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APPLICATIONS OF (p, q)-GAMMA FUNCTION TO SZÁSZ DURRMEYER OPERATORS

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Abstract. We define a (p, q) analogue of Gamma function. As an application, we propose (p, q)-Szász–Durrmeyer operators, estimate moments and establish some direct results.

1. Introduction

In the last two decades the *quantum calculus* is an active area of research among researchers. The *quantum calculus* find applications in a number of areas, including approximation theory. The relationship between approximation theory and *q-calculus* encouraged the mathematicians to give *q-analogue* of known results (see [**3**]). This rapid development of *q-calculus* has led to the discovery of new generalization of this theory. This produces some advantages like that the rate of convergence of *q*-operators is more flexible and better than the classical one. Since the *q-calculus* is based on one parameter, there is a possibility of extension of *q-calculus*. In this direction Sahai–Yadav [**14**] established some extensions to post-quantum calculus in special functions. A question arises: can we modify the operators using (p, q)-*calculus* such that our modified operator has better error estimation than the classical ones. For this purpose, we will define (p, q)-Szász–Durrmeyer operators. Several well-known operators may extend to (p, q)-*analogues*. Mursaleen et al introduced the (p, q)-*analogue* of the Bernstein operators in [**10**]. There both point-wise convergence and asymptotic formula are considered. Other important class of discrete operators has been investigated by using (p, q)-*calculus*. For example (p, q)-Bernstein–Stancu operators appeared in [**9**] (p, q) Bleimann–Butzer–Hahn and (p, q)-Szász Mirakyan operators have been studied recently in [**1**, **11**]. Very recently, in order to obtain an approximation process in the space of (p, q)-Bernstein operators, the authors [**5**] defined Durrmeyer type modification of (p, q)-Bernstein operators.

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Motivated by all the above results we propose Durrmeyer type modification of the (p,q)-Szász Mirakyan operators using an integral version of (p,q)-Gamma function (as we know it is first in literature).

The paper is organized as follows: the next section contains some basic facts regarding(p,q)-*calculus*, we also introduce (p,q)-*analogue* of Gamma function. The construction of the announced class of operators is presented in Section 3. Section 4 deals with the quantitative type estimate with a suitable modulus of continuity. The last section is devoted to weighted Korovin type theorems and we estimate the approximation of bounded functions by announced operators with the help of a Lipschitz-type maximal function.

2. Notations and Preliminaries

Following the definitions and notations of [**14**]:
Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the (p,q)-numbers are defined as

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}$$

for $n \in \mathbb{N}$. The (p,q)-factorial $[n]_{p,q}!$ of the element $n \in \mathbb{N}$ means

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, n \geqslant 1, [0]_{p,q}! = 1.$$

The (p,q)-binomial theorem is given by

$${}_1\Phi_0((a,b); -; (p,q), x) = \frac{((p,bx); -; (p,q))_\infty}{((p,ax); -; (p,q))_\infty},$$

where $((a,b); -; (p,q))_\infty = \prod_{n=0}^{\infty}(ap^n - bq^n)$. Two different (p,q)-expansions named $E_{p,q}$ and $e_{p,q}$ of the exponential function $x \mapsto e^x$ are given as follows:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2}}{[n]_{p,q}!} x^n =_1 \Phi_0((1,0); -; (p,q), x),$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_{p,q}!} x^n =_1 \Phi_0((0,1); -; (p,q), -x). \tag{2.1}$$

We know that ${}_1\Phi_0((1,0); -; (p,q), x){}_1\Phi_0((0,1); -; (p,q), x) = 1$, that is the following relation between (p,q)-exponential functions

$$e_{p,q}(x)E_{p,q}(-x) = 1 \tag{2.2}$$

holds. We mention that these (p,q)-analogues of the classical exponential functions are valid for $0 < q < p \leqslant 1$. Moreover $E_{p,q}(x)$ and $e_{p,q}(x)$ tend to e^x as $p \to 1^-$ and $q \to 1^-$.

It is obvious by the (p,q)-derivative formula $D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0$ that

$$D_{p,q}E_{p,q}(x) = E_{p,q}(qx),$$

$$D_{p,q}E_{p,q}(ax) = aE_{p,q}(aqx). \tag{2.3}$$

PROPOSITION 2.1. [**13**] *The formula of* (p,q)*-integration by part is given by*

$$\int_a^b f(px)D_{p,q}g(x)\,d_{p,q}x = f(b)g(b) - f(a)q(a) - \int_0^a g(qx)D_{p,q}f(x)\,d_{p,q}x$$

DEFINITION 2.1. For any $n \in \mathbb{N}$, we define a (p,q)-Gamma function by

$$\Gamma_{p,q}(n) = \int_0^\infty p^{(n-1)(n-2)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x.$$

LEMMA 2.1. *For any* $n \in \mathbb{N}$*, we have* $\Gamma_{p,q}(n+1) = [n]_{p,q}!$.

PROOF. From (2.1) we have $E_{p,q}(0) = 1$ and from (2.2) we have

$$\begin{aligned} E_{p,q}(\infty) &= \lim_{x\to\infty} E_{p,q}(x) = \lim_{x\to\infty} e_{p,q}(-x) = \lim_{x\to\infty} \Phi_0((1,0);-;(p,q),-x) \\ &= \lim_{x\to\infty} \frac{((p,0);-;(p,q))_\infty}{((p,-x);-;(p,q))_\infty} = 0. \end{aligned}$$

Also from (2.3) we can write

$$\begin{aligned} \Gamma_{p,q}(n+1) &= \int_0^\infty p^{n(n-1)/2}x^nE_{p,q}(-qx)\,d_{p,q}x \\ &= -\int_0^\infty p^{n(n-1)/2}x^nD_{p,q}E_{p,q}(-x)\,d_{p,q}x. \end{aligned}$$

By Proposition 2.1 using (p,q)-integration by parts for $f(x) = x^n$ and $g(x) = E_{p,q}(-x)$, we have

$$\begin{aligned} \Gamma_{p,q}(n+1) &= \frac{[n]_{p,q}}{p^{n-1}} \int_0^\infty p^{n(n-1)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x \\ &= [n]_{p,q} \int_0^\infty p^{(n-1)(n-2)/2}x^{n-1}E_{p,q}(-qx)\,d_{p,q}x = [n]_{p,q}\Gamma_{p,q}(n). \end{aligned}$$

Thus, we have

$$\Gamma_{p,q}(n+1) = [n]_{p,q}\Gamma_{p,q}(n) = [n]_{p,q}[n-1]_{p,q}\Gamma_{p,q}(n-1) = [n]_{p,q}!. \qquad \square$$

An alternate form of (p,q)-Gamma function without integral expression for n nonnegative integer, is given in [**12**] by

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)^n_{p,q}}{(p-q)^n} = [n]_{p,q}!, \quad 0 < q < p.$$

3. (p,q)-Szász–Durrmeyer Operators and Moments

In order to introduce a (p,q) Durrmeyer variant for Szasz–Mirakyan operators, we present a construction due to Acar [**1**]. The (p,q)-analogue of Szász operators for $x \in [0,\infty)$ and $0 < q < p \leqslant 1$ defined by in the following way

$$S_{n,p,q}(f;x) = \sum_{k=0}^{n} s^{p,q}_{n,k}(x) f\Big(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\Big), \tag{3.1}$$

where

$$s_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}([n]_{p,q}x)} \frac{q^{k(k-1)/2}}{[k]_{p,q}!}([n]_{p,q}x)^k.$$

In case $p = 1$, we get the q-Szász operators [**2**]. If $p = q = 1$, we get at once the well known Szász operators.

LEMMA 3.1. [**1**] *For* $x \in x \in [0, \infty)$, $0 < q < p \leqslant 1$, *we have*
(1) $S_{n,p,q}(1; x) = 1$, (2) $S_{n,p,q}(t; x) = qx$, (3) $S_{n,p,q}(t^2; x) = pqx^2 + \frac{q^2}{[n]_{p,q}}x$.

The Szász operators defined by (3.1) are discrete operators. The integral modification of these operators was proposed in [**7**]. Different variants and q-analogues in [**3**] and [**6**]. As an application of the (p, q)-Gamma function, we introduce below the Durrmeyer type (p, q) variant of the Szász operators as

DEFINITION 3.1. The (p, q)-analogue of Szász–Durrmeyer operator for $x \in [0, \infty)$ and $0 < q < p \leqslant 1$ is defined by

$$\widetilde{S}_{n,p,q}(f; x) = [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} \frac{([n]_{p,q}t)^k}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t) f(q^{1-k}p^k t)\, d_{p,q}t$$

where $s_{n,k}^{p,q}(x)$ is defined in (3.1).

It may be remarked here that for $p = q = 1$ these operators reduce to the Szász–Durrmeyer operators.

LEMMA 3.2. *For* $x \in x \in [0, \infty)$, $0 < q < p \leqslant 1$, *we have*
(1) $\widetilde{S}_{n,p,q}(1; x) = 1$
(2) $\widetilde{S}_{n,p,q}(t; x) = \frac{q}{[n]_{p,q}} + px$
(3) $\widetilde{S}_{n,p,q}(t^2; x) = \frac{p^3}{q}x^2 + \frac{[2]_{p,q}^2 x}{[n]_{p,q}} + \frac{[2]_{p,q}q^2}{p[n]_{p,q}^2}$.

PROOF. Using Definition 2.1, Lemmas 2.1 and 3.1, we have

$$\begin{aligned}\widetilde{S}_{n,p,q}(1; x) &= [n]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} \frac{([n]_{p,q}t)^k}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t)\, d_{p,q}t \\ &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{\Gamma_{p,q}(k+1)}{[k]_{p,q}!} = 1\end{aligned}$$

and next using $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we have

$$\begin{aligned}\widetilde{S}_{n,p,q}(t; x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^{\infty} p^{k(k-1)/2} q^{1-k} p^k \frac{([n]_{p,q}t)^{k+1}}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t)\, d_{p,q}t \\ &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{\Gamma_{p,q}(k+2)}{[k]_{p,q}![n]_{p,q}} = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{[k+1]_{p,q}}{[n]_{p,q}} \\ &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{1-k} \frac{(q^k + p[k]_{p,q})}{[n]_{p,q}}\end{aligned}$$

$$= \frac{q}{[n]_{p,q}} S_{n,p,q}(1;x) + \frac{p}{q} S_{n,p,q}(t;x) = \frac{q}{[n]_{p,q}} + px$$

and

$$\widetilde{S}_{n,p,q}(t^2;x) = \frac{1}{[n]_{p,q}} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)$$
$$\int_0^{\infty} p^{k(k-1)/2} q^{2-2k} p^{2k} \frac{([n]_{p,q}t)^{k+2}}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t)\, d_{p,q}t$$
$$= \frac{1}{[n]_{p,q}} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{2-2k}$$
$$\int_0^{\infty} p^{(k+1)(k+2)/2} p^{-1} \frac{([n]_{p,q}t)^{k+2}}{[k]_{p,q}!} E_{p,q}(-q[n]_{p,q}t)\, d_{p,q}t$$
$$= \frac{1}{p[n]_{p,q}^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{2-2k} \frac{\Gamma_{p,q}(k+3)}{[k]_{p,q}!}$$
$$= \frac{1}{p[n]_{p,q}^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{2-2k} [k+2]_{p,q}[k+1]_{p,q}$$
$$= \frac{1}{p[n]_{p,q}^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) q^{2-2k} (p^3[k]_{p,q}^2 + q^k(p[2]_{p,q} + p^2)[k]_{p,q} + q^{2k}[2]_{p,q})$$
$$= \frac{p^2}{q^2} S_{n,p,q}(t^2;x) + \frac{[2]_{p,q} + p}{[n]_{p,q}} S_{n,p,q}(t;x) + \frac{[2]_{p,q} q^2}{p[n]_{p,q}^2} S_{n,p,q}(1;x)$$
$$= \frac{p^2}{q^2}(pqx^2 + \frac{q^2}{[n]_{p,q}} x) + \frac{([2]_{p,q} + p)qx}{[n]_{p,q}} + \frac{[2]_{p,q} q^2}{p[n]_{p,q}^2}$$
$$= \frac{p^3}{q} x^2 + \frac{[2]_{p,q}^2 x}{[n]_{p,q}} + \frac{[2]_{p,q} q^2}{p[n]_{p,q}^2}. \qquad \square$$

REMARK 3.1. For $0 < q < p \leqslant 1$ we may write

$$\widetilde{S}_{n,p,q}((t-x),x) = \frac{q}{[n]_{p,q}} + (p-1)x,$$

$$\widetilde{S}_{n,p,q}((t-x)^2,x) = \frac{(p^3 - 2pq + q)x^2}{q} + \frac{([2]_{p,q}^2 - 2q)x}{[n]_{p,q}} + \frac{[2]_{p,q} q^2}{p[n]_{p,q}^2}. \tag{3.2}$$

4. Quantitative Estimate

By $C_B[0,\infty)$ we denote the class of all real valued continuous and bounded functions on $[0,\infty)$. The norm $\|.\|_B$ is defined by

$$\|f\|_{C_B} = \sup_{x \in [0,\infty)} |f(x)|.$$

For $f \in C_B$ the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))]\, du\, dv \tag{4.1}$$

By simple computation, it is observed that

(i) $\|f_h - f\|_{C_B} \leqslant \omega_2(f,h)$.

(ii) If f is continuous, then $f_h', f'' \in C_B$ and

$$\|f_h'\|_{C_B} \leqslant \frac{5}{h}\,\omega(f,h), \quad \|f_h''\|_{C_B} \leqslant \frac{9}{h^2}\,L\omega_2(f,h),$$

where the first and second order moduli of continuity are respectively defined by

$$\omega(f,\delta) = \sup_{\substack{x,u,v\geqslant 0\\ |u-v|\leqslant\delta}} |f(x+u) - f(x+v)|,$$
$$\omega_2(f,\delta) = \sup_{\substack{x,u,v\geqslant 0\\ |u-v|\leqslant\delta}} |f(x+2u) - 2f(x+u+v) + f(x+2v)|, \quad \delta \geqslant 0.$$

THEOREM 4.1. *Let $q \in (0,1)$ and $p \in (q,1]$ The operator $\widetilde{S}_{n,p,q}$ maps space C_B into C_B and $\|\widetilde{S}_{n,p,q}(f)\|_{C_B} \leqslant \|f\|_{C_B}$.*

PROOF. Let $q \in (0,1)$ and $p \in (q,1]$. From Lemma 3.2 we have

$$\begin{aligned}
&|\widetilde{S}_{n,p,q}(f,x)|\\
&\leqslant [n]_{p,q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\int_0^\infty p^{k(k-1)/2}\frac{([n]_{p,q}t)^k}{[k]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)|f(q^{1-k}p^k t)|d_{p,q}t\\
&\leqslant \sup_{x\in[0,\infty)}|f(x)[n]_{p,q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\int_0^\infty p^{k(k-1)/2}\frac{([n]_{p,q}t)^k}{[k]_{p,q}!}E_{p,q}(-q[n]_{p,q}t)\,d_{p,q}t\\
&= \sup_{x\in[0,\infty)}|f(x)|\widetilde{S}_{n,p,q}(1,x) = \|f\|_{C_B}.
\end{aligned}$$

□

We are going to study the degree of approximation in terms of the first and second order moduli of continuity.

THEOREM 4.2. *Let $q \in (0,1)$ and $p \in (q,1]$. If $f \in C_B[0,\infty)$, then*

$$\begin{aligned}
&|\widetilde{S}_{n,p,q}(f,x) - f(x)|\\
&\quad\leqslant 5\omega\left(f,\frac{1}{\sqrt{[n]_{p,q}}}\right)\left(\frac{q}{\sqrt{[n]_{p,q}}} + \sqrt{[n]_{p,q}}(p-1)x\right)\\
&\qquad + \frac{9}{2}\omega_2\left(f,\frac{1}{\sqrt{[n]_{p,q}}}\right)\left[2 + \frac{(p^3-2pq+q)[n]_{p,q}x^2}{q} + ([2]_{p,q}^2 - 2q)x + \frac{[2]_{p,q}q^2}{p[n]_{p,q}}\right].
\end{aligned}$$

PROOF. For $x \geqslant 0$ and $n \in \mathbb{N}$ and using the Steklov mean f_h defined by (4.1), we can write

$$|\widetilde{S}_{n,p,q}(f,x) - f(x)| \leqslant \widetilde{S}_{n,p,q}(|f-f_h|,x) + |\widetilde{S}_{n,p,q}(f_h - f_h(x),x)| + |f_h(x) - f(x)|.$$

First by Theorem 4.1 and property (i) of the Steklov mean we have

$$\widetilde{S}_{n,p,q}(|f-f_h|,x) \leqslant \|\widetilde{S}_{n,p,q}(f-f_h)\|_{C_B} \leqslant \|f-f_h\|_{C_B} \leqslant \omega_2(f,h).$$

Since $\widetilde{S}_{n,p,q}$ is a linear positive operator we get

$$|\widetilde{S}_{n,p,q}(f_h - f_h(x), x)| \leqslant |f_h'(x)|\widetilde{S}_{n,p,q}(t-x,x) + \frac{1}{2}\|f''\|_{C_B}\widetilde{S}_{n,p,q}((t-x)^2, x).$$

By Lemma 3.2, we have

$$\begin{aligned}|\widetilde{S}_{n,p,q}(f_h - f_h(x), x)| \leqslant \frac{5}{h}\omega(f,h)\bigg(\frac{q}{[n]_{p,q}} + (p-1)x\bigg) \\ + \frac{9}{2h^2}\omega_2(f,h)\widetilde{S}_{n,p,q}((t-x)^2, x),\end{aligned}$$

where $\widetilde{S}_{n,p,q}((t-x)^2,x)$ is given by (3.2). For $x \geqslant 0$, $h > 0$ and choosing $h = \sqrt{1/[n]_{p,q}}$, we get the desired result. □

REMARK 4.1. For $q \in (0,1)$ and $p \in (q,1]$ it is seen that $\lim_{n\to\infty}[n]_{p,q} = 1/(q-p)$. In order to consider the convergence of (p,q)-Szász–Durrmeyer operators, we assume $p = (p_n)$ and $q = (q_n)$ such that $0 < q_n < p_n \leqslant 1$ and for n sufficiently large $p_n \to 1, q_n \to 1$, $p_n^n \to a, q_n^n \to b$, so that $[n]_{p_n,q_n} \to \infty$. Such a sequence can always be constructed for example, we can take $p_n = 1 - 1/2n$ and $q_n = 1 - 1/n$. Clearly $\lim_{n\to\infty} p_n^n = e^{-1/2}$, $\lim_{n\to\infty} q_n^n = e^{-1}$ and $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$.

5. Direct Estimates

Let us denote by $H_{x^2}[0,\infty)$ the set of all functions f defined on the positive real axis satisfying the condition $|f(x)| \leqslant M_f(1+x^2)$, where M_f is an absolute constant depending on f. By $C_{x^2}[0,\infty)$, we mean the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ denote the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The class $C^*_{x^2}[0,\infty)$ is endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

We discuss below the weighted approximation theorem, where the approximation formula is valid for the positive real axis (see [**4**]).

THEOREM 5.1. *Let $p = p_n$ and $q = q_n$ satisfies $0 < q_n < p_n \leqslant 1$ and for n sufficiently large $p_n \to 1$, $q_n \to 1$ and $q_n^n \to a$ and $p_n^n \to b$. For each $f \in C^*_{x^2}[0,\infty)$, we have* $\lim_{n\to\infty}\|\widetilde{S}_{n,p_n,q_n}(f) - f\|_{x^2} = 0$.

PROOF. Using Korovkin's theorem, it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|\widetilde{S}_{n,p_n,q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0,1,2. \tag{5.1}$$

Since $\widetilde{S}_{n,p_n,q_n}(1,x) = 1$ the first condition of (5.1) is fulfilled for $\nu = 0$.

For $n \in \mathbb{N}$, we can write,

$$\|\widetilde{S}_{n,p_n,q_n}(t,x) - x\|_{x^2} \leqslant \frac{q_n}{[n]_{p_n,q_n}} + (p_n - 1)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$\|\widetilde{S}_{n,p_n,q_n}(t^2,x)-x^2\|_{x^2}\leqslant\left(\frac{p_n^3}{q_n}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}+\frac{[2]^2_{p_n,q_n}}{[n]_{p_n,q_n}}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{[2]_{p_n,q_n}q_n^2}{p_n[n]^2_{p_n,q_n}}\sup_{x\in[0,\infty)}\frac{1}{1+x^2},$$

which implies that for $v=1,2$ we have $\lim_{n\to\infty}\|\widetilde{S}_{n,p_n,q_n}(t^v,x)-x^v\|_{x^2}=0$. □

We give the following theorem to approximate all functions in $C_{x^2}[0,\infty)$.

THEOREM 5.2. *Let $p=p_n$ and $q=q_n$ satisfies $0<q_n<p_n\leqslant 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $q_n^n\to a$ and $p_n^n\to b$. For each $f\in C_{x^2}[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\widetilde{S}_{n,p_n,q_n}(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}=0.$$

PROOF. For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)}&\frac{|\tilde{S}_{n,p_n,q_n}(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}\\&=\sup_{x\leqslant x_0}\frac{|\tilde{S}_{n,p_n,q_n}(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\geqslant x_0}\frac{|\tilde{S}_{n,p_n,q_n}(f,x)-f(x)|}{(1+x^2)^{1+\alpha}}\\&\leqslant\|\tilde{S}_{n,p_n,q_n}(f)-f\|_{C[0,a]}\\&\quad+\|f\|_{x^2}\sup_{x\geqslant x_0}\frac{|\tilde{S}_{n,p_n,q_n}(1+t^2,x)\|}{(1+x^2)^{1+\alpha}}+\sup_{x\geqslant x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.\end{aligned}$$

By Lemma 3.2 and the well known Korovkin theorem, the first term of the above inequality tends to zero for a sufficiently large n. By Lemma 3.2, for any fixed $x_0>0$, it is easily seen that $\sup_{x\geqslant x_0}\frac{|\tilde{S}_{n,p_n,q_n}(1+t^2,x)|}{(1+x^2)^{1+\alpha}}$ tends to zero as $n\to\infty$. We can choose $x_0>0$ so large that the last part of above inequality can be made small enough. This completes the proof of the theorem. □

Now we establish some point-wise estimates of the rate of convergence of (p,q)-Szász–Durrmeyer operators. First, we give the relationship between the local smoothness of f and local approximation. A function $f\in C[0,\infty)$ is is said to satisfy the Lipschitz condition Lip_α on D, $\alpha\in(0,1]$, $D\subset[0,\infty)$ if

$$|f(t)-f(x)|\leqslant M_f|t-x|^\alpha,\quad t\in[0,\infty)\text{ and }x\in D,\tag{5.2}$$

where M_f is a constant depending only α and f.

THEOREM 5.3. *Let $f\in\mathrm{Lip}_\alpha$ on D, $D\subset[0,\infty)$ and $\alpha\in(0,1]$. We have*

$$|\tilde{S}_{n,p,q}(f,x)-f(x)|\leqslant\left(\frac{(p^3-2pq+q)x^2}{q}+\frac{([2]^2_{p,q}-2q)x}{[n]_{p,q}}+\frac{[2]_{p,q}q^2}{p[n]^2_{p,q}}\right)^{\alpha/2}+2d^\alpha(x;D)$$

where $d(x;D)$ represents the distance between x and D.

PROOF. For $x_0 \in \bar{D}$, the closure of the set D in $[0,\infty)$, we have

$$|f(t) - f(x)| \leqslant |f(t) - f(x_0)| + |f(x_0) - f(x)|, \quad x \in [0,\infty).$$

Using (5.2) we get

$$\begin{aligned} |\tilde{S}_{n,p,q}(f,x) - f(x)| &\leqslant \tilde{S}_{n,p,q}(|f(t) - f(x_0)|, x) + |f(x_0) - f(x)| \\ &\leqslant M_f \tilde{S}_{n,p,q}(|t - x_0|^\alpha, x) + M_f |x_0 - x|^\alpha. \end{aligned} \tag{5.3}$$

Then, with Hölder's inequality with $p := \frac{2}{\alpha}$ and $\frac{1}{r} := 1 - \frac{1}{p}$, we have

$$\tilde{S}_{n,p,q}(|t-x|^\alpha, x) \leqslant (\widetilde{S}_{n,p,q}(|t-x|^2, x))^{\frac{\alpha}{2}} (\tilde{S}_{n,p,q}(1,x))^{1-\frac{\alpha}{2}}. \tag{5.4}$$

Also $\widetilde{S}_{n,p,q}$ is monotone

$$\tilde{S}_{n,p,q}(|t - x_0|^\alpha, x) \leqslant (\tilde{S}_{n,p,q}(|t-x|^\alpha, x))^{\frac{\alpha}{2}} + |x_0 - x|^\alpha.$$

Using (5.3), (5.4) and (3.2), we get the desired result. □

Now, we give a local direct estimate for (p,q)-Szász–Durrmeyer operators using the Lipschitz-type maximal function of order α introduced by Lenze [**8**] as

$$\tilde{\omega}_\alpha(f,x) = \sup_{t \neq x,\ t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^\alpha}, \quad x \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{5.5}$$

THEOREM 5.4. *Let $f \in \mathrm{Lip}_\alpha$ on D and $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$, we have*

$$|\tilde{S}_{n,p,q}(f,x) - f(x)| \leqslant \tilde{\omega}_\alpha(f,\ x) \bigg(\frac{(p^3 - 2pq + q)x^2}{q} + \frac{([2]_{p,q}^2 - 2q)x}{[n]_{p,q}} + \frac{[2]_{p,q} q^2}{p[n]_{p,q}^2} \bigg)^{\frac{\alpha}{2}}$$

PROOF. From (5.5) we have

$$\begin{aligned} |f(t) - f(x)| &\leqslant \tilde{\omega}_\alpha(f,x)|t-x|^\alpha, \\ |\tilde{S}_{n,p,q}(f,x) - f(x)| &\leqslant \tilde{S}_{n,p,q}(|f(t) - f(x)|, x) \\ &\leqslant \tilde{\omega}_\alpha(f,x) \tilde{S}_{n,p,q}(|t-x|^\alpha, x). \end{aligned}$$

Applying Hölder's inequality with $p := \frac{2}{\alpha}$ and $\frac{1}{r} := 1 - \frac{1}{p}$, we have

$$|\tilde{S}_{n,p,q}(f,x) - f(x)| \leqslant \tilde{\omega}_\alpha(f,x) \tilde{S}_{n,p,q}((t-x)^2, x)^{\frac{\alpha}{2}}.$$

Using (3.2), we have our assertion. □

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