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FURTHER RESULTS ON IMAGES OF METRIC SPACES

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Abstract. We introduce the notion of an ls-π-Ponomarev-system to give necessary and sufficient conditions for $f : (M, M_0) \to X$ to be a strong wc-mapping (wc-mapping, wk-mapping) where M is a locally separable metric space. Then, we systematically get characterizations of weakly continuous strong wc-images (wc-images, wk-images) of locally separable metric spaces by means of certain networks. Also, we give counterexamples to sharpen some results on images of locally separable metric spaces in the literature.

1. Introduction

Characterizing images of metric spaces by spaces with certain networks is an interesting work in general topology [**17**,**25**]. In this field, the notion of a k-network plays an important place [**27**,**28**]. There are many modifications of k-networks such as cs-networks, cs^*-networks, wcs^*-networks, sn-networks, so-networks, etc. These notions are used to characterize continuous images of metric spaces with certain covering-properties [**2**,**5**,**7**,**10**,**19**,**21**,**26**–**29**]. In 2002, Lin [**19**] posed the following problem.

Problem. [**19**] *Find a nice mapping such that a space with a point-countable wcs^*-network can be characterized by the image of a metric space under this mapping.*

This problem has been studied by some authors in Theory of k-networks. The difficulty in solving this problem is that each member P of a wcs^*-network $\mathcal{P}$ may not contain the limit point of certain sequence which is frequently in P. Around this problem, Cai and Li [**5**] established a relation between spaces with point-countable wcs^*-networks of certain properties and images of locally separable metric spaces; Lin and Li [**20**] introduced the concepts of wks-mappings and wcs-mappings, and them to characterize spaces with point-countable k-networks. Their main result is as follows.

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THEOREM 1.1. [**20**, Lemma 14] (1) *A space with a point-countable k-network is a weakly continuous wks-image of a metric space.*
(2) *A space with point-countable wcs^*-network is a weakly continuous wcs-image of a metric space.*

By this result, Lin and Li characterized spaces with point-countable wcs^*-network by weakly continuous wcs-images of metric spaces in [**20**, Theorem 15], which give a solution to Problem 1.

By the above, it is natural to ask the following questions.

QUESTIONS. (1) What are characterizations of wks-images and wcs-images of locally separable metric spaces by spaces with certain networks?

(2) Can "weakly continuous" in Theorem 1.1 be replaced by "continuous"?

We introduce the ls-π-Ponomarev-system $(f, M, M_0, X, \mathcal{P})$ to give necessary and sufficient conditions for $f : (M, M_0) \to X$ to be a strong wc-mapping (wc-mapping, wk-mapping) where M is a locally separable metric space. Then, we systematically get characterizations of weakly continuous strong wc-images (wc-images, wk-images) of locally separable metric spaces by means of certain networks, which implies an answer to Question (1). Also, we give counterexamples to sharpen some results on images of locally separable metric spaces in [**5**, **20**], which give a negative answer for Question (2) and some more.

All spaces are Hausdorff, all mappings are onto, $\mathbb{N}$ denotes the set of all natural numbers, $\omega = \mathbb{N} \cup \{0\}$, and a convergent sequence includes its limit point. Let $f : X \to Y$ be a mapping, $\mathcal{P}$ a family of subsets of X, and K a subset of X; we write $f(\mathcal{P}) = \big\{f(P) : P \in \mathcal{P}\big\}$, $\bigcup \mathcal{P} = \bigcup \big\{P : P \in \mathcal{P}\big\}$, $\bigcap \mathcal{P} = \bigcap \big\{P : P \in \mathcal{P}\big\}$, $\mathcal{P}|_K = \big\{P \cap K : P \in \mathcal{P}\big\}$. For terms not defined here, please refer to [**11**] and [**28**].

2. The π-Ponomarev-system and images of locally separable metric spaces

First, we recall some basic notions. Let $S = \{x_n : n \in \mathbb{N}\}$ be a sequence converging to x in X, and P be a subset of X. We say that S (or $S \cup \{x\}$) is *eventually* in P [**28**], if there exists $n_0 \in \mathbb{N}$ such that $\{x_n : n \geq n_0\} \cup \{x\} \subset P$. Also, S (or $S \cup \{x\}$) is *frequently* in P [**28**] if there exists a subsequence $\{x_{k_n} : n \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ such that $\{x_{k_n} : n \in \mathbb{N}\} \cup \{x\} \subset P$.

Now, we present the notion of a network and their modifications. Let $\mathcal{P}$ be a family of subsets of a space X, a point $x \in X$, and K be a subset of X. $\mathcal{P}$ is a *network at* x in X [**23**], if $x \in P$ for every $P \in \mathcal{P}$, and whenever $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$. $\mathcal{P}$ is a *network* for X [**23**], if for each $x \in X$, there exists $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a network at x in X. $\mathcal{P}$ is a *strong network* for X [**9**] if for each $x \in X$, there exists $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a countable network at x in X. $\mathcal{P}$ is a *k-network* for X [**24**], if for each compact subset K of X and $K \subset U$ with U open in X, there exists a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$. $\mathcal{P}$ is a *cs-network* for X [**16**] (resp., *cs^*-network* for X [**12**]), if for each sequence S converging to $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $S \cup \{x\}$ is eventually (resp., frequently) in $P \subset U$.

$\mathcal{P}$ is a *wcs-network* (resp., *wcs***-network*) for K in X, if for each sequence S in K converging to $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that S is eventually (resp., frequently) in $P \subset U$. If $K = X$, then $\mathcal{P}$ is a *wcs-network* for X (resp., *wcs***-network* for X [**22**]). $\mathcal{P}$ is a *strong wcs-network* (resp., *strong wcs***-network*) for X, if for each convergent sequence S in X, there exists $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a countable *wcs*-network (resp., *wcs**-network) for S in X. $\mathcal{P}$ is a *π-network* at x in X [**20**], if for each neighborhood U of x, there exists $P \in \mathcal{P}$ such that $P \subset U$. $\mathcal{P}$ is a *strong π-network* for X, if for each $x \in X$, there exists $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a countable π-network at x in X.

Remark 2.1. (1) If $\mathcal{P}$ is a strong *wcs**-network for X, then for each $x \in X$, there exists $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a countable π-network at x in X and $\mathcal{P}(x)$ has the finite intersection property.

(2) If $\mathcal{P}$ is a network at x in X, then $\mathcal{P}$ is a π-network at x in X and $x = \bigcap \mathcal{P}$. It implies that $\mathcal{P}$ has the finite intersection property.

Let $f : X \to Y$ be a mapping, and X_0 be a subspace of X. f is an *s-mapping* [**3**] (resp., a *compact* mapping [**4**]), if f is continuous and for each $y \in Y$, $f^{-1}(y)$ is a separable (resp., compact) subset of X. f is *continuous* about X_0 [**30**], if for each $x \in X$ and each neighborhood U of $f(x)$ in Y there exists a neighborhood V of x in X such that $f(V \cap X_0) \subset U$. $f : (X, X_0) \to Y$ is a *ws-mapping* [**20**], if the restriction $f_0 = f|_{X_0} : X_0 \to Y$ is an s-mapping and f is continuous about X_0. f is a *semi-quotient* mapping [**20**], if for each subset T of Y, T is closed when and only when $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$. f is a *weakly continuous* mapping [**18**], if $f^{-1}(U) \subset \mathrm{Int}[f^{-1}(\overline{U})]$ for each open subset U of Y.

Remark 2.2. (1) Every continuous mapping is a weakly continuous mapping.

(2) There exists a weakly continuous mapping which is not continuous, see [**18**, Example 3].

Lemma 2.1. [**20**, Lemma 1] *Suppose that $f : X \to Y$ and $X_0 \subset X$. Then the following statements are equivalent.*

(1) *f is continuous about X_0.*
(2) *If a net $\{x_d\}_{d\in D}$ in X_0 converges to a point x in X, then a net $\{f(x_d)\}_{d\in D}$ converges to $f(x)$ in Y.*
(3) *If T is a subset of Y, then $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$, where $f_0 = f|_{X_0}$.*

Remark 2.3. It follows from Lemma 2.1 that a weakly continuous mapping $f : (X, X_0) \to Y$ preserves the convergence of a sequence $\{x_n : n \in \mathbb{N}\} \subset X_0$ which converges to $x \in X$.

Some properties of weakly continuous mappings are shown in the following lemma.

Lemma 2.2. [**20**] *Let $f : X \to Y$ be a mapping.*

(1) *f is weakly continuous if and only if for each $x \in X$ and each neighborhood U of $f(x)$ in Y, there exists a neighborhood V of x in X with $f(V) \subset \overline{U}$.*
(2) *If f is continuous about X_0, then the restriction $f|_{\overline{X_0}}$ is weakly continuous.*

Now we recall some notions of mappings in [**20**]. Let $f:(X,X_0)\to Y$ be a mapping which is continuous about X_0. $f:(X,X_0)\to Y$ is a *wk-mapping* [**20**] if for each compact subset K of Y and for each sequence T in K, there exists a sequence S in X_0 such that S has an accumulation in X and $f(S)$ is a subsequence of T. $f:(X,X_0)\to Y$ is a *wc-mapping* [**20**] if for each convergent sequence T in Y, there exists a sequence S in X_0 such that S has an accumulation x in X and $f(S\cup\{x\})$ is a subsequence of T. $f:(X,X_0)\to Y$ is a *wcs-mapping* (resp., *wks-mapping*) [**20**] if it is a *ws*-mapping and a *wc*-mapping (resp., *wk*-mapping).

In the same way, we call $f:(X,X_0)\to Y$ is a *strong wc-mapping* if for each convergent sequence T in Y, there exists a sequence S in X_0 such that S has an accumulation x in X and $f(S\cup\{x\})=T$. $f:(X,X_0)\to Y$ is a *strong wcs-mapping* if it is a *ws*-mapping and a strong *wc*-mapping.

Let $\mathcal{P}$ be a family of subsets of space X. $\mathcal{P}$ is *point-countable* (resp., *point-finite*) [**15**], if for each $x\in X$, $\{P\in\mathcal{P}: x\in P\}$ is countable (resp., finite).

It is similar in spirit to [**20**, Lemma 13] that the following lemma holds.

Lemma 2.3. *Let $\mathcal{B}$ be a point-countable base for a space X.*

(1) *If $f:(X,X_0)\to Y$ is a wks-mapping, then $f(\mathcal{B}|_{X_0})$ is a point-countable k-network for Y.*
(2) *If $f:(X,X_0)\to Y$ is a wcs-mapping, then $f(\mathcal{B}|_{X_0})$ is a point-countable wcs^*-network for Y.*
(3) *If $f:(X,X_0)\to Y$ is a strong wcs-mapping, then $f(\mathcal{B}|_{X_0})$ is a point-countable wcs-network for Y.*

Let $\mathcal{P}$ be a strong network for a space X. We may assume that $\mathcal{P}$ is closed under finite intersections. Put $\mathcal{P}=\{P_\alpha:\alpha\in A\}$. For every $n\in\mathbb{N}$, put $A_n=A$ and endow A_n with a discrete topology. Put

$$M=\big\{a=(\alpha_n)\in \textstyle\prod_{n\in\mathbb{N}}A_n:\{P_{\alpha_n}:n\in\mathbb{N}\}\text{ forms a network at some }x_a\text{ in }X\big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}}A_n$, is a metric space, x_a is unique, and $x_a=\bigcap_{n\in\mathbb{N}}P_{\alpha_n}$ for every $a\in M$. We define $f:M\to X$ by setting $f(a)=x_a$ for every $a\in M$. The system $(f,M,X,\mathcal{P})$ is called a *Ponomarev-system* [**13**, Definition 2.2]. This notion plays an important role in characterizing images of metric spaces by spaces with certain networks. In the spirit of Ponomarev-system $(f,M,X,\mathcal{P})$ and the proof of [**20**, Lemma 14], we introduce the π-Ponomarev-system $(f,M,M_0,X,\mathcal{P})$ as follows.

Let $\mathcal{P}$ be a strong wcs^*-network for a space X. Assume that $\mathcal{P}$ is closed under finite intersections. Put $\mathcal{P}=\{P_\alpha:\alpha\in A\}$. For every $n\in\mathbb{N}$, put $A_n=A$ and endow A_n with a discrete metric. Put

$$M=\big\{a=(\alpha_n)\in \textstyle\prod_{n\in\mathbb{N}}A_n:\{P_{\alpha_n}:n\in\mathbb{N}\}\text{ has the finite intersection property and it forms a }\pi\text{-network at some }x_a\text{ in }X\big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}}A_n$, is a metric space, x_a is unique. We define $f:M\to X$ by setting $f(a)=x_a$ for every $a\in M$. Put

$$M_0=\big\{a=(\alpha_n)\in \textstyle\prod_{n\in\mathbb{N}}A_n:\{P_{\alpha_n}:n\in\mathbb{N}\}\text{ forms a network at some }x_a\text{ in }X\big\}.$$

The system $(f, M, M_0, X, \mathcal{P})$ is called a *π-Ponomarev-system*.

REMARK 2.4. For a π-Ponomarev-system $(f, M, M_0, X, \mathcal{P})$, notations in the above definition are used in what follows unless otherwise specified. Moreover, we have

(1) $M_0 \subset M$.
(2) $(f, M_0, X, \mathcal{P})$ is a Ponomarev-system, then $x_a = \bigcap_{n\in\mathbb{N}} P_{\alpha_n}$ and $f : M_0 \to X$ is continuous and onto.

For a π-Ponomarev-system $(f, M, M_0, X, \mathcal{P})$, we have the following.

LEMMA 2.4. *Let $(f, M, M_0, X, \mathcal{P})$ be a π-Ponomarev-system. Then the following statements hold.*

(1) *$f : M \to X$ is weakly continuous.*
(2) *$f : (M, M_0) \to X$ is continuous about M_0.*

PROOF. (1) Let $a = (\alpha_n) \in M$ and U be an open neighborhood of $f(a)$ in X. Then there exists $k \in \mathbb{N}$ such that $P_{\alpha_k} \subset U$. Put $V = \{b = (\beta_n) \in M : \beta_k = \alpha_k\}$. Then V is an open neighborhood of a in M. We shall prove that $f(V \cap M_0) \subset U$. In fact, if $b = (\beta_n) \in V \cap M_0$, then $\beta_k = \alpha_k$ and $f(b) \in \bigcap_{n\in\mathbb{N}} P_{\beta_n} \subset P_{\beta_k} = P_{\alpha_k} \subset U$.

(2) Using the notations in (1) again, we shall prove that $f(V) \subset \overline{U}$; then f is weakly continuous by Lemma 2.2. In fact, for each $c = (\gamma_n) \in V$ and each open neighborhood W of $f(c)$ in X, we have that $P_{\gamma_i} \subset W$ for some $i \in \mathbb{N}$. So

$$W \cap U \supset P_{\gamma_i} \cap P_{\alpha_k} = P_{\gamma_i} \cap P_{\gamma_k} \neq \emptyset.$$

It implies that $f(c) \in \overline{U}$. □

The next technical lemma plays an important role in the arguments.

LEMMA 2.5. *Let $(f, M, M_0, X, \mathcal{P})$ be a π-Ponomarev-system. For each $a = (\alpha_n) \in M_0$ and $n \in \mathbb{N}$, put*

$$B_{a,n} = \big\{b = (\beta_i) \in M_0 : \beta_i = \alpha_i \text{ if } i \leqslant n\big\},$$

$$\mathcal{B}_a = \{B_{a,n} : n \in \mathbb{N}\}, \quad \mathcal{B}_n = \{B_{a,n} : a \in M_0\}, \quad \mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}.$$

Then the following statements hold.

(1) *$\mathcal{B}_a$ is a base at a in M_0, and $\mathcal{B}_n$ is discrete.*
(2) *$f(B_{a,n}) = \bigcap_{i=1}^{n} P_{\alpha_i}$, and $f(\mathcal{B}) = \mathcal{P}$.*

PROOF. (1). This is clear.

(2). For each $n \in \mathbb{N}$, let $x \in f(B_{a,n})$. Then $x = f(b)$ for some $b = (\beta_i) \in B_{a,n}$. This implies that $x = \bigcap_{i\in\mathbb{N}} P_{\beta_i} \subset \bigcap_{i=1}^{n} P_{\beta_i} = \bigcap_{i=1}^{n} P_{\alpha_i}$. Then $f(B_{a,n}) \subset \bigcap_{i=1}^{n} P_{\alpha_i}$.

Conversely, let $x \in \bigcap_{i=1}^{n} P_{\alpha_i}$, where $x = f(b)$ for some $b = (\beta_i) \in M$. For each $i \in \mathbb{N}$, there exists $\gamma_{n+i} \in A_{n+i}$ such that $\gamma_{n+i} = \beta_i$. Put $c = (\gamma_i)$, where $\gamma_i = \alpha_i$ if $i \leqslant n$. Then we get $c \in B_{a,n}$ and $f(c) = x$. This implies that $\bigcap_{i=1}^{n} P_{\alpha_i} \subset f(B_{a,n})$.

By above inclusions, we get $f(B_{a,n}) = \bigcap_{i=1}^{n} P_{\alpha_i}$. It follows from $\mathcal{P}$ being closed under finite intersections that $f(\mathcal{B}) = \mathcal{P}$. □

By Remark 2.4.(2) and [**13**, Proposition 2.8], we have the following proposition.

Proposition 2.1. *Let* $(f, M, M_0, X, \mathcal{P})$ *be a* π*-Ponomarev-system. Then* $f : M_0 \to X$ *is an* s*-mapping* (*resp., a compact mapping*) *if and only if* $\mathcal{P}$ *is point-countable* (*resp., point-finite*).

In [**13**,**14**], the necessary and sufficient conditions for f to be an s-mapping with covering-properties have been obtained in a Ponomarev-system $(f, M, X, \mathcal{P})$ where $\mathcal{P}$ is some network except for a point-countable wcs^*-network (k-network, wcs-network). The reason of this exception is that each member P of a wcs^*-network (k-network, wcs-network) $\mathcal{P}$ may not contain the limit point of certain sequence which is frequently in P, then the mapping f may not be a covering-mapping. In hat follows, by means of wcs-network (wcs^*-network, k-network), we obtain the necessary and sufficient conditions for $f : (M, M_0) \to X$ to be a strong wc-mapping (wc-mapping, wk-mapping) in a π-Ponomarev-system $(f, M, M_0, X, \mathcal{P})$.

Lemma 2.6. [**8**, Lemma 2.6] *Let* $\mathcal{P}$ *be a cs-network for* X *and* S *be a convergent sequence such that* $S \subset U$ *with* U *open in* X*. Then there exists* $\mathcal{F} \subset \mathcal{P}$ *satisfying*

(1) $\mathcal{F}$ *is finite.*
(2) *For each* $F \in \mathcal{F}$*,* $\emptyset \neq F \cap S \subset F \subset U$*.*
(3) *For each* $x \in S$*, there exists a unique* $F \in \mathcal{F}$ *such that* $x \in F$*.*
(4) *If* $F \in \mathcal{F}$ *contains the limit point of* S*, then* $S - F$ *is finite.*

Such an $\mathcal{F}$ is called to have *property* $cs(S, U)$.

Theorem 2.1. *Let* $(f, M, M_0, X, \mathcal{P})$ *be a* π*-Ponomarev-system. Then the following statements hold.*

(1) $f : (M, M_0) \to X$ *is a strong wc-mapping if and only if* $\mathcal{P}$ *is a strong wcs-network for* X*.*
(2) $f : (M, M_0) \to X$ *is a wc-mapping if and only if* $\mathcal{P}$ *is a strong* wcs^**-network for* X*.*
(3) $f : (M, M_0) \to X$ *is a wk-mapping if and only if* $\mathcal{P}$ *is a strong* k*-network for* X*.*

Proof. (1) *Necessity.* Let f be a strong wc-mapping. For each convergent sequence S of X, there exists a sequence L of M_0 which has an accumulation a_0 in M and $f(L \cup \{a_0\}) = S$. By using notations in Lemma 2.5 again, we have that $\mathcal{B}$ is a base for M_0. In spirit of the proof of Lemma 2.3(3), $\mathcal{P} = f(\mathcal{B})$ is a strong wcs-network for X.

Sufficiency. Let $\mathcal{P}$ be a strong wcs-network for X. We shall prove that $f : (M, M_0) \to X$ is a strong wc-mapping by the following two claims.

Claim 1. $f : (M, M_0) \to X$ *is continuous about* M_0*.*

It follows from Lemma 2.4.

Claim 2. $f : (M, M_0) \to X$ *is a strong wc-mapping.*

For each convergent sequence $S = \{x_m : m \in \omega\}$ converging to x_0 in X. We may assume that $x_n \neq x_m$ for every $n \neq m$. Since $\mathcal{P}$ is a strong wcs-network for X, there exists $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a countable wcs-network for S in X. Put $\mathcal{P}_S^0 = \{P \in \mathcal{P}_S : S \text{ is eventually in } P\}$. Then $\mathcal{P}_S^0$ is countable. Pick some $P_0 \in \mathcal{P}_S^0$, we have that $S - P_0$ is finite. For each $x \in S - P_0$, there exists $P_x \in \mathcal{P}_S$ such that

$P_x \subset X - (S - \{x\})$. Put

$$\mathcal{F} = \big\{P_x : x \in S - P_0\big\} \cup \big\{P_0\big\}, \quad \mathcal{F}^0 = \big\{P_x : x \in S - P_0\big\} \cup \big\{P_0 \cup \{x_0\}\big\}.$$

Then $\mathcal{F}^0$ has the property $cs(S, X)$ (see Lemma 2.6). It implies that

$$\big\{\mathcal{F} \subset \mathcal{P}_S : \mathcal{F}^0 \text{ has property } cs(S, X)\big\} \neq \emptyset.$$

So we can put $\big\{\mathcal{F} \subset \mathcal{P}_S : \mathcal{F}^0 \text{ has property } cs(S, X)\big\} = \big\{\mathcal{F}_n : n \in \mathbb{N}\big\}$.

For each $n \in \mathbb{N}$, put $\mathcal{F}_n = \{P_\alpha : \alpha \in B_n\}$, where B_n is a finite subset of A_n. For each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, by the definition of property $cs(S, X)$, there exists unique $\alpha_{m,n} \in B_n$ such that $x_m \in P_{\alpha_{m,n}} \in \mathcal{F}_n$, and unique $\alpha_{0,n} \in A_n$ such that S is eventually in $P_{\alpha_{0,n}}$. Put $a_m = (\alpha_{m,n})$ for every $m \in \omega$, then $a_m \in M_0$ for every $m \in \mathbb{N}$ and $a_0 \in M$. We shall prove that $a_m \to a_0$ in M. In fact, for each $n \in \mathbb{N}$, by the definition of property $cs(S, X)$, there exists $k_n \in \mathbb{N}$ such that $x_m \in P_{\alpha_{0,n}}$ for every $m \geq k_n$. Then $\alpha_{m,n} = \alpha_{0,n}$ for every $m \geq k_n$. It implies that $a_m \to a_0$ in M. To complete the proof, we need only to prove $f(a_m) = x_m$ for every $m \in \mathbb{N}$. Let $x_m \in U$ with U open in X. Then $S - \{x_m\}$ is a convergent sequence converging to x_0 in X and $S - \{x_m\} \subset X - \{x_m\}$ where $X - \{x_m\}$ is open in X. By Lemma 2.6, there exists $\mathcal{F} \subset \mathcal{P}_S$ such that $\mathcal{F}^0$ has the property $cs(S - \{x_m\}, X - \{x_m\})$. Since $U - (S - \{x_m\})$ is an open neighborhood of x_m in X, there exists $P_m \in \mathcal{P}_S$ such that $x_m \in P_m \subset U - (S - \{x_m\})$. Then $\mathcal{F}^0 \cup \{P_m\}$ has the property $cs(S, X)$. There exists $k \in \mathbb{N}$ such that $\mathcal{F}^0 \cup \{P_m\} = \mathcal{F}^0_k$; it means $\mathcal{F} \cup \{P_m\} = \mathcal{F}_k$. Then $x_m \in P_{\alpha_{m,k}} = P_m \subset U$. It implies that $\{P_{\alpha_{m,n}} : n \in \mathbb{N}\}$ forms a network at x_m in X, so $f(a_m) = x_m$ for every $m \in \mathbb{N}$.

(2) Same as the proof of (1).

(3) *Necessity.* Same as the necessity of (1).

Sufficiency. By the sufficiency of (1), we only need to prove that f is a wk-mapping. For a compact subset K of X and each sequence $T = \{x_n : n \in \mathbb{N}\}$ in K, we can assume that the sequence $\{x_n : n \in \mathbb{N}\}$ converges to a point $x \in K - \{x_n : n \in \mathbb{N}\}$ by [**20**, Lemma 12]. It follows from the proof of Claim 2 in the sufficiency of (1) above, there exists a sequence S in M such that S converges to x in M and $f(S \cup \{x\})$ is a subsequence of T. Then f is a wk-mapping. □

From Proposition 2.1 and Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $(f, M, M_0, X, \mathcal{P})$ be a π-Ponomarev-system. Then the following statements hold.*

(1) *$f : (M, M_0) \to X$ is a strong wcs-mapping if and only if $\mathcal{P}$ is a point-countable wcs-network for X.*
(2) *$f : (M, M_0) \to X$ is a wcs-mapping if and only if $\mathcal{P}$ is a point-countable wcs^*-network for X.*
(3) *$f : (M, M_0) \to X$ is a wks-mapping if and only if $\mathcal{P}$ is a point-countable k-network for X.*

In [**20**, Lemma 14], Lin and Li proved that spaces with point-countable k-networks (resp., wcs^*-network) are weakly continuous wks-(resp., wcs-)images of metric spaces. By using their method, we characterize the strong *wcs*-image of a metric space by a space with a point-countable *wcs*-network as follows.

PROPOSITION 2.2. *A space has a point-countable wcs-network if and only if it is a weakly continuous strong wcs-image of a metric space.*

In the following, we prove that "weakly continuous" in the result of Lin and Li, and Proposition 2.2 can not be replaced by "continuous".

LEMMA 2.7. [**1**, Lemma 12(3)] *Let $f : X \to Y$ be a sequentially-quotient mapping. If $\mathcal{P}$ is a cs^*-network for X, then $f(\mathcal{P})$ is a cs^*-network for Y.*

LEMMA 2.8. *Let $f : (X, X_0) \to Y$ be a weakly continuous wcs-mapping and X be a metric space. If f is continuous, then Y has a point-countable cs^*-network.*

PROOF. Let $\mathcal{B}$ be a σ-locally finite base of X. Then $\mathcal{C} = \mathcal{B}|_{\overline{X_0}}$ is a σ-locally finite base of $\overline{X_0}$. Since $f : (X, X_0) \to Y$ is a *wcs*-mapping, $f|_{\overline{X_0}}$ is a sequentially-quotient mapping. Put $\mathcal{P} = f|_{\overline{X_0}}(\mathcal{B}|_{\overline{X_0}})$, then $\mathcal{P}$ is a cs^*-network for Y by Lemma 2.7. On the other hand, $\mathcal{B}|_{\overline{X_0}} = \mathcal{B}|_{X_0}$ and $f|_{X_0}$ is an s-mapping, then $\mathcal{P}$ is point-countable. It implies that $\mathcal{P}$ is a point-countable cs^*-network for Y. □

EXAMPLE 2.1. There exists a space which has a point-countable *wcs*-network and k-network, and has not any point-countable cs^*-network.

PROOF. Let S_{ω_1} be the quotient space of the sum of uncountably many convergent sequences $S_\lambda = \{x_{\lambda,n} : n \in \omega\}$, $\lambda \in \Lambda$, by identifying all limit points $x_{\lambda,0}$ to a point x_0. Then S_{ω_1} does not have any point-countable cs^*-network. Put $\mathcal{P} = \big\{\{x_0\}\big\} \cup \big\{\{x_{\lambda,i} : i \geq n\} : n \in \mathbb{N}, \lambda \in \Lambda\big\}$. We have that $\mathcal{P}$ is a point-countable *wcs*-network and k-network for X. □

REMARK 2.5. By Example 2.1, S_{ω_1} does not have any point-countable cs^*-network. Then S_{ω_1} is not any continuous *wcs*-image of a metric space by Lemma 2.8. Note that S_{ω_1} has a point-countable *wcs*-network, then it has a point-countable wcs^*-network. Also, S_{ω_1} has a point-countable k-network. Therefore, "weakly continuous" in the result of Lin and Li, and Proposition 2.2 can not be replaced by "continuous".

3. Further results on images of locally separable metric spaces

A space X is an $\aleph_0$-*space* if X is regular and it has a countable *cs*-network [**23**]. Cai and Li [**5**] established the mapping relation between a space with a point-countable wcs^*-network and a locally separable metric space. The main result is as follows.

PROPOSITION 3.1. [**5**, Theorem 2.6] *If a space X has a point-countable wcs^*-network $\mathcal{P}$ and the closure of each element of $\mathcal{P}$ is an $\aleph_0$-subspace, then X is the pseudo-sequence-covering image of a locally separable metric space.*

By this result, it is natural to ask whether the inverse implication is true? The answer is negative by the following Example 3.1. Note that every sequence-covering mapping is a pseudo-sequence-covering mapping.

Example 3.1. There exists a sequence-covering image of a locally separable metric space which does not have any point-countable wcs^*-network.

Proof. Consider the *butterfly space* Y of McAuley [**23**, page 999], which is defined as follows: Let Y be the upper half-plane, and let $A \subset Y$ denote the x-axis. Points in $Y - A$ have ordinary plane neighborhoods. A base for the neighborhoods of a point $p \in A$ consists of all sets $N_\varepsilon(p)$ where $N_\varepsilon(p)$ consists of p together with all points $q \in Y$ having distance $< \varepsilon$ from p and lying underneath the union of the two rays in Y which emanate from p and have slopes ε and $-\varepsilon$, respectively. Such an $N_\varepsilon(p)$ is called a *butterfly neighborhood of* p.

It follows from [**23**, Example 12.1] that Y is regular and Fréchet, and not an $\aleph_0$-space. By [**15**, Theorem 5.2] and [**28**, Remark 1], Y has not any point-countable wcs^*-network. On the other hand, Y is a sequence-covering image of a metric space by [**6**, Theorem 5]. □

In [**1**], an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$ was introduced to give necessary and sufficient conditions for f to be an s-mapping with covering-properties from a locally separable metric space M onto a space X with a double cover.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X such that each X_λ has a network $\mathcal{P}_\lambda$ which is closed under finite intersections. $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ is a *double cover* for X [**1**], if each $\mathcal{P}_\lambda$ is countable. $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ is *point-countable* [**1**] (resp., *point-finite*), if $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable (resp., point-finite).

Let $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ be a double cover for a space X and $(f_\lambda, M_\lambda, X_\lambda, \mathcal{P}_\lambda)$ be a Ponomarev-system for every $\lambda \in \Lambda$. Since $\mathcal{P}_\lambda$ is countable, M_λ is a separable metric space. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and $f = \bigoplus_{\lambda\in\Lambda} f_\lambda$. Then M is a locally separable metric space and f is a mapping from M onto X. The system $(f, M, X, \{\mathcal{P}_\lambda\})$ is an *ls-Ponomarev-system* [**1**].

Now, we introduce a notion of an ls-π-Ponomarev-system as follows. Let $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ be a double cover for X and $(f_\lambda, M_\lambda, M_{\lambda,0}, X_\lambda, \mathcal{P}_\lambda)$ be the π-Ponomarev-system for every $\lambda \in \Lambda$. Since $\mathcal{P}_\lambda$ is countable, M_λ is a separable metric space. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, $M_0 = \bigoplus_{\lambda\in\Lambda} M_{\lambda,0}$, $f = \bigoplus_{\lambda\in\Lambda} f_\lambda V$. Then M is a locally separable metric space and f is a continuous mapping from M onto X. The system $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ is called an *ls-π-Ponomarev-system*.

Remark 3.1. For an ls-π-Ponomarev-system $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$, notations in the above definition are used in what follows next unless otherwise specified. Moreover, we have

(1) $M_0 \subset M$. (2) $X_\lambda = f(M_{\lambda,0}) = f(M_\lambda)$ for every $\lambda \in \Lambda$.

(3) $(f, M_0, X, \{\mathcal{P}_\lambda\})$ is an ls-Ponomarev-system, then $f : M_0 \to X$ is continuous and onto.

By Lemma 2.4, we have the following lemma.

Lemma 3.1. *Let $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ be an ls-π-Ponomarev-system. Then the following statements hold.*

(1) *$f : M \to X$ is weakly continuous.*
(2) *$f : (M, M_0) \to X$ is continuous about M_0.*

It follows from Remark 3.1 and [**1**, Theorem 2.14] that the following result holds.

Proposition 3.2. *Suppose that* $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ *is an ls-π-Ponomarev-system. Then* $f : M_0 \to X$ *is an s-mapping (resp., a compact mapping) if and only if* $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ *is point-countable (resp., point-finite).*

We give a necessary and sufficient condition for the mapping $f : (M, M_0) \to X$ to be a strong *wc*-mapping in an *ls*-π-Ponomarev-system $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$.

Proposition 3.3. *Suppose that* $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ *is an ls-π-Ponomarev-system. Then following statements are equivalent.*

(1) $f : (M, M_0) \to X$ *is a strong wc-mapping.*
(2) *For each sequence* S *converging to* x *in* X*, there exists* $\lambda \in \Lambda$ *such that*
  (a) $S \cup \{x\}$ *is eventually in* X_λ*.*
  (b) *For each open neighborhood* U *of* x *in* X_λ*, there exists* $P \in \mathcal{P}_\lambda$ *such that* $S \cap X_\lambda$ *is eventually in* $P \subset U$*.*

Proof. (1) ⇒ (2). For each sequence S converging to x in X, we need only to prove the following two claims. Assume that $x \notin S$, if necessary.

Claim 1. *There exists* $\lambda \in \Lambda$ *such that* $S \cup \{x\}$ *is eventually in* X_λ*.*

Since $f : (M, M_0) \to X$ is a strong *wc*-mapping, there exists a sequence L converging to a in M such that $f(L) = S$ and $L \subset M_0$. Since $L \cup \{a\}$ is a convergent sequence in M, there exists $\lambda \in \Lambda$ such that L is eventually in M_λ. It implies that S is eventually in X_λ.

Claim 2. *For* S *and* λ *in Claim* 1*, if* $x \in U$ *with* U *open in* X_λ*, then there exists* $P \in \mathcal{P}_\lambda$ *such that* $S \cap X_\lambda$ *is eventually in* $P \subset U$*.*

By using notations in the proof of Claim 1 again, we have that L converges to $a \in f_\lambda^{-1}(U)$ with $f_\lambda^{-1}(U)$ open in M_λ. Put $a = (\alpha_i)$, and

$$A_{a,n} = \big\{b = (\beta_i) \in M_\lambda : \beta_i = \alpha_i \text{ for every } i \leqslant n\big\},$$
$$B_{a,n} = \big\{b = (\beta_i) \in M_{\lambda,0} : \beta_i = \alpha_i \text{ for every } i \leqslant n\big\},$$

for every $n \in \mathbb{N}$. Then $\{A_{a,n} : n \in \mathbb{N}\}$ is a base at a in M_λ. So, there exists $n \in \mathbb{N}$ such that $L \cup \{a\}$ is eventually in $A_{a,n} \subset f_\lambda^{-1}(U)$. Then L is eventually in $A_{a,n} \cap M_{\lambda,0} \subset f_\lambda^{-1}(U) \cap M_{\lambda,0}$. Note that $A_{a,n} \cap M_{\lambda,0} = B_{a,n}$. So L is eventually in $B_{a,n} \subset f_\lambda^{-1}(U)$. Then S is eventually in $P \subset U$ where $P = f_\lambda(B_{a,n}) \in \mathcal{P}_\lambda$ by Lemma 2.5.

(2) ⇒ (1). We need only to prove the following two claims.

Claim 1. $f : (M, M_0) \to X$ *is continuous about* M_0*.*

This follows from Lemma 3.1.

Claim 2. $f : (M, M_0) \to X$ *is a strong wc-mapping.*

Let S be a sequence converging to x in X. Then $S \cup \{x\}$ is eventually in X_λ for some $\lambda \in \Lambda$. Put $S_\lambda = S \cap X_\lambda$ and $H_\lambda = S - X_\lambda$. Then $S_\lambda \cup \{x\}$ is a convergent sequence converging to x in X_λ and H_λ is finite. As in the proof of Claim 2 in the sufficiency of Theorem 2.1(1), there exists a sequence $L_\lambda \subset M_{\lambda,0}$ converging to a in M_λ such that $f_\lambda(L_\lambda) = S_\lambda$. On the other hand, there exists a finite subset G of

M_0 such that $H_\lambda = f(G)$. Then $L = G \cup L_\lambda$ is a sequence converging to a in M, $L \subset M_0$, and $f(L) = S$. This proves that f is a strong *wc*-mapping. □

Being similar in spirit to the proof of Proposition 3.3, we get the following propositions which are necessary and sufficient conditions for $f : (M, M_0) \to X$ to be a *wc*-mapping (*wk*-mapping) in an *ls*-π-Ponomarev-system $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$.

PROPOSITION 3.4. *Suppose that $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ is an ls-π-Ponomarev-system. Then the following statements are equivalent.*

(1) *$f : (M, M_0) \to X$ is a wc-mapping.*
(2) *For each sequence S converging to x in X, there exists $\lambda \in \Lambda$ such that*
  (a) *$S \cup \{x\}$ is frequently in X_λ.*
  (b) *For each open neighborhood U of x in X_λ, there exists $P \in \mathcal{P}_\lambda$ such that $S \cap X_\lambda$ is frequently in $P \subset U$.*

PROPOSITION 3.5. *Suppose that $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ is an ls-π-Ponomarev-system. Then the following statements are equivalent.*

(1) *$f : (M, M_0) \to X$ is a wk-mapping.*
(2) *For each sequence S in a compact subset K of X, there exists $\lambda \in \Lambda$ such that*
  (a) *$S \cup \{x\}$ is frequently in X_λ.*
  (b) *For each open neighborhood U of x in X_λ, there exists $P \in \mathcal{P}_\lambda$ such that $S \cap X_\lambda$ is frequently in $P \subset U$.*

It follows from Proposition 3.3 that we have a characterization of strong *wc*-images of locally separable metric spaces as follows.

COROLLARY 3.1. *For a space X, the following statements are equivalent.*

(1) *X is a weakly continuous strong wc-image (resp., wcs-image) of a locally separable metric space.*
(2) *X has a double cover (resp., point-countable double cover) $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ satisfying for each sequence S converging to x in X there exists $\lambda \in \Lambda$ such that*
  (a) *$S \cup \{x\}$ is eventually in X_λ.*
  (b) *For each open neighborhood U of x in X_λ, there exists $P \in \mathcal{P}_\lambda$ such that $S \cap X_\lambda$ is eventually in $P \subset U$.*

PROOF. (1) $\Rightarrow$ (2). Let $f : (M, M_0) \to X$ be a weakly continuous strong *wc*-mapping where M is a locally separable metric space. By [**11**, 4.4.F], $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ where each M_λ is a separable metric space. For each $\lambda \in \Lambda$, let $\mathcal{B}_\lambda$ be a countable base for M_λ. Put $\mathcal{P}_\lambda = f(\mathcal{B}_\lambda)$ and $X_\lambda = f(M_\lambda)$. As in the proof (1)$\Rightarrow$(2) of Proposition 3.3, we have that $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ satisfies the required conditions.

For the parenthetic part, $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ is point-countable as in the necessity's proof of Proposition 3.2.

(2) $\Rightarrow$ (1). Let $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ be a double cover for X which satisfies conditions in statement (2). Then the *ls*-π-Ponomarev $(f, M, M_0, X, \{\mathcal{P}_\lambda\})$ exists.

By Lemma 3.1 and Proposition 3.3, $f : (M, M_0) \to X$ is a weakly continuous strong *wc*-mapping and M is a locally separable metric space. Then X is a weakly continuous strong *wc*-image of a locally separable metric space.

For the parenthetic part, $f : (M, M_0) \to X$ is a *ws*-mapping by Proposition 3.2. Then X is a weakly continuous strong *wcs*-image of a locally separable metric space. □

Being similar in spirit to the proof of Corollary 3.1, we get the following corollaries on characterizations of weakly continuous *ws*-images of locally separable metric spaces.

Corollary 3.2. *For a space X, the following statements are equivalent.*

(1) *X is a weakly continuous wc-image (resp., wcs-image) of a locally separable metric space.*
(2) *X has a double cover (resp., point-countable double cover) $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ satisfying for each sequence S converging to x in X, there exists $\lambda \in \Lambda$ such that*
   (a) *$S \cup \{x\}$ is frequently in X_λ.*
   (b) *For each open neighborhood U of x in X_λ, there exists $P \in \mathcal{P}_\lambda$ such that $S \cap X_\lambda$ is frequently in $P \subset U$.*

Corollary 3.3. *For a space X, the following statements are equivalent.*

(1) *X is a weakly continuous wk-image (resp., wks-image) of a locally separable metric space.*
(2) *X has a double cover (resp., point-countable double cover) $\{(X_\lambda, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ satisfying for each sequence S in a compact subset K of X, there exists $\lambda \in \Lambda$ such that*
   (a) *$S \cup \{x\}$ is frequently in X_λ.*
   (b) *For each open neighborhood U of x in X_λ, there exists $P \in \mathcal{P}_\lambda$ such that $S \cap X_\lambda$ is frequently in $P \subset U$.*

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