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# ON ROOTS OF POLYNOMIALS WITH POSITIVE COEFFICIENTS

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Abstract. Let $\alpha$ be an algebraic number with no nonnegative conjugates over the field of the rationals. Settling a recent conjecture of Kuba, Dubickas proved that the number $\alpha$ is a root of a polynomial, say $P$, with positive rational coefficients. We give in this note an upper bound for the degree of $P$ in terms of the discriminant, the degree and the Mahler measure of $\alpha$; this answers a question of Dubickas.

## 1. Introduction

An element $\alpha$ of the set $\mathbb{C}$ of complex numbers is called an algebraic number if it is a root of a nonzero polynomial with coefficients in the field of the rationals $\mathbb{Q}$. Among nonzero elements $P$ of the ring $\mathbb{Q}[x]$ and satisfying the condition $P(\alpha) = 0$, there is only one monic polynomial having the smallest possible degree; this polynomial is called the minimal polynomial of $\alpha$ and is usually noted $\mathrm{Min}_\alpha$. The roots of $\mathrm{Min}_\alpha$ are the conjugates of $\alpha$, and the degree of $\alpha$ is the degree of $\mathrm{Min}_\alpha$. In these pages, the notions of minimal polynomial, conjugates and degree of an algebraic number are considered over $\mathbb{Q}$.

In his study of some classes of algebraic numbers on the unit circle, Kuba [3] considered the roots of polynomials with positive rational coefficients. A complex number is said to be *positively algebraic* if it is a root of a polynomial, say $P$, with positive rational coefficients [3]. In fact (as it was signaled in [2] and [3]) we may replace in this last definition the word *positive* by the sentence *nonnegative and such that* $P(0) \neq 0$, because the coefficients of the polynomial $P(x)(1 + x + \cdots + x^{\deg(P)})$, where $\deg(P)$ is the degree of $P$, are positive when the coefficients of $P$ are nonnegative and $P(0) > 0$. Clearly, a positively algebraic number is an algebraic number, and none of its conjugates is a nonnegative real number. Kuba conjectured that the converse of the last proposition is true, and verified this conjecture for some particular cases, especially when $\alpha$ is quadratic or when

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the Galois group of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is isomorphic to the symmetric group $S_{\deg(\mathrm{Min}_\alpha)}$ [3]. The question of Kuba did not remain open for a long time, since Dubickas has shown that "*an algebraic number with no nonnegative conjugates is a root of a polynomial, say again $P$, with positive rational coefficients*" [2]. At the end of his paper, Dubickas has remarked that the proof of the last mentioned proposition does not provide any estimation for the degree of $P$. In fact, replacing the arguments of the distribution modulo 1, by a simple geometrical argument we obtain the following result.

THEOREM 1.1. *Let $\alpha$ be an algebraic number with no nonnegative conjugates. Then, there is a polynomial with positive rational coefficients, vanishing at $\alpha$ and with degree less than*

$$\frac{2d\pi}{\arcsin\left(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\right)},$$

*where $d$, $\Delta$ and $M$ are the degree, the discriminant, and the Mahler measure of $\alpha$, respectively.*

Recall that if

$$\mathrm{Min}_\alpha(x) = \prod_{1 \leqslant j \leqslant d} (x - \alpha_j) = x^d + \frac{a_{d-1}}{b_{d-1}} x^{d-1} + \cdots + \frac{a_0}{b_0},$$

where the rational integers $a_0, \ldots, a_{d-1}$, and the positive rational integers $b_0, \ldots, b_{d-1}$ are so that the fractions $\frac{a_{d-1}}{b_{d-1}}, \ldots, \frac{a_0}{b_0}$ are irreducible, then

$$\Delta = \mathrm{lcm}(b_0, \ldots, b_{d-1})^{2d-2} \prod_{1 \leqslant j < k \leqslant d} (\alpha_j - \alpha_k)^2$$

and

$$M = \mathrm{lcm}(b_0, \ldots, b_{d-1}) \prod_{1 \leqslant j \leqslant d} \max\{1, |\alpha_j|\}.$$

The proof of Theorem 1.1 appears in the last section and is based on two auxiliary results, due essentially to Dubickas, and explained in the next section.

## 2. Two lemmas

The following result is an improvement of Lemma 2 of [2].

LEMMA 2.1. *Let $\omega = |\omega| e^{i\theta} \in \mathbb{C} - \{0\}$, where $i^2 = -1$, $\theta \in \left[\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}\right[$ and $n$ is a nonnegative rational integer. Then, there is $T \in \mathbb{Q}[x]$, with degree $2^{n+2} - 3$ and such that the coefficients of the polynomial $(x - \omega)(x - \bar{\omega})T(x)$, where $\bar{\omega}$ is the complex conjugate of $\omega$, are positive.*

PROOF. The scheme of the proof is identical to the one of Lemma 2 of [2] with minor modifications. We prefer to give some details of this proof. To simplify the notation set $m = 2^n$. Then, $\omega^m = |\omega|^m e^{im\theta}$, $\frac{\pi}{2} \leqslant m\theta < \pi$, $|\omega|^{2m} > 0$, $\cos m\theta \leqslant 0$ and the coefficients of the polynomial

$$(x^m - \omega^m)(x^m - \bar{\omega}^m) = x^{2m} - 2|\omega|^m (\cos m\theta) x^m + |\omega|^{2m},$$

are nonnegative real numbers. A simple calculation shows that the coefficients of

$$(x^m - \omega^m)(x^m - \bar{\omega}^m) \sum_{k=0}^{2m-1} x^k$$

are positive. For $z \in \mathbb{C}$ let

$$T_z(x) := \frac{(x^m - z^m)(x^m - \bar{z}^m)}{(x-z)(x-\bar{z})} \sum_{k=0}^{2m-1} x^k.$$

Then, the coefficients of the polynomial $T_z$ are real,

$$T_z(x) = (x^{m-1} + zx^{m-2} + \cdots + z^{m-1})(x^{m-1} + \bar{z}x^{m-2} + \cdots + \bar{z}^{m-1}) \sum_{k=0}^{2m-1} x^k$$

and $\deg(T_z) = 2^{n+2} - 3$. For each $k \in \{0, \dots, 2^{n+2}-3\}$ let $c_k(z)$ be the "coefficient" function defined by the identity

$$T_z(x) = \sum_{0 \leqslant k \leqslant 2^{n+2}-3} c_k(z) x^k.$$

Since the complex conjugation is a continuous map on $\mathbb{C}$, then so is each function $c_k$; in particular we have $\lim_{z \to \omega} c_k(z) = c_k(\omega)$, and the coefficients of

$$(x-\omega)(x-\bar{\omega})T_z(x)$$

are positive when $z$ is close to $\omega$. Finally, as the set $\mathbb{Q}(i) = \{a + ib, (a,b) \in \mathbb{Q}^2\}$ is dense in $\mathbb{C}$, and $T_z(x) \in \mathbb{Q}[x]$ when $z \in \mathbb{Q}(i)$, it is enough to choose $z$ in an appropriate neighborhood of $\omega$ in $\mathbb{C}$ which meets $\mathbb{Q}(i)$, and $T$ the corresponding polynomial $T_z$. $\square$

The following lemma is a corollary of a theorem of [1]. In [4], Mignotte obtained a slight improvement of this result.

LEMMA 2.2. *Let $\omega$ be a nonreal algebraic number. Then*

$$|\omega - \bar{\omega}| \geqslant \frac{2|\omega|\,|\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}} M^{d-1}},$$

*where $d$, $\Delta$ and $M$ are the degree, the discriminant and the Mahler measure of $\omega$, respectively.*

PROOF. See [1]. $\square$

## 3. Proof of Theorem 1.1

Let $\{\alpha_1, \dots, \alpha_r\}$ and $\{\alpha_{r+1}, \overline{\alpha_{r+1}}, \dots, \alpha_{r+s}, \overline{\alpha_{r+s}}\}$ be a partition of the set of the conjugates of $\alpha$, where the first subset is real (if it is not empty) and the second one does not meet the real line. It is clear that $r \geqslant 0$, $s \geqslant 0$, $r + 2s = d$ and the numbers $\alpha_1, \dots, \alpha_r$ are negative. Let

$$\mathrm{Min}_\alpha(x) = (x - \alpha_1) \dots (x - \alpha_r) \prod_{j=1}^{s} (x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}),$$

where $\alpha_{j+r} = |\alpha_{j+r}|\, e^{i\theta_j}$ for $j \in \{1, \ldots, s\}$, and $0 < \theta := \theta_1 \leqslant \ldots \leqslant \theta_s < \pi$ when $s \geqslant 1$. We want to show that there is a multiple, say $Q$, of $\mathrm{Min}_\alpha$ with positive rational coefficients and degree at most $C_\alpha d$, where

$$C_\alpha = \frac{2\pi}{\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr)} - \frac{1}{2} < \frac{2\pi}{\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr)}.$$

As a finite product of polynomials with nonnegative coefficients is also a polynomial with nonnegative coefficients, we obtain immediately that the coefficients of $\mathrm{Min}_\alpha$ are nonnegative when $\theta \geqslant \frac{\pi}{2}$, because

$$(x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}) = x^2 - 2|\alpha_{j+r}|(\cos\theta_{j+r})x + |\alpha_{j+r}|^2$$

and $\cos\theta_{j+r} \leqslant 0$ for each $j \in \{1, \ldots, s\}$. It follows that the polynomial

$$Q(x) := \mathrm{Min}_\alpha(x)(1 + x + \cdots + x^{d-1}),$$

has positive rational coefficients, and satisfy $Q(\alpha) = 0$. From the trivial inequality $\arcsin\bigl(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\bigr) \leqslant \frac{\pi}{2}$, we have $C_\alpha d \geqslant \frac{7}{2}d > \deg(Q) = 2d - 1$, and so Theorem 1.1 is true. Now, suppose $\theta < \frac{\pi}{2}$, and let $t$ be the largest rational integer satisfying $\theta_t < \frac{\pi}{2}$. For each $j \in \{1, \ldots, t\}$ let $n_j$ be the largest rational integer satisfying

$$\theta_j < \frac{\pi}{2^{n_j}}. \tag{3.1}$$

Then

$$n := n_1 \geqslant \cdots \geqslant n_t \tag{3.2}$$

and $\theta_j \geqslant \frac{\pi}{2^{n_j+1}}$, for each $j \in \{1, \ldots, t\}$. Lemma 2.1 asserts that there is $T_j(x) \in \mathbb{Q}[x]$ with degree $2^{n_j+2} - 3$ and such that the coefficients of the polynomial $(x - \alpha_{j+r}) \times (x - \overline{\alpha_{j+r}})T_j(x)$ are positive. Set

$$Q(x) := (x - \alpha_1)\cdots(x - \alpha_r)\Biggl(\prod_{j=1}^{t}(x - \alpha_{j+r})(x - \overline{\alpha_{j+r}})T_j(x)\Biggr)\prod_{j=t+1}^{s}(x - \alpha_{j+r})(x - \overline{\alpha_{j+r}}).$$

Then, the coefficients of $Q$ are positive, $Q(x) = \mathrm{Min}_\alpha(x)\prod_{j=1}^{t} T_j(x) \in \mathbb{Q}[x]$, and $\deg(Q) = d + \sum_{j=1}^{t}(2^{n_j+2} - 3)$. It follows by the relation (3.2) that

$$\deg(Q) \leqslant d + \sum_{j=1}^{t}(2^{n+2} - 3) \leqslant d + \sum_{j=1}^{s}(2^{n+2} - 3) \leqslant 2^{n+1}d - \frac{d}{2}, \tag{3.3}$$

since $t \leqslant s \leqslant \frac{d}{2}$. By Lemma 2.2 we have

$$|\alpha_{1+r} - \overline{\alpha_{1+r}}| = 2|\alpha_{1+r}|\sin\theta \geqslant \frac{2|\alpha_{1+r}|\,|\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}} M^{d-1}},$$

and so

$$\theta \geqslant \arcsin\Biggl(\frac{|\Delta|^{\frac{1}{2}}}{d^{\frac{d+3}{2}} M^{d-1}}\Biggr).$$

The last inequality together with the relation (3.1) (with $j = 1$) yield

$$2^{n+1} < \frac{2\pi}{\arcsin\left(|\Delta|^{\frac{1}{2}} d^{-\frac{d+3}{2}} M^{-d+1}\right)},$$

and the result follows immediately by (3.3).

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