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DISTRIBUTIVE LATTICES OF JACOBSON RINGS

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Abstract. We characterize the distributive lattices of Jacobson rings and prove that if a semiring is a distributive lattice of Jacobson rings, then, up to isomorphism, it is equal to the subdirect product of a distributive lattice and a Jacobson ring. Also, we give a general method to construct distributive lattices of Jacobson rings.

1. Introduction and preliminaries

A semigroup S is called *periodic* if each element of S has a finite order, where the order of $a \in S$ is the order of the cyclic subsemigroup of S generated by a. Periodic semigroups have been studied by many algebraists. Suppose that S is a periodic semigroup. For any $a \in S$ we all know that there exist the smallest positive integer m and the smallest positive integer r such that $a^m = a^{m+r}$. The positive integer m is referred to as the *index* and the positive integer r as the *period* of a. In particular, if the index of each $a \in S$ is equal to 1, then we call S a *strongly periodic semigroup.* Idempotent semigroups and Burnside semigroups satisfying $x^n \approx x$ are special cases of strongly periodic semigroups.

A ring $(R, +, \cdot)$ is a *Jacobson ring* if, for any $a \in R$, there exists $n \in \mathbb{N}$, $n \geqslant 2$ such that $a = a^n$. That is to say, its multiplicative reduct is a strongly periodic semigroup. It is obvious that Boolean rings are Jacobson rings. Following [**6**, Theorem 11], we have

Theorem 1.1. *Let R be a Jacobson ring. Then every element of R has finite additive order and R is commutative.*

We denote by **JR** the class of all Jacobson rings.

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Semirings are the natural generalization of rings and distributive lattices. All semirings $(S, +, \cdot)$ occurring in the literature satisfy at least the following axioms: $(S, +)$, the *additive reduct*, and $(S, \cdot)$, the *multiplicative reduct* of a semiring S are semigroups, and the multiplication distributes over addition from both sides, i.e.,

(SR1) $x + (y + z) \approx (x + y) + z$;
(SR2) $x(yz) \approx (xy)z$;
(SR3) $x(y + z) \approx xy + xz, \quad (x + y)z \approx xz + yz$.
It is, as well, often assumed that $(S, +)$ is commutative, i.e.,
(SR4) $x + y \approx y + x$.
We consider the semiring classes considered that satisfy this identity too.

By an idempotent semiring, we mean a semiring S which satisfies the additional identities $xx \approx x + x \approx x$. An idempotent semiring $(S, +, \cdot)$ is called a bisemilattice if both the multiplicative reduct $(S, \cdot)$ and the additive reduct $(S, +)$ are semilattices. Of course, a distributive lattice is a bisemilattice which satisfies the absorption law $x + xy \approx x$. The class of all distributive lattices is denoted by $\mathbf{D}$. The Mal'cev product of two classes $\mathbf{V}$ and $\mathbf{W}$ of semirings, denoted by $\mathbf{V} \circ \mathbf{W}$, we mean that the class of all semirings S on which there exists a congruence ρ such that $S/\rho \in \mathbf{W}$ and the ρ-classes which are subsemirings of S belong to $\mathbf{V}$. Thus, in this way, some classes of semirings can be constructed by considering the Mal'cev products of some given semirings.

For a semiring $(S, +, \cdot)$ we denote Green's $\mathcal{H}$ relation on the additive reduct $(S, +)$ by $\mathcal{H}^+$. Let $(S, +, \cdot)$ be a semiring whose additive reduct $(S, +)$ is a completely regular semigroup. By Theorem II.1.4 and Corollary II.1.5 in [**8**], $(S, +)$ is a commutative Clifford semigroup and $\mathcal{H}^+$ is the least semilattice congruence of the additive reduct $(S, +)$ of S, moreover, every $\mathcal{H}^+$-class is a maximal subgroup of $(S, +)$. For any $a \in S$ we denote by $\mathcal{H}_a^+$ the $\mathcal{H}^+$-class containing a and 0_a the identity of $\mathcal{H}_a^+$, respectively. It can be easily seen that $a\mathcal{H}^+b$ if and only if $0_a = 0_b$ for any $a, b \in S$.

Let $(S, +, \cdot)$ be a semiring whose additive reduct is a Clifford semigroup. We can define the natural partial order on $(S, +)$ by

$$a \leqslant_+ b \Leftrightarrow (\exists e \in E_+(S)) \ a = b + e$$

for $a, b \in S$, where $E_+(S)$ is the set of idempotents of the additive reduct $(S, +)$ of S.

The Mal'cev product of the class of Jacobson rings and the class of distributive lattices is denoted by $\mathbf{JR} \circ \mathbf{D}$. A semiring S is called a distributive lattice of Jacobson rings if it is in $\mathbf{JR} \circ \mathbf{D}$. In the following we shall study the semirings which are distributive lattices of Jacobson rings.

Some authors have studied the distributive lattices of rings(see [**1, 2, 7**]). In particular, [**1**] and [**2**] characterized the subdirect product of rings and distributive lattice, respectively. If a semiring $(S, +, \cdot)$ is isomorphic to a subdirect product of a ring and a distributive lattice, then the additive reduct $(S, +)$ of S is a sturdy semilattice of abelian groups, which means that $(S, +)$ is E-unitary. The following example shows that, in general, distributive lattices of rings are not the subdirect product of a ring and a distributive lattice.

EXAMPLE 1.1. Consider a five element semiring A_5 with operations given by

+	a	b	c	d	e
a	a	b	a	b	e
b	b	a	b	a	e
c	a	b	c	d	e
d	b	a	d	c	e
c	e	e	e	e	e

.	a	b	c	d	e
a	a	a	c	c	a
b	a	a	c	c	a
c	c	c	c	c	c
d	c	c	c	c	c
e	a	a	c	c	e

It is easy to see that $\mathcal{H}^+$ is a distributive lattice congruence and $\mathcal{H}_a^+ = \{a, b\}$, $\mathcal{H}_c^+ = \{c, d\}$ and $\mathcal{H}_e^+ = \{e\}$ are subrings of A_5. Since $e \leqslant_+ a$, $e \leqslant_+ b$, this means that $(A_5, +)$ is not E-unitary. In [**2**] and [**4**], it was proved that a distributive lattice of Boolean rings is isomorphic to subdirect product of a Boolean ring and a distributive lattice. In this paper we prove that if a semiring is distributive lattice of Jacobson rings, then, up to isomorphism, it is equal to the subdirect product of a distributive lattice and a Jacobson ring, which generalize and enrich some results from [**2, 4, 10**]. Also, we shall give a general method to construct distributive lattices of Jacobson rings.

2. Main results

LEMMA 2.1. *A semiring S is a distributive lattice of Jacobson rings, i.e., $S \in$ $\mathbf{JR} \circ \mathbf{D}$, if and only if $\mathcal{H}^+$ is the least distributive lattice congruence on S and every $\mathcal{H}^+$-class is a Jacobson ring.*

PROOF. Let a semiring S be a distributive lattice of Jacobson rings. Then there exists a semiring congruence ρ on S such that S/ρ is a distributive lattice and every ρ-class is a Jacobson ring. This also implies that ρ is a semilattice congruence on $(S, +)$. Since the additive reduct $(S, +)$ of S is a Clifford semigroup, $\mathcal{H}^+$ is the least semilattice congruence on $(S, +)$. This leads to $\mathcal{H}^+ \subseteq \rho$. On the other hand, since ρ_u (the ρ-class containing u) is a Jacobson ring for any $u \in S$, the additive reduct of ρ_u is an abelian subgroup of $(S, +)$. Thus $\rho_u \subseteq \mathcal{H}_u^+$, furthermore, $\rho \subseteq \mathcal{H}^+$. We have now shown that $\rho = \mathcal{H}^+$. That is to say that $\mathcal{H}^+$ is a distributive lattice congruence of semiring S and every $\mathcal{H}^+$-class is a Jacobson ring.

If μ is a distributive lattice congruence on S, then μ is a semilattice congruence on the additive reduct $(S, +)$. Since $\mathcal{H}^+$ is the least semilattice congruence on $(S, +)$, $\mathcal{H}^+ \subseteq \mu$, which implies that $\mathcal{H}^+$ is the least distributive lattice congruence on S.

Conversely, it is clear from definition that the semiring S is a distributive lattice of Jacobson rings since $\mathcal{H}^+$ is the least distributive lattice congruence on the semiring S and every $\mathcal{H}^+$-class is a Jacobson ring. $\square$

As a consequence of Lemma 2.1 we have the following result.

COROLLARY 2.1. *Let S be a semiring in $\mathbf{JR} \circ \mathbf{D}$. Then*

(i) *for any $a, b \in S$, $0_a + 0_b = 0_{a+b}$, $a0_b = 0_b a = 0_a 0_b = 0_{ab}$, $a + a0_b = a$;*
(ii) *$E_+(S) = \{0_a \mid a \in S\}$ is a distributive lattice.*

Let S be a semiring in $\mathbf{JR} \circ \mathbf{D}$. Define a binary relation σ^+ on S by

$$(\forall a, b \in S)\, a\, \sigma^+ b \Leftrightarrow (\exists e \in E_+(S))\ a + e = b + e.$$

It follows from Proposition 5.3.2 in [**5**] that σ^+ is the least group congruence on the additive reduct $(S, +)$ of S. Thus we have

Lemma 2.2. *Suppose that S is a semiring in $\mathbf{JR} \circ \mathbf{D}$. Then σ^+ is the least Jacobson ring congruence on S.*

Proof. Assume that $a, b \in S$ and $a\, \sigma^+ b$. Then there exists $e \in E_+(S)$ such that $a + e = b + e$. For any $c \in S$ we have that $ca + ce = cb + ce$. By Corollary 2.1 it follows that $ce \in E_+(S)$ and so $ca\, \sigma^+ cb$. Dually, we can get $ac\, \sigma^+ bc$. Thus, σ^+ is a semiring congruence on S. Since $(S/\sigma^+, +)$ is an abelian group, $(S/\sigma^+, +, \cdot)$ is a ring. For any $a \in S$ we denote by σ_a^+ the σ^+-class containing a. By Lemma 2.1 it yields that $\mathcal{H}_a^+$ is a Jacobson ring. Thus there exists a positive integer k such that $a^k = a$. Therefore, $(\sigma_a^+)^k = \sigma_{a^k}^+ = \sigma_a^+$. Hence, $(S/\sigma^+, +, \cdot)$ is a Jacobson ring and so σ^+ is a Jacobson ring congruence on S.

Suppose that θ is a Jacobson ring congruence on S. If $a, b \in S$ and $a\, \sigma^+ b$, then there exists $f \in E_+(S)$ such that $a + f = b + f$. This yields $\theta_{a+f} = \theta_{b+f}$. Thus

$$\theta_a = \theta_a + \theta_f = \theta_b + \theta_f = \theta_b$$

since $(S/\theta, +)$ a group and and θ_f is the identity of $(S/\theta, +)$. This implies $a\, \theta\, b$ and so $\sigma^+ \subseteq \theta$. This shows that σ^+ is the least Jacobson ring congruence on S. □

Now we are able to obtain the decomposition theorem of distributive lattice of Jacobson rings.

Theorem 2.1. *Suppose that S is a semiring. Then S is a distributive lattice of Jacobson rings if and only if S is (isomorphic to) the subdirect product of a distributive lattice and a Jacobson ring.*

Proof. Suppose that $a \in S$, $e \in E_+(S)$ and $a + e \in E_+(S)$. Thus there is $f \in E_+(S)$ such that $a + e = f$. This yields $a + e + f = e + f$, and (left-)multiplying it by a, we have $a^2 + a(e + f) = a(e + f)$, which implies

$$a^2 + a + a(e + f) = a + a(e + f). \tag{1}$$

Since the $\mathcal{H}^+$-class containing $a(e + f)$ is a Jacobson ring, by Theorem 1.1, there exists a positive integer k such that $k \cdot (a(e+f)) = 0_{a(e+f)}$. Adding $(k-1) \cdot a(e+f)$ to the both sides of (1), we get $a^2 + a + k \cdot a(e + f) = a + k \cdot a(e + f)$. This implies $a^2 + a + 0_{a(e+f)} = a + 0_{a(e+f)}$. By Corollary 2.1(i), it follows that $a^2 + a = a$, and, multiplying it by a, we have $a^3 + a^2 = a^2$, which implies $a + a^3 + a^2 = a + a^2$. Thus $a^3 + a = a$. By induction, it can be easily shown that $a^m + a = a$ for any positive integer $m \geqslant 2$. Since the $\mathcal{H}^+$-class containing a is also a Jacobson ring, there exists a positive integer $l \geqslant 2$ such that $a^l = a$. Thus, $a = a^l + a = a + a$, i.e., $a \in E_+(S)$. Therefore the additive reduct $(S, +)$ is E-unitary. By Proposition 5.9.1 in [**5**] we have $\sigma^+ \cap \mathcal{H}^+ = 1_S$, which, by Lemma I.4.18 in [**8**], implies that S is the subdirect product of $S/\mathcal{H}^+$ and S/σ^+.

The converse is trivial. □

Let $F_1, \dots, F_k$ be a fixed list of finite fields with different characteristics $p_1, ..., p_k$ and respective sizes $q_1 = {p_1}^{n_1}, \dots, q_k = {p_k}^{n_k}$, for some positive integers $n_1, \dots, n_k$. Let $c = p_1 \cdots p_k$, and let n be a positive integer such that $n-1$ is the least common multiple of $q_1 - 1, \dots, q_k - 1$. It was proved in [**10**] that the semiring variety $\mathbf{V} = \mathbf{HSP}\{B_2, F_1, \dots, F_k\}$ generated by two-element distributive lattice B_2 and finite fields $F_1, \dots, F_k$ is determined by (SR1-4) and the following identities:

(DFSR1) $(c+1) \cdot x \approx x$; (DFSR4) $x + c \cdot xy \approx x$;

(DFSR2) $x^n \approx x$; (DFSR5) $xy \approx yx$;

(DFSR3) $c \cdot x^2 \approx c \cdot x$; (DFSR6) $\dfrac{c}{p_i} \cdot x^{q_i} \approx \dfrac{c}{p_i} \cdot x \ \ (1 \leqslant i \leqslant k)$.

Suppose that S is a semiring in $\mathbf{V}$. From (SR4) and (DFSR1) we have that the additive reduct $(S, +)$ is a commutative Clifford semigroup. It follows by Theorem 2.1 in [**10**] that S is isomorphic to the subdirect product of the distributive lattice $S/\mathcal{H}^+$ and Jacobson ring S/σ^+. Thus, by Theorem 2.1, S belongs to $\mathbf{JR} \circ \mathbf{D}$ and so $\mathbf{V} \subseteq \mathbf{JR} \circ \mathbf{D}$. This shows that the above theorem generalizes Theorem 2.1 in [**10**].

In the rest of this section we give a method to construct distributive lattices of Jacobson rings. Assume that $(D, +, \cdot)$ is a distributive lattice. Define a binary relation $\leqslant$ on D by

$$(\forall \alpha, \beta \in D)\, \alpha \leqslant \beta \Leftrightarrow \alpha = \alpha + \beta.$$

It is easy to check that $\leqslant$ is a partial order on D. For any α, $\beta \in D$ it is easy to see that $\alpha + \beta \leqslant \alpha$. Similarly, we have $\alpha + \beta \leqslant \beta$. It is well known that $\alpha \leqslant \alpha\beta$, $\beta \leqslant \alpha\beta$ and $\alpha + \beta \leqslant \alpha\beta$.

In order to discuss the structure of S, we have to recall the following concept from [**9**] and [**11**].

Let $\{(S_\alpha, +, \cdot) \mid \alpha \in D\}$ be a family of disjoint semirings $(S_\alpha, +, \cdot)$ which are indexed by a distributive lattice D together with a family of monomorphisms $\varphi_{\alpha,\beta} \colon S_\alpha \to S_\beta (\beta \leqslant \alpha)$ satisfying the following conditions: for any $\alpha, \beta, \gamma \in D$,

(i) $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$;
(ii) If $\gamma \leqslant \beta \leqslant \alpha$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$;
(iii) If $\gamma \leqslant \alpha + \beta$ then
$S_\alpha\varphi_{\alpha,\gamma} + S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha+\beta}\varphi_{\alpha+\beta,\gamma}$.
$S_\alpha\varphi_{\alpha,\gamma} \cdot S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$.

On the set $S = \bigcup_{\alpha \in D} S_\alpha$ define addition and multiplication by

$$a + b = a\varphi_{\alpha,\alpha+\beta} + b\varphi_{\beta,\alpha+\beta},$$
$$a \cdot b = (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta},$$

for any $a \in S_\alpha$, $b \in S_\beta$. Then we can check that $(S, +, \cdot)$ is a semiring, denoted by $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$. We call the constructed semiring $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$ the *sturdy distributive lattice* of semirings S_α.

If all semirings S_α are in a class of semirings $\mathbf{C}$, we call $S = [D; S_\alpha, \varphi_{\alpha,\beta}]$ the sturdy distributive lattice of $\mathbf{C}$-semirings.

THEOREM 2.2. *Suppose that S is a semiring. Then S is a distributive lattice of Jacobson rings if and only if S is a sturdy distributive lattice of Jacobson rings.*

PROOF. Let a semiring S belong to $\mathbf{JR} \circ \mathbf{D}$. By Lemma 2.1, we can assume that S is a distributive lattice D of Jacobson rings R_α's, where $D \cong S/\mathcal{H}^+$ and each R_α is an $\mathcal{H}^+$-class of S. For convenience, for any $\alpha \in D$ we denote by 0_α the unique idempotent of abelian group $(R_\alpha, +)$. Thus, $E_+(S) = \{0_\alpha \mid \alpha \in D\}$. From Lemma 2.2 we have that σ^+ is the least Jacobson ring congruence on S, which means that $(S/\sigma^+, +, \cdot)$ is a Jacobson ring. By Theorem 2.1 it follows that S is isomorphic to the subdirect product of the distributive lattice $S/\mathcal{H}^+$ and Jacobson ring S/σ^+. This implies that the additive reduct $(S, +)$ of S is isomorphic to the subdirect product of the semilattice $(S/\mathcal{H}^+, +)$ and abelian group $(S/\sigma^+, +)$. Thus, $(S, +)$ is a sturdy semilattice $(D, +)$ of abelian groups $(R_\alpha, +)(\alpha \in D)$. Then, by Theorems IV.1.3, IV.1.6 and IV.1.7 in [**8**], we can express $(S, +) = [(D, +); (R_\alpha, +); \varphi_{\alpha,\beta}]$ as a sturdy semilattice of additive abelian groups $R_\alpha(\alpha \in D)$, where $(D, +)[(R_\alpha, +)]$ denotes the additive semigroup of distributive lattice D [of Jacobson rings R_α] and $\varphi_{\alpha,\beta}$ is defined by

$$(\forall a \in R_\alpha)\, a\varphi_{\alpha,\beta} = a + 0_\beta.$$

From $(S, +) = [(D, +); (R_\alpha, +); \varphi_{\alpha,\beta}]$ we have that $\varphi_{\alpha,\beta}(\beta \leqslant \alpha)$ is a group monomorphism from $(R_\alpha, +)$ to $(R_\beta, +)$. In the following, we are going to show that $\varphi_{\alpha,\beta}(\beta \leqslant \alpha)$ is a semiring homomorphism.

For $a, b \in R_\alpha$, we have $a\varphi_{\alpha,\beta} = a + 0_\beta$ and $b\varphi_{\alpha,\beta} = b + 0_\beta$. Then, by Corollary 2.1, we have

$$(ab)\varphi_{\alpha,\beta} = ab + 0_\beta = a + a0_\beta + b0_\beta + 0_\beta = (a + 0_\beta)(b + 0_\beta) = (a\varphi_{\alpha,\beta})(b\varphi_{\alpha,\beta}).$$

This shows that $\varphi_{\alpha,\beta}$ is a semigroup homomorphism from $(R_\alpha, \cdot)$ to $(R_\beta, \cdot)$ and so $\varphi_{\alpha,\beta}$ is a semiring monomorphism.

For any $\alpha, \beta \in D$, since R_α and R_β are $\mathcal{H}^+$-classes and $\mathcal{H}^+$ is a distributive lattice congruence, $R_\alpha \cdot R_\beta \subseteq R_{\alpha\beta}$. Thus, for any $a \in S_\alpha, b \in S_\beta$, we have

$$a\varphi_{\alpha,\alpha+\beta} = a + 0_{\alpha+\beta}, \quad b\varphi_{\beta,\alpha+\beta} = b + 0_{\alpha+\beta}, \quad (ab)\varphi_{\alpha\beta,\alpha+\beta} = ab + 0_{\alpha+\beta}.$$

By Corollary 2.1 we have

$$\begin{aligned} ab + 0_{\alpha+\beta} &= a + a0_{\alpha+\beta} + b0_{\alpha+\beta} + 0_{\alpha+\beta} \\ &= (a + 0_{\alpha+\beta})(b + 0_{\alpha+\beta}) = (a\varphi_{\alpha,\alpha+\beta})(b\varphi_{\beta,\alpha+\beta}). \end{aligned}$$

Thus, $(ab)\varphi_{\alpha\beta,\alpha+\beta} = (a\varphi_{\alpha,\alpha+\beta})(b\varphi_{\beta,\alpha+\beta})$.

Let $\gamma \in D$ and $\gamma \leqslant \alpha + \beta$. Since $\varphi_{\alpha+\beta,\gamma}$ is a semiring homomorphism, we have

$$\begin{aligned} a\varphi_{\alpha,\gamma} b\varphi_{\beta,\gamma} &= a\varphi_{\alpha,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} b\varphi_{\beta,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} \\ &= (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi_{\alpha+\beta,\gamma} = (ab)\varphi_{\alpha\beta,\alpha+\beta}\varphi_{\alpha+\beta,\gamma} = (ab)\varphi_{\alpha\beta,\gamma}. \end{aligned}$$

This shows $R_\alpha\varphi_{\alpha,\gamma} \cdot R_\beta\varphi_{\beta,\gamma} \subseteq R_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$. Hence,

$$ab = ((ab)\varphi_{\alpha\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta} = (a\varphi_{\alpha,\alpha+\beta} b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta}.$$

Since $a + b = a\varphi_{\alpha,\alpha+\beta} + b\varphi_{\beta,\alpha+\beta}$ is evident, by the above definition, S is a sturdy distributive lattice D of Jacobson rings R_α's, where $D \cong S/\mathcal{H}^+$ and each R_α is a $\mathcal{H}^+$-class of semiring S.

Conversely, if the semiring S is a sturdy distributive lattice D of Jacobson rings R_α $(\alpha \in D)$, then $S = [D; R_\alpha, \varphi_{\alpha,\beta}]$. Define a binary relation η on S by

$$(a, b \in S)\, a\,\eta\, b \Leftrightarrow (\exists \alpha \in D)\, a, b \in R_\alpha.$$

It is a routine matter to verify that η is a distributive lattice congruence and that every η-class is a Jacobson ring. That is to say, $S \in \mathbf{JR} \circ \mathbf{D}$. $\square$

By Theorems 2.1 and 2.2 the following corollary is directly obtained.

Corollary 2.2. *Let S be a semiring. Then the following statements are equivalent:*

(i) *S is a distributive lattice of Jacobson rings;*
(ii) *S is the subdirect product of a distributive lattice and a Jacobson ring;*
(iii) *S is a sturdy distributive lattice of Jacobson rings.*

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