PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 102(116) (2017), 93–105 DOI: https://doi.org/10.2298/PIM1716093Y

f-KENMOTSU MANIFOLDS WITH THE SCHOUTEN–VAN KAMPEN CONNECTION

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Abstract. We study 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. With the help of such a connection, we study projectively flat, conharmonically flat, Ricci semisymmetric and semisymmetric 3-dimensional f-Kenmotsu manifolds. Finally, we give an example of 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection.

1. Introduction

The Schouten–van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [**2**, **4**, **11**]. Solov'ev investigated hyperdistributions in Riemannian manifolds using the Schouten–van Kampen connection [**12**–**15**]. Then Olszak studied the Schouten–van Kampen connection to an almost contact metric structure [**8**]. He characterized some classes of almost contact metric manifolds with the Schouten–van Kampen connection and found certain curvature properties of this connection on these manifolds.

On the other hand, let M be an almost contact manifold, i.e., M is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) [**1**]. Denote by Φ the fundamental 2-form of M, $\Phi(X,Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on M.

For further use, we recall the following definitions [**1**, **3**, **10**]. The manifold M and its structure (ϕ, ξ, η, g) is said to be:

i) normal, if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),
ii) almost cosymplectic, if $d\eta = 0$ and $d\Phi = 0$,

2010 *Mathematics Subject Classification:* 53C15, 53C25, 53C50.
Key words and phrases: Schouten-van Kampen connection, f-Kenmotsu manifolds, Ricci-semisymmetric, semisymmetric, Einstein manifold, η-Einstein manifold.
Communicated by Stevan Pilipović.

iii) cosymplectic, if it is normal and almost cosymplectic (equivalently, $\nabla\phi = 0$, ∇ being covariant differentiation with respect to the Levi-Civita connection).

The manifold M is called locally conformal, cosymplectic (respectively almost cosymplectic), if M has an open covering $\{U_t\}$ endowed with differentiable functions $\sigma_t : U_i \to \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \quad \xi_t = e^{\sigma_t}\xi, \quad \eta_t = e^{-\sigma_t}\eta, \quad g_t = e^{-2\sigma_t}g$$

is cosymplectic (respectively almost cosymplectic).

Also, Olszak and Rosca [**9**] studied normal locally conformal almost cosymplectic manifolds. They given a geometric interpretation of f-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

By an f-Kenmotsu manifold, we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic manifold.

In the present paper we study some curvature properties of 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. The paper is organized as follows: after introduction, we give the Schouten–van Kampen connection and f-Kenmotsu manifolds. Then we adapt the Schouten–van Kampen connection on 3-dimensional f-Kenmotsu manifolds. In section 5, we study projectively flat 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. In section 6, we consider conharmonically flat 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. Section 7 is devoted to study Ricci semisymmetric 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection and we prove that if a 3-dimensional f-Kenmotsu manifold is Ricci semisymmetric, then it is an η-Einstein manifold. In section 8, we study semisymmetric 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. Finally, we give an example of a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection which verifies Theorem 5.1 and Theorem 6.1.

2. The Schouten–van Kampen connection

Let M be a connected pseudo-Riemannian manifold of an arbitrary signature $(p, n-p)$, $0 \leqslant p \leqslant n$, $n = \dim M \geqslant 2$. By g and ∇ we denote the pseudo-Riemannian metric and Levi-Civita connection induced from the metric g on M respectively. Assume that H and V are two complementary, orthogonal distributions on M such that $\dim H = n-1$, $\dim V = 1$, and the distribution V is non-null. Thus $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. Assume that ξ is a unit vector field and η is a linear form such that $\eta(\xi) = 1$, $g(\xi, \xi) = \varepsilon = \pm 1$ and

$$H = \ker \eta, \quad V = \operatorname{span}\{\xi\}.$$

We can always choose such ξ and η at least locally (in a certain neighborhood of an arbitrarily chosen point of M). We also have $\eta(X) = \varepsilon g(X, \xi)$. Moreover, it holds that $\nabla_X \xi \in H$.

For any $X \in TM$, by X^h and X^v we denote the projections of X onto H and V, respectively. Thus, we have $X = X^h + X^v$ with

$$X^h = X - \eta(X)\xi, \quad X^v = \eta(X)\xi. \tag{2.1}$$

The Schouten–van Kampen connection $\tilde{\nabla}$ associated to the Levi-Civita connection ∇ and adapted to the pair of the distributions (H, V) is defined by [**2**]

$$\tilde{\nabla}_X Y = (\nabla_X Y^h)^h + (\nabla_X Y^v)^v, \tag{2.2}$$

and the corresponding second fundamental form B is defined by $B = \nabla - \tilde{\nabla}$. Note that condition (2.2) implies the parallelism of the distributions H and V with respect to the Schouten–van Kampen connection $\tilde{\nabla}$.

From (2.1), one can compute

$$(\nabla_X Y^h)^h = \nabla_X Y - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi,$$
$$(\nabla_X Y^v)^v = (\nabla_X \eta)(Y)\xi + \eta(\nabla_X Y)\xi,$$

which enables us to express the Schouten–van Kampen connection with help of the Levi-Civita connection in the following way [**12**]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{2.3}$$

Thus, the second fundamental form B and the torsion $\tilde{T}$ of $\tilde{\nabla}$ are [**12, 13**]

$$B(X, Y) = \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi,$$
$$\tilde{T}(X, Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X, Y)\xi.$$

With the help of the Schouten–van Kampen connection (2.3), many properties of some geometric objects connected with the distributions H, V can be characterized [**12–14**]. Probably, the most spectacular is the following statement: g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla} g = 0$, $\tilde{\nabla}\eta = 0$.

3. f-Kenmotsu manifolds

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfying

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \end{gathered} \tag{3.1}$$

for any vector fields $X, Y \in \chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is a f-Kenmotsu manifold if the Levi-Civita connection of g satisfy [**7**]

$$(\nabla_X \phi)(Y) = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = \text{constant} \neq 0$, then the manifold is an α-Kenmotsu manifold [**5**]. 1-Kenmotsu manifold is a Kenmotsu manifold [**6**]. If $f = 0$, then the manifold is cosymplectic [**5**]. An f-Kenmotsu manifold is said to be *regular* if $f^2 + f' \neq 0$, where $f' = \xi(f)$.

For an f-Kenmotsu manifold from (3.1) it follows that

$$\nabla_X \xi = f\{X - \eta(X)\xi\}. \tag{3.2}$$

Then using (3.2), we have

$$(\nabla_X \eta)(Y) = f\{g(X,Y) - \eta(X)\,\eta(Y)\}. \tag{3.3}$$

The condition $df \wedge \eta = 0$ holds if $\dim M \geqslant 5$. This does not hold in general if $\dim M = 3$ [**9**].

As is well known, in a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ -\frac{\tau}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned}$$

In a 3-dimensional f-Kenmotsu manifold M, we have [**9**]

$$\begin{aligned} R(X,Y)Z &= \Big(\frac{\tau}{2} + 2f^2 + 2f'\Big)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad - \Big(\frac{\tau}{2} + 3f^2 + 3f'\Big)\big\{g(Y,Z)\,\eta(X)\xi - g(X,Z)\,\eta(Y)\xi \\ &\qquad + \eta(Y)\,\eta(Z)X - \eta(X)\,\eta(Z)Y\big\}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} S(X,Y) &= \Big(\frac{\tau}{2} + f^2 + f'\Big)\,g(X,Y) - \Big(\frac{\tau}{2} + 3f^2 + 3f'\Big)\,\eta(X)\,\eta(Y), \\ QX &= \Big(\frac{\tau}{2} + f^2 + f'\Big)X - \Big(\frac{\tau}{2} + 3f^2 + 3f'\Big)\,\eta(X)\xi, \end{aligned} \tag{3.5}$$

where R denotes the curvature tensor, S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature of M.

From (3.4) and (3.5), we obtain

$$R(X,Y)\xi = -(f^2 + f')\{\eta(Y)X - \eta(X)Y\}, \tag{3.6}$$

$$S(X,\xi) = -2(f^2 + f')\,\eta(X). \tag{3.7}$$

4. 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection

Let M be a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection. Then using (3.2) and (3.3) in (2.3), we get

$$\tilde{\nabla}_X Y = \nabla_X Y + f(g(X,Y)\xi - \eta(Y)X). \tag{4.1}$$

Let R and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten–van Kampen connection $\tilde{\nabla}$,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \tilde{R}(X,Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}.$$

Using (4.1), by direct calculations, we obtain the following formula connecting R and $\tilde{R}$ on a 3-dimensional f-Kenmotsu manifold M,

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + f^2\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + f'\big\{g(Y,Z)\,\eta(X)\xi - g(X,Z)\,\eta(Y)\xi \\ &\qquad + \eta(Y)\,\eta(Z)X - \eta(X)\,\eta(Z)Y\big\}. \end{aligned} \tag{4.2}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{S}, S$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\tau}, \tau$ of the connections $\tilde{\nabla}$ and ∇ are given by

(4.3)
$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ f^2\{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\}\\ &+ f'\{g(Y,Z)\,\eta(X)\,\eta(W) - g(X,Z)\,\eta(Y)\,\eta(W)\\ &\quad + g(X,W)\,\eta(Y)\,\eta(Z) - g(Y,W)\,\eta(X)\,\eta(Z)\},\end{aligned}$$

$$\tilde{S}(Y,Z) = S(Y,Z) + (2f^2 + f')\,g(Y,Z) + f'\eta(Y)\,\eta(Z)), \tag{4.4}$$

$$\tilde{Q}X = QX + (2f^2 + f')X + f'\eta(X)\xi, \tag{4.5}$$
$$\tilde{\tau} = \tau + 6f^2 + 4f',$$

respectively, where

$$\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W) \text{ and } R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

5. Projectively flat 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study projectively flat 3-dimensional f-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional f-Kenmotsu manifold, the projective curvature tensor with respect to the Schouten–van Kampen connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \tfrac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{5.1}$$

If $\tilde{P} = 0$, then the manifold M is called *projectively flat* manifold with respect to the Schouten–van Kampen connection.

Let M be a projectively flat manifold with respect to the Schouten–van Kampen connection. From (5.1), we have

$$\tilde{R}(X,Y)Z = \tfrac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{5.2}$$

Using (4.3) and (4.4) in (5.2), we get

$$\begin{aligned}&g(R(X,Y)Z,W) + f^2\{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\}\\ &\quad + f'\{g(Y,Z)\,\eta(X)\,\eta(W) - g(X,Z)\,\eta(Y)\,\eta(W)\\ &\quad + g(X,W)\,\eta(Y)\,\eta(Z) - g(Y,W)\,\eta(X)\,\eta(Z)\}\\ &= \tfrac{1}{2}\{S(Y,Z)\,g(X,W) - S(X,Z)\,g(Y,W)\\ &\quad + [2f^2 + f'][g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)]\\ &\quad + f'[\eta(Y)\,\eta(Z)\,g(X,W) - \eta(X)\,\eta(Z)\,g(Y,W)]\}.\end{aligned} \tag{5.3}$$

Now putting $W = \xi$ in (5.3), we obtain

$$\begin{aligned}(f^2 + f')\{g(X,Z)\,\eta(Y) - g(Y,Z)\,\eta(X)\} &+ (f^2+f')\{g(Y,Z)\,\eta(X) - g(X,Z)\,\eta(Y)\}\\ = \tfrac{1}{2}\{S(Y,Z)\,\eta(X) - S(X,Z)\,\eta(Y) &+ (2f^2+f')[g(Y,Z)\,\eta(X) - g(X,Z)\,\eta(Y)]\},\end{aligned}$$

which gives

$$S(Y,Z)\,\eta(X)-S(X,Z)\,\eta(Y)+(2f^2+f')\big[g(Y,Z)\,\eta(X)-g(X,Z)\,\eta(Y)\big]=0. \tag{5.4}$$

Again putting $X=\xi$ in (5.4), we get

$$S(Y,Z)=-(2f^2+f')\,g(Y,Z)-f'\eta(Y)\,\eta(Z). \tag{5.5}$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection.

Also, using (5.5) in (4.4), we obtain $\tilde{S}(Y,Z)=0$. Hence the manifold M is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Then from (5.2) the manifold M is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let M be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold M is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (5.1), we get $\tilde{P}(X,Y)Z=0$, that is, the manifold M is a projectively flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

THEOREM 5.1. *Let M be a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:*

i) *M is projectively flat with respect to the Schouten–van Kampen connection,*
ii) *M is Ricci flat with respect to the Schouten–van Kampen connection,*
iii) *M is flat with respect to the Schouten–van Kampen connection.*

6. Conharmonically flat 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study conharmonically flat 3-dimensional f-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional f-Kenmotsu manifold the conharmonic curvature tensor with respect to the Schouten–van Kampen connection is given by

$$\begin{aligned}\tilde{K}(X,Y)Z&=\tilde{R}(X,Y)Z\\&\quad-\big\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y+g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y\big\}.\end{aligned} \tag{6.1}$$

If $\tilde{K}=0$, then the manifold M is called *conharmonically flat* manifold with respect to the Schouten–van Kampen connection.

Let M be a conharmonically flat manifold with respect to the Schouten–van Kampen connection. From (6.1), we have

$$\tilde{R}(X,Y)Z=\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y+g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y. \tag{6.2}$$

Using (4.3), (4.4) and (4.5) in (6.2), we get

$$\begin{aligned}&R(X,Y)Z+f^2\big\{g(Y,Z)X-g(X,Z)Y\big\}\\&\quad+f'\big\{g(Y,Z)\,\eta(X)\xi-g(X,Z)\,\eta(Y)\xi+\eta(Y)\,\eta(Z)X-\eta(X)\,\eta(Z)Y\big\}\\&=S(Y,Z)X-S(X,Z)Y\\&\quad+\Big(4f^2+2f'+\frac{\tau}{2}+f^2+f'\Big)\big\{g(Y,Z)X-g(X,Z)Y\big\}\end{aligned} \tag{6.3}$$

$$+ f'\big\{\eta(Y)\,\eta(Z)X - \eta(X)\,\eta(Z)Y\big\}$$
$$+ \Big(f' - \frac{\tau}{2} - 3f^2 - 3f'\Big)\big\{g(Y,Z)\,\eta(X)\xi - g(X,Z)\,\eta(Y)\xi\big\}.$$

Now putting $X = \xi$ in (6.3), we obtain

$$\begin{aligned}
(6.4)\qquad & R(\xi,Y)Z + (f^2+f')\{g(Y,Z)\xi - \eta(Z)Y\}\\
&= S(Y,Z)\xi - S(\xi,Z)Y\\
&\quad + \Big(4f^2 + 2f' + \frac{\tau}{2} + f^2 + f'\Big)\{g(Y,Z)\xi - \eta(Z)Y\}\\
&\quad + f'\{\eta(Y)\,\eta(Z)\xi - \eta(Z)Y\}\\
&\quad + \Big(f' - \frac{\tau}{2} - 3f^2 - 3f'\Big)\{g(Y,Z)\xi - \eta(Z)\,\eta(Y)\xi\}.
\end{aligned}$$

Using (3.4) and (3.7) in (6.4), we get

$$\begin{aligned}
(6.5)\qquad & S(Y,Z)\xi - S(\xi,Z)Y + \Big(4f^2 + 2f' + \frac{\tau}{2} + f^2 + f'\Big)\big\{g(Y,Z)\xi - \eta(Z)Y\big\}\\
&\quad + f'\big\{\eta(Y)\,\eta(Z)\xi - \eta(Z)Y\big\}\\
&\quad + \Big(f' - \frac{\tau}{2} - 3f^2 - 3f'\Big)\big\{g(Y,Z)\xi - \eta(Z)\,\eta(Y)\xi\big\} = 0.
\end{aligned}$$

Taking the inner product with ξ in (6.5), we have

$$S(Y,Z) + 2(f^2+f')\,\eta(Y)\,\eta(Z) + (2f^2+f')\big\{g(Y,Z) - \eta(Y)\,\eta(Z)\big\} = 0,$$

which gives

$$(6.6)\qquad S(Y,Z) = -(2f^2+f')\,g(Y,Z) - f'\eta(Y)\,\eta(Z).$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection.

Using (6.6) in (4.4), we obtain $\tilde{S}(Y,Z) = 0$. Hence the manifold M is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Then from (6.2) the manifold M is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let M be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold M is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (6.1), we get $\tilde{K}(X,Y)Z = 0$. i.e., the manifold M is a conharmonically flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

THEOREM 6.1. *Let M be a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:*

i) *M is conharmonically flat with respect to the Schouten–van Kampen connection,*
ii) *M is Ricci flat with respect to the Schouten–van Kampen connection,*
iii) *M is flat with respect to the Schouten–van Kampen connection.*

7. Ricci semisymmetric 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection

A f-Kenmotsu manifold with the Schouten–van Kampen connection is called *Ricci semisymmetric* if $\tilde{R}(X,Y)\cdot\tilde{S}=0$, where $\tilde{R}(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vectors X,Y. Then

$$\tilde{S}(\tilde{R}(X,Y)Z,W)+\tilde{S}(Z,\tilde{R}(X,Y)W)=0. \tag{7.1}$$

Using (4.3) and (4.4) in (7.1), we get

$$\begin{aligned}
&S(R(X,Y)Z,W)+S(Z,R(X,Y)W)+f'\big\{\eta(R(X,Y)Z)\,\eta(W)\\
&+f'\eta(R(X,Y)W)\,\eta(Z)\big\}+f^2\big\{S(X,W)\,g(Y,Z)-S(Y,W)\,g(X,Z)\\
&+S(X,Z)\,g(Y,W)-S(Y,Z)\,g(X,W)\big\}\\
&-f'(f^2+f')\big\{g(Y,Z)\,\eta(X)\,\eta(W)-g(X,Z)\,\eta(Y)\,\eta(W)+g(Y,W)\,\eta(X)\,\eta(Z)\\
&-g(X,W)\,\eta(Y)\,\eta(Z)\big\}+f'\big\{S(X,W)\,\eta(Y)\,\eta(Z)-S(Y,W)\,\eta(X)\,\eta(Z)\\
&\qquad\qquad+S(X,Z)\,\eta(Y)\,\eta(W)-S(Y,Z)\,\eta(X)\,\eta(W)\big\}=0.
\end{aligned}$$

Let M be Ricci semisymmetric with respect to the Levi-Civita connection. Then we have

$$\begin{aligned}
&f'\{\eta(R(X,Y)Z)\,\eta(W)+f'\eta(R(X,Y)W)\,\eta(Z)\}+f^2\big\{S(X,W)\,g(Y,Z)\\
&-S(Y,W)\,g(X,Z)+S(X,Z)\,g(Y,W)-S(Y,Z)\,g(X,W)\big\}\\
&-f'(f^2+f')\big\{g(Y,Z)\,\eta(X)\,\eta(W)-g(X,Z)\,\eta(Y)\,\eta(W)+g(Y,W)\,\eta(X)\,\eta(Z)\\
&-g(X,W)\,\eta(Y)\,\eta(Z)\big\}+f'\big\{S(X,W)\,\eta(Y)\,\eta(Z)-S(Y,W)\,\eta(X)\,\eta(Z)\\
&\qquad\qquad+S(X,Z)\,\eta(Y)\,\eta(W)-S(Y,Z)\,\eta(X)\,\eta(W)\big\}=0.
\end{aligned} \tag{7.2}$$

Putting $W=\xi$ in (7.2), we obtain

$$\begin{aligned}
&f'\eta(R(X,Y)Z)+f^2\big\{S(X,\xi)\,g(Y,Z)-S(Y,\xi)\,g(X,Z)\\
&\quad+S(X,Z)\,\eta(Y)-S(Y,Z)\,\eta(X)\big\}\\
&\quad-f'(f^2+f')\big\{g(Y,Z)\,\eta(X)-g(X,Z)\,\eta(Y)\big\}+f'\big\{S(X,\xi)\,\eta(Y)\,\eta(Z)\\
&\qquad\qquad-S(Y,\xi)\,\eta(X)\,\eta(Z)+S(X,Z)\,\eta(Y)-S(Y,Z)\,\eta(X)\big\}=0.
\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
&2(f^2+f')^2\big\{g(Y,Z)\,\eta(X)-g(X,Z)\,\eta(Y)\big\}\\
&-(f^2+f')\big\{S(Y,Z)\,\eta(X)-S(X,Z)\,\eta(Y)\big\}=0.
\end{aligned} \tag{7.3}$$

Again putting $X=\xi$ in (7.3), we have

$$2(f^2+f')^2\big\{g(Y,Z)-\eta(Y)\,\eta(Z)\big\}-(f^2+f')\big\{S(Y,Z)+2(f^2+f')\,\eta(Y)\,\eta(Z)\big\}=0,$$

which gives

$$(f^2+f')\big\{S(Y,Z)+4(f^2+f')\,\eta(Y)\,\eta(Z)-2(f^2+f')\,g(Y,Z)\big\}=0. \tag{7.4}$$

Let $f^2 + f' \neq 0$, then from (7.4), we get

$$S(Y,Z) = 2(f^2 + f')\, g(Y,Z) - 4(f^2 + f')\, \eta(Y)\, \eta(Z). \tag{7.5}$$

Hence the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.

Using (7.5) in (4.4), we obtain

$$\tilde{S}(Y,Z) = (4f^2 + 3f')\, g(Y,Z) - (4f^2 + 3f')\, \eta(Y)\, \eta(Z).$$

Thus we have the following:

THEOREM 7.1. *Let M be a Ricci semisymmetric 3-dimensional regular f-Kenmotsu manifold with the Schouten–van Kampen connection. If M is a Ricci semisymmetric 3-dimensional f-Kenmotsu manifold with respect to the Levi-Civita connection, then M is an η-Einstein manifold with respect to the Schouten–van Kampen connection.*

8. Semisymmetric 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study a semisymmetric regular 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection. If a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection is *semisymmetric* then we can write

$$(\tilde{R}(X,Y)\cdot \tilde{R})(Z,U)W = 0,$$

which gives

$$\tilde{R}(X,Y)\tilde{R}(Z,U)W - \tilde{R}(\tilde{R}(X,Y)Z,U)W - \tilde{R}(Z,\tilde{R}(X,Y)U)W - \tilde{R}(Z,U)\tilde{R}(X,Y)W = 0. \tag{8.1}$$

Using (4.2) in (8.1), we have

$$\tilde{R}(X,Y)R(Z,U)W - R(\tilde{R}(X,Y)Z,U)W - R(Z,\tilde{R}(X,Y)U)W - R(Z,U)\tilde{R}(X,Y)W = 0,$$

which gives

$$(\tilde{R}(X,Y)\cdot R)(Z,U)W = 0. \tag{8.2}$$

Again using (4.2) in (8.2), we obtain

(8.3)

$$\begin{aligned}
&R(X,Y)R(Z,U)W - R(R(X,Y)Z,U)W - R(Z,R(X,Y)U)W \\
&- R(Z,U)R(X,Y)W + f^2\{g(R(Z,U)W,Y)X - g(R(Z,U)W,X)Y \\
&- g(Y,Z)R(X,U)W + g(X,Z)R(Y,U)W - g(Y,U)R(Z,X)W \\
&+ g(X,U)R(Z,Y)W - g(Y,W)R(Z,U)X + g(X,W)R(Z,U)Y\} \\
&+ f'\{g(R(Z,U)W,Y)\,\eta(X)\xi - g(R(Z,U)W,X)\,\eta(Y)\xi + \eta(R(Z,U)W)\,\eta(Y)X \\
&- \eta(R(Z,U)W)\,\eta(X)Y - g(Y,Z)\,\eta(R(X,U)W)\xi + g(X,Z)\,\eta(R(Y,U)W)\xi \\
&- \eta(Y)\,\eta(Z)R(X,U)W + \eta(X)\,\eta(Z)R(Y,U)W - g(Y,U)\,\eta(R(Z,X)W)\xi
\end{aligned}$$

$$+ g(X,U)\,\eta(R(Z,Y)W)\xi - \eta(Y)\,\eta(U)R(Z,X)W + \eta(X)\,\eta(U)R(Z,Y)W$$
$$- g(Y,W)\,\eta(R(Z,U)X)\xi + g(X,W)\,\eta(R(Z,U)Y)\xi$$
$$- \eta(Y)\,\eta(W)R(Z,U)X + \eta(X)\,\eta(W)R(Z,U)Y\big\} = 0.$$

Now from (8.3), we can say:

If $0 \neq f =$ constant (say $f = \alpha$), then $f' = 0$. Hence we get $R\cdot R = -\alpha^2 Q(g,R)$. Therefore the manifold M is a pseudosymmetric α-Kenmotsu manifold.

If f is not constant, then using $X = \xi$ in (8.3), we get

$$\begin{aligned}(8.4)\quad & R(\xi,Y)R(Z,U)W - R(R(\xi,Y)Z,U)W - R(Z,R(\xi,Y)U)W \\ & - R(Z,U)R(\xi,Y)W + f^2\{g(R(Z,U)W,Y)\xi - g(R(Z,U)W,\xi)Y \\ & - g(Y,Z)R(\xi,U)W + g(\xi,Z)R(Y,U)W - g(Y,U)R(Z,\xi)W \\ & + g(\xi,U)R(Z,Y)W - g(Y,W)R(Z,U)\xi + g(\xi,W)R(Z,U)Y\} \\ & + f'\{g(R(Z,U)W,Y)\xi - g(R(Z,U)W,\xi)\,\eta(Y)\xi + \eta(R(Z,U)W)\,\eta(Y)\xi \\ & - \eta(R(Z,U)W)Y - g(Y,Z)\,\eta(R(\xi,U)W)\xi + g(\xi,Z)\,\eta(R(Y,U)W)\xi \\ & - \eta(Y)\,\eta(Z)R(\xi,U)W + \eta(Z)R(Y,U)W - g(Y,U)\,\eta(R(Z,\xi)W)\xi \\ & + g(\xi,U)\,\eta(R(Z,Y)W)\xi - \eta(Y)\,\eta(U)R(Z,\xi)W + \eta(U)R(Z,Y)W \\ & - g(Y,W)\,\eta(R(Z,U)\xi)\xi + g(\xi,W)\,\eta(R(Z,U)Y)\xi \\ & \qquad - \eta(Y)\,\eta(W)R(Z,U)\xi + \eta(W)R(Z,U)Y\} = 0.\end{aligned}$$

Taking the inner product with ξ in (8.4), we obtain

$$\begin{aligned}(8.5)\quad & \eta(R(\xi,Y)R(Z,U)W) - \eta(R(R(\xi,Y)Z,U)W) - \eta(R(Z,R(\xi,Y)U)W) \\ & - \eta(R(Z,U)R(\xi,Y)W) + f^2\big\{g(R(Z,U)W,Y) - g(R(Z,U)W,\xi)\,\eta(Y) \\ & - g(Y,Z)\,\eta(R(\xi,U)W) + g(\xi,Z)\,\eta(R(Y,U)W) - g(Y,U)\,\eta(R(Z,\xi)W) \\ & + g(\xi,U)\,\eta(R(Z,Y)W) - g(Y,W)\,\eta(R(Z,U)\xi) + g(\xi,W)\,\eta(R(Z,U)Y)\big\} \\ & + f'\big\{g(R(Z,U)W,Y) - g(R(Z,U)W,\xi)\,\eta(Y) + \eta(R(Z,U)W)\,\eta(Y) \\ & - \eta(R(Z,U)W)\,\eta(Y) - g(Y,Z)\,\eta(R(\xi,U)W) + g(\xi,Z)\,\eta(R(Y,U)W) \\ & - \eta(Y)\,\eta(Z)\,\eta(R(\xi,U)W) + \eta(Z)\,\eta(R(Y,U)W) - g(Y,U)\,\eta(R(Z,\xi)W) \\ & + g(\xi,U)\,\eta(R(Z,Y)W) - \eta(Y)\,\eta(U)\,\eta(R(Z,\xi)W) + \eta(U)\,\eta(R(Z,Y)W) \\ & - g(Y,W)\,\eta(R(Z,U)\xi) + g(\xi,W)\,\eta(R(Z,U)Y) \\ & - \eta(Y)\,\eta(W)\,\eta(R(Z,U)\xi) + \eta(W)\,\eta(R(Z,U)Y)\big\} = 0.\end{aligned}$$

Let $\{e_i\}$ $(1 \leqslant i \leqslant 3)$ be an orthonormal basis of the tangent space at any point of M. Then the sum for $1 \leqslant i \leqslant 3$ of the relation (8.5) for $Y = Z = e_i$ gives

$$\begin{aligned}& \eta(R(\xi,e_i)R(e_i,U)W) - \eta(R(R(\xi,e_i)e_i,U)W) - \eta(R(e_i,R(\xi,e_i)U)W) \\ & - \eta(R(e_i,U)R(\xi,e_i)W) + f^2\big\{g(R(e_i,U)W,e_i) - g(R(e_i,U)W,\xi)\,\eta(e_i) \\ & - g(e_i,e_i)\,\eta(R(\xi,U)W) + g(\xi,e_i)\,\eta(R(e_i,U)W) - g(e_i,U)\,\eta(R(e_i,\xi)W) \\ & + g(\xi,U)\,\eta(R(e_i,e_i)W) - g(e_i,W)\,\eta(R(e_i,U)\xi) + g(\xi,W)\,\eta(R(e_i,U)e_i)\big\}\end{aligned}$$

$$
\begin{aligned}
&+ f'\{g(R(e_i,U)W,e_i) - g(R(e_i,U)W,\xi)\,\eta(e_i) + \eta(R(e_i,U)W)\,\eta(e_i)\\
&- \eta(R(e_i,U)W)\,\eta(e_i) - g(e_i,e_i)\,\eta(R(\xi,U)W) + g(\xi,e_i)\,\eta(R(e_i,U)W)\\
&- \eta(e_i)\,\eta(e_i)\,\eta(R(\xi,U)W) + \eta(e_i)\,\eta(R(e_i,U)W) - g(e_i,U)\,\eta(R(e_i,\xi)W)\\
&+ g(\xi,U)\,\eta(R(e_i,e_i)W) - \eta(e_i)\,\eta(U)\,\eta(R(e_i,\xi)W) + \eta(U)\,\eta(R(e_i,e_i)W)\\
&- g(e_i,W)\,\eta(R(e_i,U)\xi) + g(\xi,W)\,\eta(R(e_i,U)e_i)\\
&- \eta(e_i)\,\eta(W)\,\eta(R(e_i,U)\xi) + \eta(W)\,\eta(R(e_i,U)e_i)\} = 0.
\end{aligned}
$$

After some calculations, we obtain

$$
\begin{aligned}
&2(f^2+f')\{S(U,W) - 2g(R(\xi,W)U,\xi)\}\\
&- f^2\{S(U,W) - 2g(R(\xi,W)U,\xi) - 2(f^2+f')\,\eta(U)\,\eta(W)\}\\
&- f'\{S(U,W) - 2g(R(\xi,W)U,\xi) - 2(f^2+f')\,\eta(U)\,\eta(W)\} = 0,
\end{aligned}
$$

which gives

$$(f^2+f')\{S(U,W) - 2g(R(\xi,W)U,\xi) + 2(f^2+f')\,\eta(U)\,\eta(W)\} = 0. \tag{8.6}$$

Let $f^2+f' \neq 0$. Then from (8.6), we get

$$S(U,W) - 2g(R(\xi,W)U,\xi) + 2(f^2+f')\,\eta(U)\,\eta(W) = 0. \tag{8.7}$$

Using (3.6) in (8.7), we obtain $S(U,W) = -2(f^2+f')\,g(U,W)$.

Thus we have the following:

THEOREM 8.1. *Let M be a 3-dimensional regular f-Kenmotsu manifold with the Schouten–van Kampen connection. If M is semisymmetric with respect to the Schouten–van Kampen connection, then:*

i) *If* $0 \neq f = \alpha =$ constant, *then the manifold M is a pseudosymmetric α-Kenmotsu manifold, or,*
ii) *If f is not constant, then the manifold M* is an Einstein manifold.

9. An example of a 3-dimensional f-Kenmotsu manifold with the Schouten–van Kampen connection

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z^2\frac{\partial}{\partial x}, \quad e_2 = z^2\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z,\phi W) = g(Z,W) - \eta(Z)\,\eta(W),$$

for any $Z, W \in \chi(M)$. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z}e_2, \quad [e_1, e_3] = -\frac{2}{z}e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{9.1}$$

Using (9.1), we have

$$2g(\nabla_{e_1} e_3, e_1) = 2g\Big(-\frac{2}{z}e_1, e_1\Big), \quad 2g(\nabla_{e_1} e_3, e_2) = 0 \quad \text{and} \quad 2g(\nabla_{e_1} e_3, e_3) = 0.$$

Hence $\nabla_{e_1} e_3 = -\frac{2}{z}e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z}e_2$ and $\nabla_{e_3} e_3 = 0$. (9.1) further yields

$$\begin{aligned} &\nabla_{e_1} e_2 = 0, \qquad \nabla_{e_2} e_2 = \frac{2}{z}e_3, \quad \nabla_{e_3} e_2 = 0, \\ &\nabla_{e_1} e_1 = \frac{2}{z}e_3, \quad \nabla_{e_2} e_1 = 0, \qquad \nabla_{e_3} e_1 = 0. \end{aligned} \tag{9.2}$$

From (9.2), we see that the manifold satisfies $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that M is an f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu manifold [**16**].

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{9.3}$$

With the help of the above formula and using (9.3), it can be easily verified that

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, && R(e_2, e_3)e_3 = -\frac{6}{z^2}e_2, \\ &R(e_1, e_3)e_3 = -\frac{6}{z^2}e_1, && R(e_1, e_2)e_2 = -\frac{4}{z^2}e_1, \\ &R(e_3, e_2)e_2 = -\frac{6}{z^2}e_3, && R(e_1, e_3)e_2 = 0, \\ &R(e_1, e_2)e_1 = \frac{4}{z^2}e_2, && R(e_2, e_3)e_1 = 0, \\ &R(e_1, e_3)e_1 = \frac{6}{z^2}e_3. \end{aligned} \tag{9.4}$$

Now the Schouten–van Kampen connection on M is given by

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_3 = \Big(-\frac{2}{z} - f\Big)e_1, && \tilde{\nabla}_{e_2} e_3 = \Big(-\frac{2}{z} - f\Big)e_2, \\ &\tilde{\nabla}_{e_3} e_3 = -f(e_3 - \xi), && \tilde{\nabla}_{e_1} e_2 = 0, \\ &\tilde{\nabla}_{e_2} e_2 = \frac{2}{z}(e_3 - \xi), && \tilde{\nabla}_{e_3} e_2 = 0, \\ &\tilde{\nabla}_{e_1} e_1 = \frac{2}{z}(e_3 - \xi), && \tilde{\nabla}_{e_2} e_1 = 0 \\ &\tilde{\nabla}_{e_3} e_1 = 0. \end{aligned} \tag{9.5}$$

From (9.5), we can see that $\tilde{\nabla}_{e_i} e_j = 0$ $(1 \leqslant i, j \leqslant 3)$ for $\xi = e_3$ and $f = -\frac{2}{z}$. Hence M is a 3-dimensional f-Kenmotsu manifold with respect to the Schouten–van Kampen connection. Also using (9.4), it can be seen that $\tilde{R} = 0$. Thus the manifold M is a flat manifold with respect to the Schouten–van Kampen connection. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten–van Kampen connection, the manifold M is both a projectively flat and a conharmonically flat 3-dimensional f-Kenmotsu manifold with respect to the Schouten–van Kampen connection. So, from Theorems 5.1 and 6.1, M is an η-Einstein manifold with respect to the Levi-Civita connection.

Acknowledgement. The author is grateful to the referees for their comments and valuable suggestions for improvement of this work.

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(Received 10 12 2015)
(Revised 08 03 2017)