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# ON THE EXTREME POINTS AND STRONGLY EXTREME POINTS IN KÖTHE–BOCHNER SPACES

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ABSTRACT. We give the necessary conditions of extreme points and strongly extreme points in the unit ball of Köthe–Bochner spaces. The conditions have been shown to be sufficient earlier.

## 1. Introduction

Let $(\Omega, \Sigma, \mu)$ denote a measure space with $\sigma$-finite and complete measure $\mu$ and $L^0 = L^0(\Omega)$ denote the space of all (equivalence classes of) $\Sigma$-measurable real-valued functions, equipped with the topology of convergence in measure on $\mu$-finite sets. In what follows, if $x, y \in L^0$, then $x \leqslant y$ means $x(t) \leqslant y(t)$ $\mu$-almost everywhere in $\Omega$. A Banach subspace $E$ of $L^0$ is said be a Köthe function space (over $(\Omega, \Sigma, \mu)$) if

(i) $|x| \leqslant |y|$, $x \in L^0$, $y \in E$ imply $x \in E$ and $\|x\| \leqslant \|y\|$.

(ii) $\operatorname{supp} E := \bigcup\{\operatorname{supp} x : x \in E\} = \Omega$, where $\operatorname{supp} x = \{t \in \Omega : x(t) \neq 0\}$.

Köthe spaces are Banach lattices and detailed studies of Banach lattices and Köthe spaces may be found in [**4**]. A Köthe function space $E$ is said to be order continuous provided $x_n \searrow 0$ implies $\|x_n\| \to 0$. $E$ is said to be strictly monotone if $x \leqslant y$ and $x \neq y$ implies $\|x\| < \|y\|$.

In this paper we always denote by $E$ a Köthe function space on $(\Omega, \Sigma, \mu)$, $X$ a Banach space. By $E(X)$ we denote the Banach space of all equivalence classes of strongly measurable function $x : \Omega \to X$ such that $\tilde{x} = \|x(\cdot)\|_X \in E$ equipped with the norm $|||x||| = \|\tilde{x}\|_E$.

The space $(E(X), |||\cdot|||)$ is called the Köthe–Bochner space. The most important class of Köthe–Bochner function spaces $E(X)$ are the Lebesgue–Bochner spaces $L^p(X)$, $(1 \leqslant p < \infty)$ and their generalization the Orlicz–Bochner spaces $L^{\phi}(X)$.

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They have been studied by many authors. The geometric properties of the Köthe–Bochner spaces have been also studied by many authors.

For any Banach space $X$, let $S(X)$ and $B(X)$ stand for the unit sphere and unit ball of $X$, respectively. An $x \in S(X)$ is called an extreme point of $B(X)$ if $2x = y+z$, $y, z \in B(X)$ implies $y = z$. Denote the set of all extreme points of $B(X)$ by $\operatorname{Ext} B(X)$. An $X$ is called rotund (R) provided that $S(X) = \operatorname{Ext} B(X)$. An $x \in S(X)$ is said to be strongly extreme point of $B(X)$ if for every sequence $\{x_n\}$ in $X$, $\lim_n \|x_n \pm x\| = 1$ implies $\lim_n x_n = 0$, or equivalently, for $\{x_n\}, \{y_n\} \subset X$, $\lim_n \|x_n\| = \lim_n \|y_n\| = 1$ and $2x = x_n + y_n$ $(n = 1, 2, \dots)$ implies $\lim_n \|x_n - y_n\| = 0$. An $X$ is called mid-point locally uniformly rotund (MLUR) if the set of all strongly extreme points of $B(X)$ is equal to $S(X)$. A point $x \in S(X)$ is said to be a point of local uniform rotundity (or a LUR-point) if for arbitrary sequence $\{x_n\} \subset B(X)$, $\lim_n \|x_n + x\| = 2$ and $\lim_n \|x_n\| = 1$ imply $\lim_n x_n = x$. A space $X$ is locally uniformly uniformly rotund (LUR) if every point of $S(X)$ is a LUR-point. Clearly, (LUR) $\Rightarrow$ (MLUR) $\Rightarrow$ (R).

In 2006, Ren, Feng and Wu [**5**] obtained the sufficient conditions for extreme points of $B(E(X))$. The sufficient or necessary conditions for strongly extreme points of $B(E(X))$ were discussed by Hudzik and Mastylo [**2**] and Cerdá, Hudzik, and Mastylo [**1**]. We list the results here:

Proposition 1.1. [**5**] *Let $E$ be a strictly monotone Köthe function space and $X$ a real Banach space. Assume that $x \in S(E(X))$ have the following properties:*

(a) *$\tilde{x} = \|x(\cdot)\|_X$ is an extreme point of $B(X)$.*

(b) *$\frac{x(t)}{\|x(t)\|_X}$ is an extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*

*Then $x$ is an extreme point of $B(E(X))$.*

Proposition 1.2. [**2**] *Let $E$ be a locally uniformly rotund Köthe function space and $X$ a real Banach space. If $x \in S(E(X))$ is such that $\frac{x(t)}{\|x(t)\|_X}$ is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$, then $x$ is a strongly extreme point of $B(E(X))$.*

Proposition 1.3. [**2**] *Let $E$ be as in Proposition 1.2 and $X$ a separable Banach space. If $x \in S(E(X))$ is a strongly extreme point, then $\frac{x(t)}{\|x(t)\|_X}$ is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*

Proposition 1.4. [**1**] *Let $E$ be an order continuous Banach function lattice and $X$ a real Banach space. Assume that $x \in S(E(X))$ have the following properties:*

(a) *$\tilde{x} = \|x(\cdot)\|_X$ is a strongly extreme point of $B(X)$.*

(b) *$\frac{x(t)}{\|x(t)\|_X}$ is a strongly extreme point of $B(X)$ for a.e. $t \in \operatorname{supp} x$.*

*Then $x$ is a strongly extreme point of $B(E(X))$.*

In this paper, we will show that the two properties in Proposition 1 and Proposition 4 are exactly the necessary conditions for $x \in S(E(X))$ to be extreme point and strongly extreme point of $B(E(X))$, respectively.

## 2. Main Results

Theorem 2.1. *Let $E$ be a Köthe function space and $X$ a real Banach space. Assume that $x \in S(E(X))$ is an extreme point of $B(E(X))$. Then we have:*

(a) $\tilde{x} = \|x(\cdot)\|_X$ *is an extreme point of* $B(X)$.

(b) $\frac{x(t)}{\|x(t)\|_X}$ *is an extreme point of* $B(X)$ *for a.e.* $t \in \operatorname{supp} x$.

Proof. (a) Suppose that $\|x(\cdot)\|_X$ is not an extreme point of $B(E)$. Then there exist two real measurable functions $y(t)$, $z(t)$, such that $y(\cdot)$, $z(\cdot) \in S(E)$ and $2\|x(t)\|_X = y(t) + z(t)$ but $y \neq z$. Define

$$g(t) = \begin{cases} \frac{x(t)}{\|x(t)\|_X}, & t \in \operatorname{supp} x, \\ x_0, & \text{otherwise}, \end{cases}$$

where $x_0$ is a fixed point in $S(X)$. Obviously $g(\cdot)$ is measurable. Let $f_1(t) = y(t)g(t)$, $f_2(t) = z(t)g(t)$. Then $|||f_1||| = \big\|\|y(t)g(t)\|_X\big\|_E = \|y(t)\|_E = 1$.

Similarly, $|||f_2||| = 1$. Moreover, $f_1(t) + f_2(t) = (y(t) + z(t))g(t) = 2x(t)$ but $f_1 \neq f_2$. This contradicts the assumption that $x$ is an extreme point.

(b) Observe that $\frac{x(t)}{\|x(t)\|_X} \in S(X)$. Let $A = \big\{t : \frac{x(t)}{\|x(t)\|_X} \subsetneq \operatorname{Ext} S(X)\big\}$. Suppose that $\mu A > 0$. Then there are $y_1, z_1 \in S(X)$ such that $2\frac{x(t)}{\|x(t)\|_X} = y_1(t) + z_1(t)$, $(\in A)$ but $y_1 \neq z_1$. Define

$$(y(t), z(t)) = \begin{cases} (x(t), x(t)), & t \in \Omega \setminus A, \\ \big(\|x(t)\|_X y_1(t), \|x(t)\|_X z_1(t)\big), & t \in A. \end{cases}$$

Therefore, $2x(t) = y(t) + z(t)$ a.e. $t \in \Omega$ but $z \neq y$. Moreover,

$$\begin{aligned} |||y||| &= \big\|\|y(t)\|_X\big\|_E \\ &= \big\|\|y(t)\|_X \chi_A + \|y(t)\|_X \chi_{\Omega \setminus A}\big\|_E \\ &= \big\|\big\|\|x(t)\|_X y_1(t)\big\|_X \chi_A + \|x(t)\|_X \chi_{\Omega \setminus A}\big\|_E \\ &= \big\|\|x(t)\|_X \chi_A + \|x(t)\|_X \chi_{\Omega \setminus A}\big\|_E \\ &= \big\|\|x(t)\|_X\big\|_E = |||x||| = 1. \end{aligned}$$

Similarly, $|||z||| = 1$. This means that $x$ is not an extreme point of $B(E(X))$, a contradiction. So $\mu A = 0$, which finishes the proof. □

Theorem 2.2. *Let $E$ be a Köthe function space and $X$ a real Banach space. Assume that $x$ is a strongly extreme point of $B(E(X))$. Then*

(a) $\|x(\cdot)\|_X$ *is a strongly extreme point of* $B(E)$;

(b) $\frac{x(t)}{\|x(t)\|_X}$ *is a strongly extreme point of* $B(X)$ *a.e.* $t \in \operatorname{supp} x$.

Proof. (a) Assume $\lim_{n\to\infty} \big\|\|x(t)\|_X \pm x_n(t)\big\|_E = 1$, where $\{x_n\}$ is a sequence of real valued functions. Define

$$g(t) = \begin{cases} \frac{x(t)}{\|x(t)\|_X}, & t \in \operatorname{supp} x, \\ x_0, & \text{otherwise}, \end{cases}$$

where $x_0 \in S(X)$ is a fixed point. Then $g(t)$ is measurable. Moreover, $\|g(t)\|_X \equiv 1$ and $\|x(t)\|_X g(t) \equiv x(t)$ for every $t \in \Omega$. Observe that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \big\| \|x(t)\|_X \pm x_n(t) \big\|_E \\ &= \lim_{n\to\infty} \big|\big\| \big[ \|x(t)\|_X \pm x_n(t) \big] g(t) \big\|\big| \\ &= \lim_{n\to\infty} |\|x(t) \pm x_n(t) g(t) \||\,. \end{aligned}$$

Since $x$ is a strongly extreme point in $B(E(X))$, we have

$$\lim_{n\to\infty} |\|x_n(t)g(t)\|| = \lim_{n\to\infty} \|x_n(t)\|_E = 0$$

which implies that $\|x(\cdot)\|_X$ is a strongly extreme point of $B(E)$.

(b) Denote by $z(t) = \frac{x(t)}{\|x(t)\|_X}$. Define

$$A = \{t \in \operatorname{supp} x : z(t) \text{ is not a strongly extreme point of } B(X)\}.$$

We should prove $\mu A = 0$. Suppose it is not the case, e.g., $\mu A > 0$. Define

$$\begin{aligned} A_{mk} = \big\{ t \in A : &\text{ there exists a sequence } \{z_n(t)\} \text{ such that} \\ &1 - \tfrac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \tfrac{1}{n},\ \|z_n(t)\|_X \geqslant \tfrac{1}{m} \text{ for } n \geqslant k \big\}. \end{aligned}$$

Then $\bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} A_{mk} = A$. There are $m_0, k_0 \geqslant 1$ such that $\mu(A_{m_0 k_0}) > 0$. Let $\delta = \frac{1}{m_0}$. We deduce that in the set $B = A_{m_0 k_0}$, if $n \geqslant k_0$, then $1 - \frac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \frac{1}{n}$ and $\|z_n(t)\|_X \geqslant \delta > 0$.

However the functions $z_n(\cdot)$ need not be measurable, so we have to make some modification.

$z(\cdot)$ is measurable, it is almost a uniform limit of a sequence of countably valued functions. That is, there is a subset $C \subset B$ with $\mu C = \mu B$ and a sequence of countably valued functions $\{y_n(\cdot)\}$ such that $y_n(t) \to z(t)$ uniformly on $C$. There exist a natural number $k_1$ satisfying $\|y_n(t) - z(t)\|_X < \frac{1}{n}$ whenever $n > k_1$. Denote by $y_n(t) = \sum_{i=1}^{\infty} e_{in} \chi_{C_{in}}, (\mu(C_{in}) > 0)$. $y_n$ is single valued on $C_{in}$, i.e., $y_n(t) \equiv e_{in} (t \in C_{in})$. Let $k = \max\{k_0, k_1\}$. When $n > k$, we have $1 - \frac{1}{n} < \|z_n(t) \pm z(t)\|_X < 1 + \frac{1}{n}$, and $\|z_n(t)\|_X \geqslant \delta$. Select a point $t_{i_0} \in C_{in}$, then define $x_n(t) = \sum_{i=1}^{\infty} z_n(t_{i_0}) \chi_{C_{in}}$. Then $\{x_n(\cdot)\}$ is a sequence of measurable functions since they are countably valued.

For every $t \in C_{in}$ $(i = 1, 2, \dots)$, if $n > k$, then

$$\begin{aligned} \|x_n(t) \pm z(t)\|_X &\geqslant \|x_n(t) \pm z(t_{i_0})\|_X - \|z(t_{i_0}) - y_n(t_{i_0})\|_X - \|y_n(t_{i_0}) - z(t)\|_X \\ &= \|z_n(t_{i_0}) \pm z(t_{i_0})\|_X - \|z(t_{i_0}) - y_n(t_{i_0})\|_X - \|y_n(t) - z(t)\|_X \\ &> 1 - \tfrac{1}{n} - \tfrac{1}{n} - \tfrac{1}{n} = 1 - \tfrac{3}{n} \to 1. \end{aligned}$$

Analogously, $\|x_n(t) \pm z(t)\|_X \leqslant 1 + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = 1 + \frac{3}{n} \to 1$. Moreover, $\|x_n(t)\|_X = \|z_n(t_{i_0})\|_X \geqslant \delta$.

Now we define a sequence of measurable functions on $G$ as follows:

$$x'_n(t) = \begin{cases} 0, & t \in G \setminus C \\ \|x(t)\|_X x_n(t), & t \in C. \end{cases}$$

Thus, for $n > k$,

$$\begin{aligned}\big|\|x_n'(t)\|_X\big| &= \big\|\|x_n'(t)\|_X\big\|_E\\ &= \big\|\|\|x(t)\|_X x_n(t)\|_X\chi_C\big\|_E = \big\|\|x(t)\|_X\|x_n(t)\|_X\chi_C\big\|_E\\ &= \big\|\|x(t)\|_X\chi_C\|x_n(t)\|_X\big\|_E \geqslant \delta\big\|\|x(t)\|_X\chi_C\big\|_E = \delta' > 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\|x_n'(t) \pm x(t)\|\| &= \big\|\|x_n'(t) \pm x(t)\|_X\big\|_E\\ &= \big\|\|x(t)\|_X\chi_{G\setminus C} + \big\|x(t) \pm \|x(t)\|_X x_n(t)\big\|_X\chi_C\big\|_E\\ &= \Big\|\|x(t)\|_X\chi_{G\setminus C} + \|x(t)\|_X\Big\|\frac{x(t)}{\|x(t)\|_X} \pm x_n(t)\Big\|_X\chi_C\Big\|_E\\ &\geqslant \big\|\|x(t)\|_X\chi_{G\setminus C} + (1-\tfrac{3}{n})\|x(t)\|_X\chi_C\big\|_E\\ &> (1-\tfrac{3}{n})\big\|\|x(t)\|_X\big\|_E = (1-\tfrac{3}{n})\ \|\|\ x(t)\|\| = 1-\tfrac{3}{n} \to 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\|x_n'(t) \pm x(t)\|\| &\leqslant \big\|\|x(t)\|_X\chi_{G\setminus C} + (1+\tfrac{3}{n})\|x(t)\|_X\chi_C\big\|_E\\ &\leqslant (1+\tfrac{3}{n})\big\|\|x(t)\|_X\big\|_E = (1+\tfrac{3}{n})\ \|\|\ x(t)\|\| = 1+\tfrac{3}{n} \to 1.\end{aligned}$$

This contradicts the fact that $x$ is a strongly extreme point of $B(E(X))$. The proof is completed. □

## References


1. J. Cerda, H. Hudzik, M. Mastylo, *Geometric properties of Köthe–Bocher spaces*, Math. Proc. Camb. Philos. Soc. **120**(3) (1996), 521–533.
2. H. Hudzik, M. Mastylo, *Strongly extreme points in Köthe–Bochner spaces*, Rocky Mt. J. Math. **23** (1993), 899–909.
3. M. Khandaqji, F. Awawdeh, *Uniform convexity of Köthe–Bochner function spaces*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **24** (2008), 385–390.
4. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces* (I), Springer, Berlin, 1977.
5. L. W. Ren, G. C. Feng, C. X. Wu, *Rotundity in Köthe spaces*, Adv. Math., Beijing **35**(3) (2006), 350–360.



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