0$ and

$$L''(0) = p\,(q-1)\,\Big(|y_1|^{r-2} \cdot z_1^2 + |y_2|^{r-2} \cdot z_2^2\Big) \ .$$

Because $L''(0) > 0$ for $\mathbf{z} \neq \mathbf{0}$, we obtain $\mathbf{z} = \mathbf{0}$. ∎

Lemma 3. *Let $p > 2$, and $S(l_2^q) \ni \mathbf{z}$ be such a vector that $\mathbf{z}_i \neq 0$ for all $i \in \mathbb{N}$. Then the operator $T = \mathbf{z} \otimes \mathbf{e}_i$ is an extreme contraction in $\mathcal{L}(l_2^p, l_2^q)$.*

Proof: Let, for example, assume that $T = \mathbf{z} \otimes \mathbf{e}_1$. Similarly to the case of the proof of lemma 3 we obtain:

$$(T \pm R)\,(\mathbf{x} + \lambda \mathbf{x}^{\perp}) = \mathbf{e}_1 \pm \lambda \mathbf{z} \ .$$

This means that the following inequality should be fulfilled:

$$|1 \pm \lambda z_1|^q + |\lambda z_2|^q \le \Big(|x_1 - \lambda x_2^{p-1}|^p + |x_2 + \lambda x_1^{p-1}|^p\Big)^{q/p} \ .$$

Denoting left-hand side of the above inequality by $L(\lambda)$, the right-hand side by $R(\lambda)$ and differentiatig both expressions by λ we obtain $P'(0) = 0$ and $L''(0) = q \cdot z_1$, hence $z_1 = 0$. The rest of the proof is similar to the case 2) before lemma 1 (with $\mu = z_2$). Hence $z_2 = 0$ and $\operatorname{Im}(R) = \{\mathbf{0}\}$. ∎

Therefore we can formulate the following theorem:

Theorem 1. *Let $2 < p < \infty$, and let q be such a number for which $(1/p) + (1/q) = 1$. Then $T \in L(l_2^p, l_2^q)$ is an extreme contraction if and only if*

$\|T\| = 1$ *and:* **either** T *attains its norm on two linearly independent vectors in* l_2^p **or** T *is of one of the following forms:*

a) $T = [\,y_1 \quad 0 y_2 \quad 0\,]$ *or* $T = [\,0 \quad y_1 0 \quad y_2\,]$, $\quad \mathbf{y} \in S(l_2^q)$.

b) $T = [\,0 \quad 0 x_1^{p-1} \quad x_2^{p-1}\,]$ *or* $T = [\,x_1^{p-1} \quad x_2^{p-1} 0 \quad 0\,]$, $\quad \mathbf{x} \in S(l_2^p)$.

Proof: We have just proved that an extreme contraction T has the form described in the theorem.

If $\|T\| = 1$ and T attains its norm on two linearly independent vectors or $T = \mathbf{e}_i \otimes \mathbf{e}_j$, then T is evidently an extreme contraction. We obtain the remaining part from lemmas 2 and 3. ∎

3 – Remarks on the case $1 < p < 2$

Let us recall the following inequality:

Lemma 4([16])**.** *Let* $1 < p < r < \infty$ *and let* $\gamma = \sqrt{(p-1)/(r-1)}$*. Then, for all* $\lambda \in \mathbb{R}$*, we have*

$$\left(\frac{|1+\gamma\lambda|^r + |1-\gamma\lambda|^r}{2}\right)^{1/r} \le \left(\frac{|1+\lambda|^p + |1-\lambda|^p}{2}\right)^{1/p} ,$$

(*) moreover, if $\lambda \neq 0$ *then the strict inequality proves to be true.*

Remark. In [16] the lemma is formulated without (*), but (*) is a direct corollary from the proof (cf. [16] p.75).

Put $\mathbf{f} = (1,1)$ and $\mathbf{f}^\perp = (-1,1)$. The above inequality can be considered as the inequality

$$\|T_\gamma(\mathbf{f} + \lambda \cdot \mathbf{f}^\perp)\|_r \le \|\mathbf{f} + \lambda \cdot \mathbf{f}^\perp\|_p$$

for $T_\gamma \in \mathcal{L}(l_2^p, l_2^r)$ of the form

$$T_\gamma = 2^{((1/p)-(1/r))}\Big(\mathbf{f} \otimes \mathbf{f} + \gamma \cdot \mathbf{f}^\perp \otimes \mathbf{f}^\perp\Big) .$$

Let us note that T_γ has the form (1). Let Φ_γ be the function of the form (2) for $\mu = \gamma$. It is easy to check that $\Phi_\gamma''(0) = 0$ for T_γ, hence $\mu = \gamma$ is the maximal number and T_γ is an extreme contraction. This means that γ is the best possible constant in the above inequality. Hence we have:

Proposition. *Let* $1 < p < r < \infty$, $p < 2$, $\gamma = \sqrt{(p-1)/(r-1)}$ *and let*

$$T_\gamma = 2^{((1/p)-(1/r))}\,[\,1 \pm \gamma \quad 1 \mp \gamma 1 \mp \gamma \quad 1 \pm \gamma\,] \ .$$

Then $T_\gamma \in \operatorname{ext} B(\mathcal{L}(l_2^p, l_2^r))$ *and* T_γ *attains its norm only on one-dimensional space.*

Note that (with the use of computer calculations) for $p \in (1;2)$, taking such μ that $\Phi''_\mu(0) = 0$ some of corresponding operators T_μ are contractions and some of them have the norm greater than one. It means that for $p \in (1;2)$, $\mathbf{x} \neq \mathbf{e}_i$, $\mathbf{y} \neq \mathbf{e}_i$, $i = 1,2$, there are also extreme operators which are two-dimensional and which attain their norm only at one independent vector.

4 – Exposed points

We recall that a point q_0 of a convex set Q is called *exposed* if there exists such a linear functional ξ for which $\xi(q_0) > \xi(q)$ for all $q \in Q \backslash \{q_0\}$.

Lemma 5. *Let* $p, r \in (1;\infty)$ *and let* $\mathbf{x} \in S(l_2^p)$, $\mathbf{y} \in S(l_2^r)$ *with* $\mathbf{x} \neq \mathbf{e}_i$, $\mathbf{y} \neq \mathbf{e}_i$, $i = 1,2$. *Then* $\mathbf{x}^{p-1} \otimes \mathbf{y}$ *is not an extreme point of* $B(\mathcal{L}(l_2^p, l_2^r))$.

Proof: Considering the function $\Phi_\mu(\lambda)$ for $\mathbf{x}^{p-1} \otimes \mathbf{y} \in \mathcal{L}(l_2^p, l_2^r)$ it is easy to see that $\Phi''_{\mu_0} > 0$ for sufficietly small $\mu_0 > 0$. It means that in some neighbourhood of $\mathbf{x}$ the operator $\mathbf{x}^{p-1} \otimes \mathbf{y} \pm \mathbf{x}^\perp \otimes (\mathbf{y}^{r-1})^\perp$ does not extend norm one. Therefore, for sufficiently small $\mu_0 > 0$, we have

$$\left\| \mathbf{x}^{p-1} \otimes \mathbf{y} \pm \mu_0 \mathbf{x}^\perp \otimes (\mathbf{y}^{r-1})^\perp \right\| \le 1 ,$$

i.e. $\mathbf{x}^{p-1} \otimes \mathbf{y}$ is not an extreme contraction. ∎

Theorem 2. *Let* $p \in (2,\infty)$. *Then all extreme points of* $B(\mathcal{L}(l_2^p, l_2^r))$ *except the two dimensional operators which attain their norms only on one-dimensional subspace, are exposed points.*

Proof: Let a contraction T attains its norm at two linearly independent vectors $\mathbf{x}_1$, $\mathbf{x}_2$ with $\|\mathbf{x}_i\| = 1$, $i = 1,2$. Then the functional ξ defined by

$$\xi(R) = \frac{1}{2}\Big(\Big\langle R\mathbf{x}_1, (T\mathbf{x}_1)^{q-1}\Big\rangle + \Big\langle R\mathbf{x}_2, (T\mathbf{x}_2)^{q-1}\Big\rangle\Big)$$

exposes $B(\mathcal{L}(l_2^p, l_2^r))$ at T. Indeed, let $\|\xi\| = \xi(T) = 1$. Suppose that $\xi(R) = 1$ for some $R \in B(\mathcal{L}(l_2^p, l_2^r))$. Then $\langle R\mathbf{x}_i, (T\mathbf{x}_i)^{q-1}\rangle = 1$, $i = 1,2$, and by strict convexity of l_2^q we have $R\mathbf{x}_i = T\mathbf{x}_i$ $(i = 1,2)$. Since $\mathbf{x}_1$, $\mathbf{x}_2$ generate l_2^p , we obtain $R = T$, i.e. T is exposed.

If an extreme operator T has the form $\mathbf{e}_i \otimes \mathbf{y}$ $(\mathbf{y} \neq \mathbf{e}_1, \mathbf{e}_2)$, then T is exposed by the functional ξ defined by $\xi(R) = \langle R\mathbf{e}_1, \mathbf{y}^{q-1}\rangle$, $R \in \mathcal{L}(l_2^p, l_2^q)$. Indeed, we have $\|\xi\| = \xi(T) = 1$. Moreover, for $R \in B(\mathcal{L}(l_2^p, l_2^r))$ with $\xi(R) = 1$ we have $R\mathbf{e}_1 = \mathbf{y}$. Because $\|R\| \le 1$ and $\|R\mathbf{e}_1\| = 1$, we have $R\mathbf{e}_2 = 0$. Hence $R = T$ and T is exposed. We use analogous arguments for the operator of the form $\mathbf{x}^{p-1} \otimes \mathbf{e}_i$, $i = 1, 2$. In accordance with the lemma 5 there are no other extreme one-dimensional operators. Let T be the extreme two-dimensional operator, which attains its norm only on a one-dimensional subspace. Then T assumes the form

$$T = \mathbf{x}^{p-1} \otimes \mathbf{y} + \mu_0\, \mathbf{x}^\perp \otimes (\mathbf{y}^{q-1})^\perp\,, \qquad \mu_0 \neq 0\ ,$$

i.e. T attains its norm only at $\mathbf{x}$. We define the set of functionals

$$\mathcal{A} = \Big\{ \xi \in B(\mathcal{L}(l_2^p, l_2^q)^*) \colon \xi(T) = \|\xi\| = 1 \Big\}\ .$$

The set $\mathcal{A}$ is a closed convex subset of the $B(\mathcal{L}(l_2^p, l_2^r))$. In fact: $\mathcal{A}$ is a compact face of $B(\mathcal{L}(l_2^p, l_2^r)^*)$. Hence $\operatorname{ext} \mathcal{A} \subset \operatorname{ext} B(\mathcal{L}(l_2^p, l_2^r)^*)$. From the Ruess–Stegall results, we know that each element $\xi \in \operatorname{ext} B(\mathcal{L}(l_2^p, l_2^r)^*)$ has the form $\xi(R) = \langle R\mathbf{x}_0, \mathbf{u}_0 \rangle = (\mathbf{x}_0 \otimes \mathbf{u}_0)(R)$ for some $\mathbf{x}_0, \mathbf{u}_0 \in l_2^p$ with $\|\mathbf{x}_0\| = \|\mathbf{u}_0\| = 1$. The condition $\xi(T) = 1$ implies that $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{u}_0 = \mathbf{y}^{q-1}$. Hence $\operatorname{ext} \mathcal{A}$ has only one element $\xi_0 = \mathbf{x} \otimes \mathbf{y}^{q-1}$ than $\mathcal{A} = \{\xi_0\}$, as well. Therefore, there exists only one functional which supports $B(\mathcal{L}(l_2^p, l_2^r))$ at T. It is easy to see that ξ_0 does not expose $B(\mathcal{L}(l_2^p, l_2^r))$ at T, at least for the simple reason that $\xi_0(\mathbf{x}^{p-1} \otimes \mathbf{y}) = 1$. Hence T is not an exposed point. ∎

We point out that all elements of the unit sphere of $\mathcal{L}(l_2^p, l_2^q)$ are smooth, except for these (extreme) operators, which attain their norms at two linearly independent vectors (see Heinrich [9]).

Remark 1. Theorem 1 remains valid for **every** $p > 2$ and $1 < q < 2$. We can prove this using methods simillar to used in the proof of theorem 1.

Remark 2. On the figure 1 we can see the unit ball for $p = 3$ and its image by the extreme operator for $q = 3/2$. This operator is an operator corresponding to inequality formulated in lemma 4.

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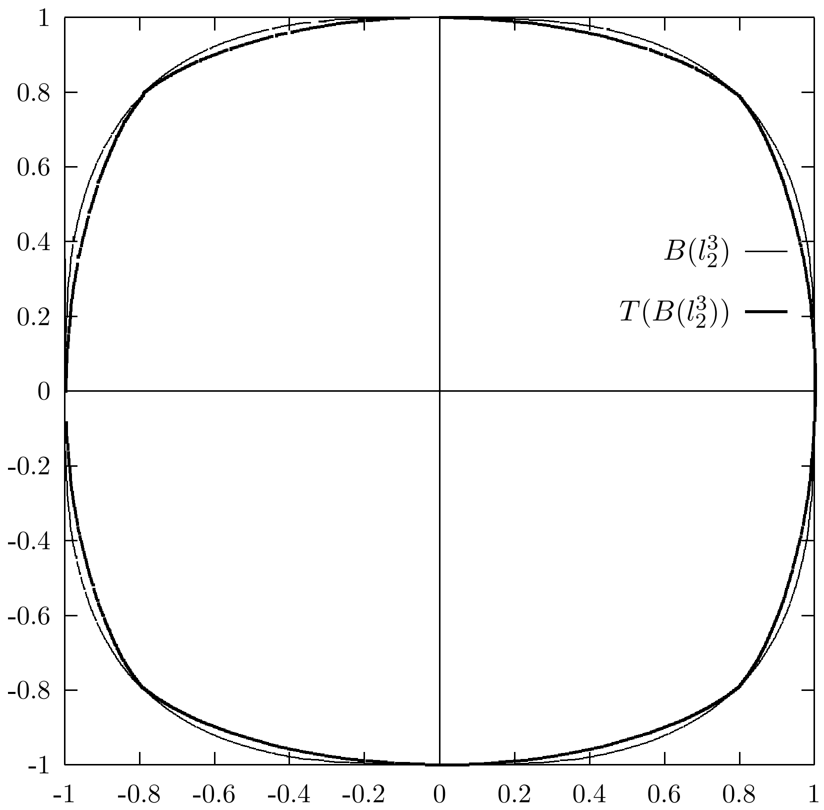

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