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SPACES OF ENTIRE FUNCTIONS OF SLOW GROWTH REPRESENTED BY DIRICHLET SERIES

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1 – Let

$$f(s)=\sum_{n=1}^{\infty} a_n\, e^{s.\lambda_n}\ , \tag{1.1}$$

where $0<\lambda_1<\lambda_2<...<\lambda_n<...$, $\lambda_n\to\infty$ as $n\to\infty$, $s=\sigma+it$ (σ, t being reals) and $\{a_n\}_1^\infty$ any sequence of complex numbers, be a Dirichlet series. Further, let

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=D<\infty\ , \tag{1.2}$$

$$\limsup_{n\to\infty}(\lambda_{n+1}-\lambda_n)=h>0\ , \tag{1.3}$$

and

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\lambda_n}=-\infty\ . \tag{1.4}$$

Then the series in (1.1) represents an entire function $f(s)$. We denote by X the set of all entire functions $f(s)$ having representation (1.1) and satisfying the conditions (1.2)–(1.4). By giving different topologies on the set X, Kamthan [4] and Hussain and Kamthan [2] have studied various topological properties of these spaces. Hence we define, for any nondecreasing sequence $\{r_i\}$ of positive numbers, $r_i\to\infty$,

$$\|f\|_{r_i}=\sum|a_n|\,e^{r_i\lambda_n}\ ,\qquad i=1,2,...\ , \tag{1.5}$$

where $f\in X$. Then from (1.4), $\|f\|_{r_i}$ exists for each i and is a norm on X. Further, $\|f\|_{r_i}\le\|f\|_{r_{i+1}}$. With these countable number of norms, a metric d is defined on X as:

$$d(f,g)=\sum_{i=1}^{\infty}\frac{1}{2^i}\,\frac{\|f-g\|_{r_i}}{1+\|f-g\|_{r_i}}\ ,\qquad f,g\in X\ . \tag{1.6}$$

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Further, following functions are defined for each $f \in X$, namely

$$p(f) = \sup_{n\geq 1} |a_n|^{1/\lambda_n} \;; \tag{1.7}$$

$$\|f\|_i = \sup_{n\leq i}\Big(|a_n|^{1/\lambda_n}\Big) \;. \tag{1.8}$$

Then $p(f)$ and $\|f\|_i$ are para-norms on X. Let

$$s(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i}\,\frac{\|f-g\|_i}{1+\|f-g\|_i} \;. \tag{1.9}$$

It was shown [2, Lemma 1] that the three topologies induced by d, s and p on X are equivalent. Many other properties of these spaces were also obtained (see [2], pp. 206–209).

For the space of entire functions of finite Ritt order [6] and type, yet another norm $\|f\|_q$ and hence a metric λ was introduced and the properties of this space X_λ were studied.

Let, for $f \in X$,

$$M(\sigma,f) \equiv M(\sigma) = \sup_{-\infty<t<\infty} |f(\sigma+it)| \;,$$

then $M(\sigma)$ is called the maximum modulus of $f(s)$. The Ritt order of $f(s)$ is defined as

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \rho\,, \qquad 0 \leq \rho \leq \infty \;. \tag{1.10}$$

For $\rho < \infty$, the entire function f is said to be of finite order. A function $\rho(\sigma)$ is said to be proximate order [3] if

$$\rho(\sigma) \to \rho \quad \text{as} \;\; \sigma \to \infty, \;\; 0 < \rho < \infty \;, \tag{1.11}$$

$$\sigma\,\rho'(\sigma) \to 0 \quad \text{as} \;\; \sigma \to \infty \;. \tag{1.12}$$

For $f \in X$, define

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{e^{\sigma\rho(\sigma)}} \leq A < \infty \;. \tag{1.13}$$

Then it was proved [3] that (1.13) holds if and only if

$$\limsup_{n\to\infty} \phi(\lambda_n)\, |a_n|^{1/\lambda_n} \leq (A.e\,\rho)^{1/\rho} \;, \tag{1.14}$$

where $\phi(t)$ is the unique solution of the equation $t = \exp[\sigma.\rho(\sigma)]$.

(Apparently the inequality (4.1) and the definition of $\phi(t)$ contain some misprints in [2, pp. 209–210]).

For each $f \in X$, define

$$\|f\|_q = \sum_{n=1}^{\infty} |a_n| \left\{ \frac{\phi(\lambda_n)}{[(A+\frac{1}{q})\, e\, \rho]^{1/\rho}} \right\}^{\lambda_n} ,$$

where $q = 1, 2, \ldots$. For $q_1 \le q_2$, $\|f\|_{q_1} \le \|f\|_{q_2}$. It was proved that $\|f\|_q$, $q = 1, 2, \ldots$, induces on X a unique topology such that X becomes a convex topological vector space, where this topology is given by the metric λ,

$$\lambda(f,g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f-g\|_q}{1+\|f-g\|_q} .$$

This space was denoted by X_λ. Various properties of this space were studied [2, pp. 209–216].

It is evident that if $\rho = 0$, then the definition of the norm $\|f\|_q$ and proximate order $\rho(\sigma)$ is not possible. It is the aim of this paper to give a metric on the space of entire functions of zero order thereby studying some properties of this space.

2 – For an entire function $f(s)$ represented by (1.1), for which ρ defined by (1.10) is equal to zero, we define following Rahman [5]

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\log \sigma} = \rho^* , \qquad 1 \le \rho^* \le \infty . \tag{2.1}$$

Then ρ^* is said to be the logarithmic order of $f(s)$. For $1 < \rho^* < \infty$, we define the logarithmic proximate order [1] $\rho^*(\sigma)$ as a continuous piecewise differentiable function for $\sigma \ge \sigma_0$ such that

$$\rho^*(\sigma) \to \rho^* \quad \text{as} \quad \sigma \to \infty , \tag{2.2}$$

$$\sigma . \log \sigma . \rho'^*(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty . \tag{2.3}$$

Then the logarithmic type T^* of f with respect to proximate order $\rho^*(\sigma)$ is defined as [7]:

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}} = T^* , \qquad 0 < T^* < \infty . \tag{2.4}$$

It was proved by one of the authors [7] that $f(s)$ is of logarithmic order ρ^*, $1 < \rho^* < \infty$, and logarithmic type T^*, $0 < T^* < \infty$, if and only if

$$\limsup_{n\to\infty} \frac{\lambda_n\, \phi(\lambda_n)}{\log |a_n|^{-1}} = \frac{\rho^*}{(\rho^*-1)} \, (\rho^*\, T^*)^{1/(\rho^*-1)} , \tag{2.5}$$

where $\phi(t)$ is the unique solution of the equation $t = \sigma^{\rho^*(\sigma)-1}$.

We now denote by X the set of all entire functions $f(s)$ given by (1.1), satisfying (1.2) to (1.4), for which

$$\limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}} \le T^* < \infty\,, \qquad 1 < \rho^* < \infty\ . \tag{2.6}$$

Then from (2.5), we have

$$\limsup_{n\to\infty} \frac{\lambda_n\,\phi(\lambda_n)}{\log|a_n|^{-1}} \le \left(\frac{\rho^*}{\rho^*-1}\right).(\rho^*\,T^*)^{1/(\rho^*-1)}\ . \tag{2.7}$$

In all our further discussion, we shall denote $(\rho^*/(\rho^*-1))^{(\rho^*-1)}$ by the constant K. Then from (2.7) we have

$$|a_n| < \exp\left[-\frac{\lambda_n.\phi(\lambda_n)}{\{K.\rho^*(T^*+\varepsilon)\}^{1/(\rho^*-1)}}\right]\,, \tag{2.8}$$

where $\varepsilon > 0$ is arbitrary and $n > n_0$.

Now, for each $f \in X$, let us define

$$\|f\|_q = \sum_{n=1}^{\infty} |a_n| \exp\left[\frac{\lambda_n\,\phi(\lambda_n)}{\{K\,\rho^*(T^*+\frac{1}{q})\}^{1/(\rho^*-1)}}\right]\,,$$

where $q = 1, 2, 3, \ldots$. In view of (2.8), $\|f\|_q$ exists and for $q_1 \le q_2$, $\|f\|_{q_1} \le \|f\|_{q_2}$. This norm induces a metric topology on X.

We define

$$\lambda(f,g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \cdot \frac{\|f-g\|_q}{1+\|f-g\|_q}\ .$$

The space X with the above metric λ will be denoted by X_λ.

Now we prove

Theorem 1. *The space X_λ is a Fréchet space.*

Proof: It is sufficient to show that X_λ is complete. Hence, let $\{f_\alpha\}$ be a λ-Cauchy sequence in X. Therefore, for any given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$\|f_\alpha - f_\beta\|_q < \varepsilon \quad \forall\,\alpha,\beta > n_0,\ \ q \ge 1\ .$$

Denoting $f_\alpha(s) = \sum_1^\infty a_n^{(\alpha)}\, e^{s.\lambda_n}$, $f_\beta(s) = \sum_1^\infty a_n^{(\beta)}\, e^{s.\lambda_n}$, we have therefore

$$\sum_{n=1}^{\infty} |a_n^{(\alpha)} - a_n^{(\beta)}|.\exp\left[\frac{\lambda_n\,\phi(\lambda_n)}{\{K.\rho^*(T^*+\frac{1}{q})\}^{1/(\rho^*-1)}}\right] < \varepsilon \tag{2.9}$$

for $\alpha, \beta > n_0$, $q \geq 1$. Hence we obviously have

$$|a_n^{(\alpha)} - a_n^{(\beta)}| < \varepsilon \quad \forall\, \alpha, \beta > n_0 \ ,$$

i.e., $\{a_n^{(\alpha)}\}$ is a Cauchy sequence of complex numbers for each fixed $n = 1, 2, \ldots$. Hence

$$\lim_{\alpha \to \infty} a_n^{(\alpha)} = a_n \ , \qquad n = 1, 2, \ldots \ .$$

Now letting $\beta \to \infty$ in (2.9), we have for $\alpha > n_0$,

$$\sum_{n=1}^{\infty} |a_n^{(\alpha)} - a_n| . \exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}} \Big] < \varepsilon \ . \tag{2.10}$$

Taking $\alpha = n_0$, we get for a fixed q,

$$|a_n| . \exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}} \Big] \leq |a_n^{(n_0)}| . \exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}} \Big] + \varepsilon \ .$$

Now, $f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} . e^{s.\lambda_n} \in X_\lambda$, hence the condition (2.8) is satisfied. For arbitrary $p > q$, we have

$$|a_n^{(n_0)}| < \exp\Big[\frac{-\lambda_n \, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{p})\}^{1/(\rho^*-1)}} \Big] \quad \text{for arbitrarily large } n \ .$$

Hence we have

$$|a_n| \exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}} \Big] <$$
$$< \exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{(K\, \rho^*)^{1/(\rho^*-1)}} \Big\{ \frac{1}{(T^* + \frac{1}{q})^{1/(\rho^*-1)}} - \frac{1}{(T^* + \frac{1}{p})^{1/(\rho^*-1)}} \Big\} \Big] + \varepsilon \ .$$

Since $\varepsilon > 0$ is arbitrary and the first term on the R.H.S. $\to 0$ as $n \to \infty$, we find that the sequence $\{a_n\}$ satisfies (2.8). Then $f(s) = \sum_{n=1}^{\infty} a_n \, e^{s\lambda_n}$ belongs to X_λ. ∎

Now, from (2.10), we have for $q = 1, 2, \ldots$, $\|f_\alpha - f\|_q < \varepsilon$. Hence

$$\lambda(f_\alpha, f) = \sum_{q=1}^{\infty} \frac{1}{2^q} \, \frac{\|f_\alpha - f\|_q}{1 + \|f_\alpha - f\|_q} \leq \frac{\varepsilon}{(1+\varepsilon)} \sum_{q=1}^{\infty} \frac{1}{2^q} = \frac{\varepsilon}{(1+\varepsilon)} < \varepsilon \ .$$

Since the above inequality holds for all $\alpha > n_0$, we finally get $f_\alpha \to f$ where $f \in X_\lambda$. Hence X_λ is complete. This proves Theorem 1. ∎

Now, we characterize the linear continuous functionals on X_λ. We prove

Theorem 2. *A continuous linear functional ψ on X_λ is of the form*

$$\psi(f) = \sum_{n=1}^{\infty} a_n\, C_n$$

if and only if

$$|C_n| \le A.\exp\left[\frac{\lambda_n\,\phi(\lambda_n)}{\{K\,\rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}}\right] \tag{2.11}$$

for all $n \ge 1$, $q \ge 1$, where A is a finite, positive number, $f = f(s) = \sum_{n=1}^{\infty} a_n\, e^{s.\lambda_n}$ and λ_1 is sufficiently large.

Proof: Let $\psi \in X'_\lambda$. Then for any sequence $\{f_m\} \in X_\lambda$ such that $f_m \to f$, we have $\psi(f_m) \to \psi(f)$ as $m \to \infty$. Now let

$$f(s) = \sum_{n=1}^{\infty} a_n\, e^{s.\lambda_n}\ ,$$

where a_n's satisfy (2.8). Then $f \in X_\lambda$. Also, let

$$f_m(s) = \sum_{n=1}^{m} a_n\, e^{s.\lambda_n}\ .$$

Then $f_m \in X_\lambda$ for $m = 1, 2, \ldots$. Let q be any fixed positive integer and let $0 < \varepsilon < \frac{1}{q}$. From (2.8), we can find an integer m such that

$$|a_n| < \exp\left[\frac{-\lambda_n\,\phi(\lambda_n)}{\{K\,\rho^*(T^* + \varepsilon)\}^{1/(\rho^*-1)}}\right], \qquad n > m\ .$$

Then

$$\begin{aligned}
\Big\| f - \sum_{n=1}^{m} a_n\, e^{s.\lambda_n} \Big\|_q &= \Big\| \sum_{n=m+1}^{\infty} a_n\, e^{s.\lambda_n} \Big\|_q = \\
&= \sum_{n=m+1}^{\infty} |a_n|.\exp\left[\frac{\lambda_n\,\phi(\lambda_n)}{\{K\,\rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}}\right] \\
&< \sum_{n=m+1}^{\infty} \exp\left[\frac{\lambda_n\,\phi(\lambda_n)}{(K\,\rho^*)^{1/(\rho^*-1)}}\left\{\left(T^* + \frac{1}{q}\right)^{-1/(\rho^*-1)} - (T^* + \varepsilon)^{-1/(\rho^*-1)}\right\}\right] \\
&< \varepsilon \ \text{ for sufficiently large values of } m\ .
\end{aligned}$$

Hence

$$\lambda(f,f_m) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f-f_m\|_q}{1+\|f-f_m\|_q} \le \frac{\varepsilon}{(1+\varepsilon)} < \varepsilon \ ,$$

i.e., $f_m \to f$ as $m \to \infty$ in X_λ. Hence by assumption that $\psi \in X'_\lambda$, we have

$$\lim_{m\to\infty} \psi(f_m) = \psi(f) \ .$$

Let us denote by $C_n = \psi(e^{s.\lambda_n})$. Then

$$\psi(f_m) = \sum_{n=1}^{m} a_n \, \psi(e^{s.\lambda_n}) = \sum_{n=1}^{m} a_n \, C_n \ .$$

Also $|C_n| = |\psi(e^{s.\lambda_n})|$. Since ψ is continuous on X_λ, it is continuous on $X_{\| \, \|_q}$ for each $q = 1, 2, 3, \ldots$. Hence there exists a positive constant A independent of q such that

$$|\psi(e^{s.\lambda_n})| = |C_n| \le A\|\alpha\|_q \, , \qquad q \ge 1 \ ,$$

where $\alpha(s) = e^{s.\lambda_n}$. Now using the definition of the form for $\alpha(s)$, we get

$$|C_n| \le A.\exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K \, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}}\Big] , \qquad n \ge 1, \ \ q \ge 1 \ .$$

Hence we get $\psi(f) = \sum_{n=1}^{\infty} a_n \, C_n$, where C_n's satisfy (2.11).

Conversely, suppose that $\psi(f) = \sum_1^{\infty} a_n \, C_n$ and C_n satisfies (2.11). Then for $q \ge 1$,

$$|\psi(f)| \le A \sum_{n=1}^{\infty} |a_n|.\exp\Big[\frac{\lambda_n \, \phi(\lambda_n)}{\{K \, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}}\Big]$$

i.e. $|\psi(f)| \le A.\|f\|_q \, , \qquad q \ge 1$,

i.e. $\psi \in X'_{\| \, \|_q} \, , \qquad q \ge 1$.

Now, since

$$\lambda(f,g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f-g\|_q}{1+\|f-g\|_q} \ ,$$

therefore $X'_\lambda = \bigcup_{q=1}^{\infty} X'_{\| \, \|_q}$. Hence $\psi \in X'_\lambda$.

This completes the proof of Theorem 2. ∎

Lastly, we give the construction of total sets in X_λ. Following [2], we give

Definition. Let X be a locally convex topological vector space. A set $E \subset X$ is said to be total if and only if for any $\psi \in X'$ with $\psi(E) = 0$, we have $\psi = 0$.

Now, we prove

Theorem 3. *Consider the space X_λ defined before and let $f(s)=\sum_{n=1}^{\infty} a_n e^{s.\lambda_n}$, $a_n \neq 0$, for $n = 1,2,...$, $f \in X_\lambda$. Suppose G is a subset of the complex plane having at least one limit point in the complex plane. Define, for $\mu \in G$,*

$$f_\mu(s) = \sum_{n=1}^{\infty} (a_n\, e^{\mu.\lambda_n}).e^{s.\lambda_n} \ .$$

Then $E = \{f_\mu \colon \mu \in G\}$ is total in X_λ.

Proof: Since $f \in X_\lambda$, from (2.7) we have

$$\limsup_{n\to\infty} \frac{\lambda_n\, \phi(\lambda_n)}{\log |a_n\, e^{\mu.\lambda_n}|^{-1}} = \limsup_{n\to\infty} \frac{\phi(\lambda_n)}{\log |a_n|^{-1/\lambda_n} - R(\mu)}$$
$$\leq \left(\frac{\rho^*}{\rho^*-1}\right) (\rho^* T^*)^{1/(\rho^*-1)}, \quad \text{since } R(\mu) < \infty \ .$$

Hence, if we denote by $M_\mu(\sigma) = \sup_{-\infty<t<\infty} |f_\mu(\sigma+it)|$, then from (2.6),

$$\limsup_{\sigma\to\infty} \frac{\log M_\mu(\sigma)}{\sigma^{\rho^*(\sigma)}} \leq T^* < \infty \ .$$

Therefore, $f_\mu \in X_\lambda$ for each $\mu \in G$. Thus $E \subset X_\lambda$.

Now, let ψ be a linear continuous functional on X_λ and suppose that $\psi(f_\mu)=0$. From Theorem 2, there exists a sequence $\{C_n\}$ of complex numbers such that

$$\psi(g) = \sum_{n=1}^{\infty} b_n\, C_n \,, \qquad g(s) = \sum_{n=1}^{\infty} b_n\, e^{s\lambda_n} \ \in \ X_\lambda \ ,$$

where

$$|C_n| < A.\exp\left[\frac{\lambda_n\, \phi(\lambda_n)}{\{K\, \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^*-1)}}\right], \qquad n \geq 1, \quad q \geq 1 \ , \tag{2.12}$$

A being a constant and λ_1 is sufficiently large.

Hence

$$\psi(f_\mu) = \sum_{n=1}^{\infty} a_n\, C_n\, e^{\mu\lambda_n} = 0 \,, \qquad \mu \in G \ .$$

Let us consider the function $F(s) = \sum_{n=1}^{\infty} a_n\, C_n\, e^{s.\lambda_n}$.

Then from (2.8) and (2.12), for any ε, $0 < \varepsilon < \frac{1}{q}$,

$$|a_n C_n|^{1/\lambda_n} < A^{1/\lambda_n} \exp\left[\phi(\lambda_n) \left\{ \left(K\rho^* \left(T^* + \frac{1}{q}\right)^{-1/(\rho^*-1)} \right) - (K\rho^*(T^*+\varepsilon)^{-1/(\rho^*-1)}) \right\} \right]$$

for all $n > n_0$. By definition of $\phi(t)$, $\phi(\lambda_n) \to \infty$ as $n \to \infty$ and $\lambda_n \to \infty$. Hence we get

$$\limsup_{n\to\infty} \frac{\log |a_n\, C_n|}{\lambda_n} = -\infty \ ,$$

i.e., $F(s)$ satisfies (1.4). Hence $F \in X$.

Also, $F(\mu) = 0\ \forall\, \mu \in G$. Thus the entire function $F(s) \equiv 0$ in the entire complex plane. But this implies that $a_n\, C_n = 0$, $\forall\, n = 1, 2, \ldots$. Since we have started with $a_n \neq 0$, thus we get $C_n = 0$, $n = 1, 2, \ldots$. Hence $\psi \equiv 0$. This proves Theorem 3. ∎

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