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SOME NEW BITOPOLOGICAL NOTIONS

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Abstract. We introduce strong separation axioms in bitopological spaces. We also introduce the notion of strong pairwise compactness.

1. Introduction

Kelly [**3**] introduced the notion of bitopological spaces: the triple $(X, \mathcal{P}_1, \mathcal{P}_2)$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two topologies on X, is called a bitopological space. Due to presence of two topologies in a bitopological space, it is always possible to consider the interior of a $(\mathcal{P}_i)$open set with respect to the topology $\mathcal{P}_j$, where $i, j \in \{1, 2\}$, $j \neq i$. Now it is interesting to note, for a nontrivial $(\mathcal{P}_i)$open set U, $(\mathcal{P}_j)\operatorname{int} U$ may even be an empty set. Even if $(\mathcal{P}_j)\operatorname{int} U \neq \emptyset$, it is obvious that $(\mathcal{P}_j)\operatorname{int} U \subset U$. This observation leads us to define the notion of strong pairwise compactness (Definition 3.3). In subsequent endeavors, we introduce the strong separation axioms: strong pairwise Hausdorffness, strong pairwise regularity, strong pairwise normality. In fact, the results of bitopological spaces are generalization of the results of topological spaces. But the notions we introduce here are not generalization of any result of topological spaces since nontrivial similar concepts in topological spaces are absurd. However, when the two topologies in a bitopological space coincide, these notions reduce to equivalent conventional concepts of topological spaces.

Throughout the paper, a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is simply denoted by X. R denotes the set of real numbers and N, the set of natural numbers. $(\mathcal{T})\operatorname{int} A$ denotes the interior and $(\mathcal{T})\operatorname{cl} A$, the closure of a set A in the topological space $(X, \mathcal{T})$. For a topological space $(X, \mathcal{T})$ and $A \subset X$, we write $(A, \mathcal{T}_A)$ to denote the subspace on A of $(X, \mathcal{T})$. Always $i, j \in \{1, 2\}$ and whenever i, j appear together,

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The author dedicates this work to Prof. M. K. Bose, Department of Mathematics, University of North Bengal on his fruitful completion of 60 years.

$j \neq i$. To avoid any confusion, we also write $\mathcal{U}^{(X)}$ to denote a pairwise open cover of X and $A^{(X)}$ to denote a $(\mathcal{P}_i)$open set in X.

To make the article self-explanatory, we recall the following known definitions.

Definition 1.1 (Kelly [**3**]). A bitopological space X is said to be pairwise Hausdorff if for each pair of distinct points x and y of X, there exist $U \in \mathcal{P}_i$ and $V \in \mathcal{P}_j$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 1.2 (Kelly [**3**]). In a bitopological space X, $\mathcal{P}_i$ is said to be regular with respect to $\mathcal{P}_j$ if for each $x \in X$ and each $(\mathcal{P}_i)$closed set A with $x \notin A$, there exist $U \in \mathcal{P}_i$ and $V \in \mathcal{P}_j$ such that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. X is said to be pairwise regular if $\mathcal{P}_i$ is regular with respect to $\mathcal{P}_j$ for both $i = 1$ and $i = 2$.

Definition 1.3 (Kelly [**3**]). A bitopological space X is said to be pairwise normal if for any pair of a $(\mathcal{P}_i)$closed set A and a $(\mathcal{P}_j)$closed set B with $A \cap B = \emptyset$, there exist $U \in \mathcal{P}_j$ and $V \in \mathcal{P}_i$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Definition 1.4 (Fletcher et al. [**2**]). A cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of X is said to be a pairwise open cover of X if $\mathcal{U} \subset \mathcal{P}_1 \cup \mathcal{P}_2$ and for each $i \in \{1, 2\}$, $\mathcal{U} \cap \mathcal{P}_i$ contains a nonempty set.

Definition 1.5 (Fletcher et al. [**2**]). A bitopological space X is said to be pairwise compact if every pairwise open cover of X has a finite subcover.

Definition 1.6 (Pahk and Choi [**4**]). A family $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ of subsets of X is said to be pairwise closed if $\{X - F_\alpha \mid \alpha \in A\}$ is pairwise open.

Definition 1.7 (Swart [**6**, p. 136]). , Let $(X, \mathcal{P}_1, \mathcal{P}_2)$ and $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathcal{P}_1, \mathcal{P}_2) \to (Y, \mathcal{Q}_1, \mathcal{Q}_2)$ is said to be continuous if $f : (X, \mathcal{P}_1) \to (Y, \mathcal{Q}_1)$ and $f : (X, \mathcal{P}_2) \to (Y, \mathcal{Q}_2)$ are continuous.

Definition 1.8 (Romaguera and Marin [**5**, p. 237]). Let $(X, \mathcal{P}_1, \mathcal{P}_2)$ and $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathcal{P}_1, \mathcal{P}_2) \to (Y, \mathcal{Q}_1, \mathcal{Q}_2)$ is said to be open if $f : (X, \mathcal{P}_1) \to (Y, \mathcal{Q}_1)$ and $f : (X, \mathcal{P}_2) \to (Y, \mathcal{Q}_2)$ are open.

2. Bitopological strong separation axioms

In this section, we introduce the notions of strong separation axioms in bitopological spaces.

Definition 2.1. A bitopological space X is said to satisfy the pairwise intersection property if for each pair of a $(\mathcal{P}_i)$open set $A(\neq X)$ and a $(\mathcal{P}_j)$open set $B(\neq X)$ with $A \cap B \neq \emptyset$, we have a $(\mathcal{P}_j)$open set U and a $(\mathcal{P}_i)$open set V such that $A \cap B \subset U \subset A$ and $A \cap B \subset V \subset B$.

Example 2.1. For any $a \in R$, we define

$$\mathcal{P}_1 = \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\},$$
$$\mathcal{P}_2 = \{\emptyset, R, \{a\}, (a, \infty), [a, \infty)\}.$$

Here the two topologies $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent, and the bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ satisfies the pairwise intersection property.

Definition 2.2. A bitopological space X is said to be strongly pairwise Hausdorff if for each $x, y \in X$ with $x \neq y$, there exist a $(\mathcal{P}_i)$open set U and a $(\mathcal{P}_j)$open set V such that $x \in (\mathcal{P}_j) \operatorname{int} U$, $y \in (\mathcal{P}_i) \operatorname{int} V$ and $U \cap V = \emptyset$.

It readily follows, a strongly pairwise Hausdorff space X is pairwise Hausdorff and for each $i \in \{1, 2\}$, the space $(X, \mathcal{P}_i)$ is Hausdorff.

Example 2.2. Let R be the set of reals, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the usual topology and the upper limit topology respectively. Suppose $x, y \in R$ with $x \neq y$. If $x < y$, then there exists $a, b, c \in R$ such that $a < x < b < y < c$. We choose $U_1 = (a, b)$, $V_1 = (b, c]$. Then $x \in U_1 \in \mathcal{P}_1$, $y \in V_1 \in \mathcal{P}_2$ and $U_1 \cap V_1 = \emptyset$. Also $(a, x] \subset U_1$, $(b, c) \subset V_1$, and $(a, x] \in \mathcal{P}_2$ and $(b, c) \in \mathcal{P}_1$. Thus $x \in (\mathcal{P}_2) \operatorname{int} U_1$ and $y \in (\mathcal{P}_1) \operatorname{int} V_1$. Similarly on choosing $V_2 = (a, b]$, $U_2 = (b, c)$, we obtain $x \in (\mathcal{P}_1) \operatorname{int} V_2$, $y \in (\mathcal{P}_2) \operatorname{int} U_2$ and $V_2 \cap U_2 = \emptyset$. Thus the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise Hausdorff.

Example 2.3 ([**2**, Example 4, p. 330]). Let X be the set of nonnegative reals, and let $\mathcal{P}$ be the usual topology on X and $\mathcal{Q} = \{\emptyset\} \cup \{U \cup (x, \infty) \mid U \in \mathcal{P} \text{ and } x \in X\}$. Then the bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is pairwise Hausdorff but it is not strongly pairwise Hausdorff.

Theorem 2.1. *A pairwise Hausdorff bitopological space with pairwise intersection property is strongly pairwise Hausdorff.*

Proof. Let X be a pairwise Hausdorff bitopological space with pairwise intersection property. Suppose $x, y \in X$ with $x \neq y$. Now by pairwise Hausdorffness, there exist a $(\mathcal{P}_1)$open set U_1 and a $(\mathcal{P}_2)$open set V_1 such that $x \in U_1$, $y \in V_1$ with $U_1 \cap V_1 = \emptyset$. Also there exist a $(\mathcal{P}_2)$open set V_2 and a $(\mathcal{P}_1)$open set U_2 such that $x \in V_2$, $y \in U_2$ with $U_2 \cap V_2 = \emptyset$. Now $U_i \cap V_j \neq \emptyset$. So we obtain a $(\mathcal{P}_j)$open set G_j and a $(\mathcal{P}_i)$open set H_i such that $U_i \cap V_j \subset G_j \subset U_i$ and $U_i \cap V_j \subset H_i \subset V_j$. Thus we get, $x \in (\mathcal{P}_2) \operatorname{int} U_1$ and $y \in (\mathcal{P}_1) \operatorname{int} V_1$ with $U_1 \cap V_1 = \emptyset$. Also we obtain, $x \in (\mathcal{P}_1) \operatorname{int} V_2$ and $y \in (\mathcal{P}_2) \operatorname{int} U_2$ with $U_2 \cap V_2 = \emptyset$. Thus the space is strongly pairwise Hausdorff. □

In the same fashion as of strong pairwise Hausdorffness, we may have the following definition.

Definition 2.3. A bitopological space X is said to be pairwise strongly regular if for each $x \in X$ and each $(\mathcal{P}_i)$closed set F with $x \notin F$, there exist a $(\mathcal{P}_i)$open set U and a $(\mathcal{P}_j)$open set V such that $x \in (\mathcal{P}_j) \operatorname{int} U$, $F \subset (\mathcal{P}_i) \operatorname{int} V$ and $U \cap V = \emptyset$.

Unfortunately, with this definition of pairwise strong regularity, we obtain $\mathcal{P}_1 = \mathcal{P}_2$.

Definition 2.4. A bitopological space X is said to be strongly pairwise regular if for each $x \in X$ and each $(\mathcal{P}_i)$closed set F with $x \notin F$, there exist a $(\mathcal{P}_i)$open set U and a $(\mathcal{P}_j)$open set V such that $x \in U$, $F \subset (\mathcal{P}_i) \operatorname{int} V$ and $U \cap V = \emptyset$.

It readily follows, if a bitopological space X is strongly pairwise regular, then X is pairwise regular and each of $(X, \mathcal{P}_i)$ is regular.

EXAMPLE 2.4. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the upper limit topology and the lower limit topology respectively on R. We consider here the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$.

Let $x \in R$ and F be $(\mathcal{P}_1)$closed with $x \notin F$. Now we have a collection $\mathcal{A}$ of intervals of the form $(a, b]$ such that $R - F = \bigcup\{(a, b] \mid (a, b] \in \mathcal{A}\}$. Since $x \notin F$, we obtain an interval $(a, b] \subset R$ such that $x \in (a, b]$. We choose $\alpha \in R$ such that $a < \alpha < x \leqslant b$. We put $U = (\alpha, b]$ and $V = (-\infty, \alpha) \cup (b, \infty)$. So $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. Also V is $(\mathcal{P}_i)$open for each $i \in \{1, 2\}$. Thus considering V as a $(\mathcal{P}_2)$open set, we have $F \subset (\mathcal{P}_1)\operatorname{int} V$. Similarly if F is $(\mathcal{P}_2)$closed, we may obtain a $(\mathcal{P}_2)$open set U and a $(\mathcal{P}_1)$open set V such that $x \in U$, $F \subset (\mathcal{P}_2)\operatorname{int} V$ and $U \cap V = \emptyset$. Hence $(R, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise regular.

THEOREM 2.2. *A bitopological space X with pairwise intersection property is strongly pairwise regular if X is pairwise regular, and for each $i \in \{1, 2\}$, $(X, \mathcal{P}_i)$ is regular.*

PROOF. Let $x \in X$ and F be $(\mathcal{P}_i)$closed with $x \notin F$. Hence by pairwise regularity, there exist a $(\mathcal{P}_i)$open set U' and a $(\mathcal{P}_j)$open set V' such that $x \in U'$, $F \subset V'$ with $U' \cap V' = \emptyset$. Also $(X, \mathcal{P}_i)$ is regular. Hence there exist $(\mathcal{P}_i)$open sets U'' and V'' such that $x \in U''$, $F \subset V''$ with $U'' \cap V'' = \emptyset$. Thus $x \in U' \cap U'' \in \mathcal{P}_i$ and $F \subset V' \cap V''$. Now by pairwise intersection property, there exists a $(\mathcal{P}_i)$open set H such that $V' \cap V'' \subset H \subset V'$. We put $U = U' \cap U''$ and $V = V'$. Then $x \in U$ and $F \subset (\mathcal{P}_i)\operatorname{int} V$. It is easy to see that $U \cap V = \emptyset$. Hence X is strongly pairwise regular. □

DEFINITION 2.5. A bitopological space X is said to be strongly pairwise normal if for each $(\mathcal{P}_i)$closed set E and each $(\mathcal{P}_j)$closed set F with $E \cap F = \emptyset$, there exist a $(\mathcal{P}_j)$open set U and a $(\mathcal{P}_i)$open set V such that $E \subset (\mathcal{P}_i)\operatorname{int} U$, $F \subset (\mathcal{P}_j)\operatorname{int} V$ and $U \cap V = \emptyset$.

It follows that a strongly pairwise normal space X is pairwise normal but for each $i \in \{1, 2\}$, the space $(X, \mathcal{P}_i)$ need not be normal.

EXAMPLE 2.5. [**1**, Example 2.3, p. 300]. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two topologies on R defined by

$$\mathcal{P}_1 = \{R, \emptyset, (-\infty, a], (a, \infty)\},$$
$$\mathcal{P}_2 = \{R, \emptyset, R - \{a\}, (-\infty, a), (-\infty, a], (a, \infty)\}.$$

where $a \in R$. The bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise normal.

EXAMPLE 2.6. [**1**, Example 2.4, p. 301]. Let X be any set with $a, b \in X$. Suppose

$$\mathcal{P}_1 = \{\emptyset, X\} \cup \{A \subset X \mid a \in A\},$$
$$\mathcal{P}_2 = \{\emptyset, X\} \cup \{A \subset X \mid a \notin A,\ b \in A\}.$$

The bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise normal but it is not strongly pairwise normal.

THEOREM 2.3. *If a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise normal, and for each $i \in \{1, 2\}$, $(X, \mathcal{P}_i)$ satisfies the axiom T_1, then $\mathcal{P}_1 = \mathcal{P}_2$.*

PROOF. Straightforward. □

We recall that a topological space $(X, \mathcal{T})$ is normal if for nonempty $(\mathcal{T})$closed sets E, F with $E \cap F = \emptyset$, there exist $(\mathcal{T})$open sets U, V such that $E \subset U$, $F \subset V$ and $U \cap V = \emptyset$. In this fashion, we say that a topological space $(X, \mathcal{T})$ possesses 'covering properties of closed sets by open sets' if for each nontrivial $(\mathcal{T})$closed set E there exists a $(\mathcal{T})$open set U $(\neq X)$ such that $E \subset U$.

THEOREM 2.4. *A bitopological space X with pairwise intersection property is strongly pairwise normal if X is pairwise normal and for each $i \in \{1, 2\}$, $(X, \mathcal{P}_i)$ possesses covering properties of closed sets by open sets.*

PROOF. Let E be $(\mathcal{P}_i)$closed and F be $(\mathcal{P}_j)$closed with $E \cap F = \emptyset$. Hence by pairwise normality, there exist a $(\mathcal{P}_j)$open set U and a $(\mathcal{P}_i)$open set V such that $E \subset U$, $F \subset V$ with $U \cap V = \emptyset$. Since each of $(X, \mathcal{P}_i)$ and $(X, \mathcal{P}_j)$ possesses covering properties of closed sets by open sets, we obtain a $(\mathcal{P}_i)$open set G_i and a $(\mathcal{P}_j)$open set H_j such that $E \subset G_i$ and $F \subset H_j$. Thus there exist a $(\mathcal{P}_i)$open set G and a $(\mathcal{P}_j)$open set H such that $U \cap G_i \subset G \subset U$ and $V \cap H_j \subset H \subset V$. So we have $E \subset (\mathcal{P}_i)\operatorname{int} U$ and $F \subset (\mathcal{P}_j)\operatorname{int} V$ with $U \cap V = \emptyset$. Thus the space is strongly pairwise normal. □

3. Strong pairwise compactness

In this section, we introduce the concept of strong pairwise compactness.

DEFINITION 3.1. A subset A of a bitopological space X is said to be $(\mathcal{P}_i, \mathcal{P}_j)$dually open if there exists a $(\mathcal{P}_j)$open set U such that $A = (\mathcal{P}_i)\operatorname{int} U$.

The complement of a $(\mathcal{P}_i, \mathcal{P}_j)$dually open set is said to be $(\mathcal{P}_i, \mathcal{P}_j)$dually closed. So a subset F of X is $(\mathcal{P}_i, \mathcal{P}_j)$dually closed iff there exists a $(\mathcal{P}_j)$closed set E such that $F = (\mathcal{P}_i)\operatorname{cl} E$.

Obviously, the union of an arbitrary collection of $(\mathcal{P}_i, \mathcal{P}_j)$dually open sets is $(\mathcal{P}_i)$open and the intersection of a finite number of $(\mathcal{P}_i, \mathcal{P}_j)$dually closed sets is $(\mathcal{P}_i)$closed.

EXAMPLE 3.1. For $a \in R$, we define

$$\begin{aligned} \mathcal{P}_1 &= \{\emptyset, R\} \cup \{(-\infty, a), (a, \infty), R - \{a\}\}, \\ \mathcal{P}_2 &= \{\emptyset, R\} \cup \{(-\infty, a), [a, \infty)\}. \end{aligned}$$

In the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$, we have $(\mathcal{P}_1)\operatorname{int}(-\infty, a) = (-\infty, a)$ and $(\mathcal{P}_1)\operatorname{int}[a, \infty) = (a, \infty)$. Thus $(-\infty, a)$ and (a, ∞) are $(\mathcal{P}_1, \mathcal{P}_2)$dually open. But $(-\infty, a) \cup (a, \infty) \neq (\mathcal{P}_1)\operatorname{int} V$ for any $(\mathcal{P}_2)$open set V.

Thus the union of some $(\mathcal{P}_i, \mathcal{P}_j)$dually open sets may not be $(\mathcal{P}_i, \mathcal{P}_j)$dually open. Also, if $\{U_\alpha \mid \alpha \in A\}$ is an arbitrary collection of $(\mathcal{P}_i, \mathcal{P}_j)$dually open sets such that $U_\alpha = (\mathcal{P}_i)\operatorname{int} V_\alpha$ where V_α is $(\mathcal{P}_j)$open, then in general, $\bigcup_\alpha U_\alpha \neq (\mathcal{P}_i)\operatorname{int}(\bigcup_\alpha V_\alpha)$.

Example 3.1 also shows the intersection of finitely many $(\mathcal{P}_i, \mathcal{P}_j)$dually closed sets may not be $(\mathcal{P}_i, \mathcal{P}_j)$dually closed. And if $\{E_k \mid k = 1, 2, \ldots, n\}$ is a finite collection of $(\mathcal{P}_i, \mathcal{P}_j)$dually closed sets such that $E_k = (\mathcal{P}_i) \operatorname{cl} F_k$ where F_k is $(\mathcal{P}_j)$closed, then in general, $\bigcap_{k=1}^n E_k \neq (\mathcal{P}_i) \operatorname{cl}(\bigcap_{k=1}^n F_k)$.

Definition 3.2. A collection $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of subsets of X is said to be pairwise dually open if each $U_\alpha \in \mathcal{U}$ is $(\mathcal{P}_i, \mathcal{P}_j)$dually open for some $i \in \{1, 2\}$ and for each $i \in \{1, 2\}$, $\mathcal{U} \cap \mathcal{P}_i$ contains a nonempty set. $\mathcal{U}$ is said to be a pairwise dually open cover of X if it covers X.

Definition 3.3. A bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is said to be strongly pairwise compact if each pairwise open cover $\mathcal{U}$ of X has a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\{(\mathcal{P}_j) \operatorname{int} V \mid V \in \mathcal{V} \cap \mathcal{P}_i,\ i \in \{1, 2\}\}$ covers X.

Clearly, a strongly pairwise compact space is pairwise compact. But converse is not true. For, we consider Example 3.1. Here we consider the pairwise open cover $\{(-\infty, a), [a, \infty)\}$ of R. Now $(\mathcal{P}_2) \operatorname{int}(-\infty, a) = (-\infty, a)$ and $(\mathcal{P}_1) \operatorname{int}[a, \infty) = (a, \infty)$. So $\{(\mathcal{P}_2) \operatorname{int}(-\infty, a), (\mathcal{P}_1) \operatorname{int}[a, \infty)\}$ is not a cover of R. The bitopological space of Example 2.5 is strongly pairwise compact.

Definition 3.4. A cover $\mathcal{V}$ of X consisting of $(\mathcal{P}_1)$ or $(\mathcal{P}_2)$open sets is said to be a step refinement of a pairwise open cover $\mathcal{U}$ of X if each $(\mathcal{P}_j)$open set of $\mathcal{V}$ is contained in some $(\mathcal{P}_i)$open set of $\mathcal{U}$.

In the above definition, $\mathcal{V}$ may not be a pairwise open cover of X.

On a strongly pairwise compact space, each pairwise open cover has a finite step refinement.

Theorem 3.1. *If X is strongly pairwise compact and $F \subset X$ is $(\mathcal{P}_j)$closed, then each $(\mathcal{P}_i)$open cover of F has a $(\mathcal{P}_j)$open finite subcover.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a $(\mathcal{P}_i)$open cover of F. Hence $\mathcal{U} \cup \{X - F\}$ is a pairwise open cover of X. So there exists a finite subcover $\mathcal{V}$ of $\mathcal{U} \cup \{X - F\}$ such that $\bigcup\{(\mathcal{P}_j) \operatorname{int} V \mid V \in \mathcal{V} \cap \mathcal{P}_i,\ i \in \{1, 2\}\} = X$. Now on noting $(\mathcal{P}_i) \operatorname{int}(X - F) \subset X - F$, we may obtain the required $(\mathcal{P}_j)$open finite subcover from $\mathcal{V}$. $\square$

Theorem 3.2. *For a bitopological space X, the following statements are equivalent:*

(i) *X is strongly pairwise compact.*
(ii) *Each pairwise open cover of X has a finite step refinement.*
(iii) *Each pairwise closed collection $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ of subsets of X with empty intersection has a finite subcollection $\mathcal{E}$ such that* $\bigcap\{(\mathcal{P}_j) \operatorname{cl} E \mid E \in \mathcal{E} \text{ whenever } X - E \in \mathcal{P}_i, i \in \{1, 2\}\} = \emptyset$.

Proof. (i)$\Rightarrow$(ii): Obvious.

(ii)$\Rightarrow$ (iii): Let $\{F_\alpha \mid \alpha \in A\}$ be a pairwise closed collection of subsets of X with $\bigcap_\alpha\{F_\alpha \mid \alpha \in A\} = \emptyset$. Then $\mathcal{G} = \{X - F_\alpha \mid \alpha \in A\}$ is a pairwise open cover of X. So using (ii), we obtain a finite step refinement $\mathcal{H} = \{H_k \mid k = 1, 2, \ldots, n\}$ of $\mathcal{G}$. For $H_k \in \mathcal{H} \cap \mathcal{P}_i$, there exists a $X - F_{\alpha_k} \in \mathcal{G} \cap \mathcal{P}_j$ such that $H_k \subset X - F_{\alpha_k}$.

Thus $(\mathcal{P}_i)\operatorname{cl} F_{\alpha_k} \subset X - H_k$. Also, $\bigcap_{k=1}^n (X - H_k) = \emptyset$ which in turn implies $\bigcap_{k=1}^n (\mathcal{P}_i)\operatorname{cl} F_{\alpha_k} = \emptyset$.

(iii)$\Rightarrow$(i): Let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ be a pairwise open cover of X. So $\mathcal{F} = \{X - U_\gamma \mid \gamma \in \Gamma\}$ is a pairwise closed collection of subsets of X with $\bigcap_\gamma (X - U_\gamma) = \emptyset$. Hence by (iii), there exists a finite subcollection $\mathcal{E} = \{X - U_{\gamma_k} \mid k = 1, 2, \ldots, n\}$ of $\mathcal{F}$ with $\bigcap_{k=1}^n \{(\mathcal{P}_j)\operatorname{cl}(X - U_{\gamma_k}) \mid U_{\gamma_k} \in \mathcal{P}_i,\ i \in \{1, 2\}\} = \emptyset$. Now $X - (\mathcal{P}_j)\operatorname{cl}(X - U_{\gamma_k}) = (\mathcal{P}_j)\operatorname{int} U_{\gamma_k}$. Thus X is strongly pairwise compact. □

Theorem 3.3. *If a bitopological space X is strongly pairwise compact, then each pairwise dually open cover of X has a finite subcover.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a pairwise dually open cover of X. For definiteness, we suppose that U_α is $(\mathcal{P}_i, \mathcal{P}_j)$dually open. Hence there exists a $(\mathcal{P}_j)$open set $G_{\gamma(\alpha)}$ such that $U_\alpha = (\mathcal{P}_i)\operatorname{int} G_{\gamma(\alpha)}$. Now it follows that $\mathcal{G} = \{G_{\gamma(\alpha)} \mid \alpha \in A\}$ is a pairwise open cover of X. Hence strong pairwise compactness of X ensures the existence of a finite subcollection $\{G_{\gamma(\alpha_k)} \mid k = 1, 2, \cdots, n\}$ of $\mathcal{G}$ such that $X = \bigcup_{i=1}^n (\mathcal{P}_i)\operatorname{int} G_{\gamma(\alpha_k)}$. Thus $\{U_{\alpha_k} \mid k = 1, 2, \ldots, n\}$ is a finite subcover of $\mathcal{U}$. □

Converse of Theorem 3.3 is also true if each pairwise open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of X is such that for each $U_\alpha \in \mathcal{U}$, $(\mathcal{P}_j)\operatorname{int} U_\alpha \neq \emptyset$ whenever U_α is $(\mathcal{P}_i)$open.

Theorem 3.4. *If G is open in both $\mathcal{P}_1$ and $\mathcal{P}_2$ senses and H is $(\mathcal{P}_i, \mathcal{P}_j)$dually open in a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$, then $G \cap H$ is $(\mathcal{P}_i, \mathcal{P}_j)$dually open.*

Proof. The proof is straightforward and hence omitted. □

Corollary 3.1. *The union of a set which is closed in both $\mathcal{P}_1$ and $\mathcal{P}_2$ senses, and a $(\mathcal{P}_i, \mathcal{P}_j)$dually closed set is $(\mathcal{P}_i, \mathcal{P}_j)$dually closed.*

Proof. Obvious. □

Theorem 3.5. *If X is strongly pairwise compact and $A \subset X$ is $(\mathcal{P}_i)$closed for some $i \in \{1, 2\}$ and $(\mathcal{P}_i)$open for each $i \in \{1, 2\}$, then A is strongly pairwise compact.*

Proof. Suppose $\mathcal{U}^{(A)} = \{U_\alpha^{(A)} \mid \alpha \in \Delta\}$ is a pairwise open cover of $(A, \mathcal{P}_{1A}, \mathcal{P}_{2A})$. For each $U_\alpha^{(A)} \in \mathcal{U}^{(A)} \cap \mathcal{P}_{iA}$, we obtain a $(\mathcal{P}_i)$open set $V_\alpha^{(X)}$ such that $U_\alpha^{(A)} = A \cap V_\alpha^{(X)}$. So $\mathcal{V} = \{V_\alpha^{(X)} \mid \alpha \in \Delta\} \cup \{X - A\}$ is a pairwise open cover of X. Since X is strongly pairwise compact, we get a finite subcollection $\mathcal{V}_1^{(X)}$ of $\mathcal{V}$ such that $X = \bigcup\{(\mathcal{P}_j)\operatorname{int} V \mid V \in \mathcal{V}_1^{(X)} \cap \mathcal{P}_i,\ i \in \{1, 2\}\}$. Then we obtain a finite subcollection $\mathcal{V}_1^{(A)} = \mathcal{V}_1^{(X)} - \{X - A\} = \{V_{\alpha_k}^{(X)} \mid \alpha_k \in \Delta,\ k = 1, 2, \ldots, m\}$ such that $A \subset \bigcup_{k=1}^m \{(\mathcal{P}_j)\operatorname{int} V_{\alpha_k}^{(X)} \mid V_{\alpha_k}^{(X)} \in \mathcal{V}_1^{(A)} \cap \mathcal{P}_i,\ k = 1, 2, \ldots, m\}$. Thus $\bigcup_{k=1}^m \{A \cap (\mathcal{P}_j)\operatorname{int} V_{\alpha_k}^{(X)} \mid V_{\alpha_k}^{(X)} \in \mathcal{V}_1^{(A)} \cap \mathcal{P}_i,\ k = 1, 2, \ldots, m\}$ is a cover of A. Now $U_{\alpha_k}^{(A)} = A \cap V_{\alpha_k}^{(X)} \supset A \cap (\mathcal{P}_j)\operatorname{int} V_{\alpha_k}^{(X)} = (\mathcal{P}_j)\operatorname{int}(A \cap V_{\alpha_k}^{(X)})$. Hence $(\mathcal{P}_j)\operatorname{int} U_{\alpha_k}^{(A)} \supset (\mathcal{P}_j)\operatorname{int}(A \cap V_{\alpha_k}^{(X)})$. Thus $\{U_{\alpha_k}^{(A)} \mid k = 1, 2, \ldots, m\}$ is a finite subcollection of $\mathcal{U}^{(A)}$ such that $\{(\mathcal{P}_j)\operatorname{int} U_{\alpha_k}^{(A)} \mid U_{\alpha_k}^{(A)} \in \mathcal{U}^{(A)} \cap \mathcal{P}_{iA},\ k = 1, 2, \ldots, m\}$ covers A. Hence A is strongly pairwise compact. □

THEOREM 3.6. *Strong pairwise compactness is preserved under continuous, open and onto mappings.*

PROOF. Let X and Y be the bitopological spaces $(X, \mathcal{P}_1, \mathcal{P}_2)$ and $(Y, \mathcal{Q}_1, \mathcal{Q}_2)$ respectively. Also, let a function $f : X \to Y$ be a continuous, open and onto mapping and X be strongly pairwise compact. Suppose $\mathcal{U}^{(Y)} = \{U_\alpha \mid \alpha \in A\}$ is a pairwise open cover of Y. Then using continuity of f, we see $\mathcal{U}^{(X)} = \{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is a pairwise open cover of X. Since X is strongly pairwise compact, there exists a finite subcollection $\mathcal{V}^{(X)} = \{f^{-1}(U_{\alpha_k}) \mid \alpha_k \in A,\ k = 1, 2, \dots, n\}$ such that $\{(\mathcal{P}_j)\operatorname{int}(f^{-1}(U_{\alpha_k})) \mid U_{\alpha_k} \in \mathcal{U}^{(Y)} \cap \mathcal{Q}_i,\ k = 1, 2, \dots, n\}$ covers X. Now we have

$$\begin{aligned}
Y &= f(X) \quad \text{(since } f \text{ is onto)}\\
&= f\Big(\bigcup_{k=1}^{n} \Big\{(\mathcal{P}_j)\operatorname{int}(f^{-1}(U_{\alpha_k})) \mid f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i,\ i \in \{1,2\}\Big\}\Big)\\
&= \bigcup_{k=1}^{n} \Big\{f\big((\mathcal{P}_j)\operatorname{int}(f^{-1}(U_{\alpha_k}))\big) \mid f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i,\ i \in \{1,2\}\Big\}\\
&\subset \bigcup_{k=1}^{n} \Big\{(\mathcal{Q}_j)\operatorname{int}\big(f(f^{-1}(U_{\alpha_k}))\big) \mid f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i,\ i \in \{1,2\}\Big\}\\
&= \bigcup_{k=1}^{n} \Big\{(\mathcal{Q}_j)\operatorname{int} U_{\alpha_k} \mid f^{-1}(U_{\alpha_k}) \in \mathcal{V}^{(X)} \cap \mathcal{P}_i,\ i \in \{1,2\}\Big\} \quad \text{(since } f \text{ is onto).}
\end{aligned}$$

Hence Y is strongly pairwise compact. □

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